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Enumerative combinatorics, vol. I, by Richard P. Stanley. Wadsworth and Brooks/Cole, Monterey, 1986, xi + 306 pp., $42.95. ISBN 0-534-06546-5

Professor Stanley begins his preface with a brief suggestion of what enumerative combinatorics is all about:

> Enumerative combinatorics is concerned with counting the number of elements of a finite set S. This definition, as it stands, tells us little about the subject since virtually any mathematical problem can be cast in these terms. In a genuine enumerative problem, the elements of S will usually have a rather simple combinatorial definition and very little additional structure. It will be clear that S has many elements, and the main issue will be to count (or estimate) them all and not, for example, to find a particular element.

While this is surely accurate, to the outsider it is probably unclear why anyone else cares about this topic, let alone why it should be a center of widespread mathematical research. Of course, mathematical research is often motivated by the intrinsic appeal of problems, and this is surely the case for enumerative combinatorics. However, it should be remarked that there is a large consumer market for enumerative combinatorics also. For example, the mathematical analysis of the running time of computer algorithms is precisely a problem in enumerative combinatorics. Indeed the books on analysis of algorithms by D. Knuth [**8**], D. Knuth and R. Greene [**9**] and P. Purdom and C. Brown [**12**] are really books devoted to numerous aspects of enumerative combinatorics. In physics, we find that the problems of statistical mechanics are in the final analysis asymptotic estimates for the enumerations of combinatorial structures. In this regard, the recent work on exactly solved models has relied heavily on enumerative combinatorics [**2**, Chapter 13], [**15**]. Finally it is not uncommon to find interesting enumerative combinatorics arising in many branches of mathematics from group representation theory [**10**] to analysis [**1**].

Perhaps the best way to view modern combinatorics is to return to the observations of one of its pioneers, Major Percy Alexander MacMahon. The following is taken from MacMahon's 1901 Presidential Address to the British Association [**11**, pp. 889–890]: MacMahon is vigorously describing how undeveloped areas of mathematics (he is surely thinking of combinatorics) profit from and, in turn, enrich the classical major branches of mathematics.

> A subject of study may acquire the reputation of being narrow either because it has for some reason or other not attracted workers, and is in reality virgin soil only awaiting the arrival of a husbandman with the necessary skill; or because it is an extremely difficult

subject which has resisted previous attempts to elucidate it. ...Though the subject may be a narrow one, it by no means follows that the appropriate or possible methods of research are prescribed within narrow limits. I will instance the Theory of Numbers which, in comparatively recent times, was a subject of small extent and of restricted application to other branches of science. The problems that presented themselves naturally, or were brought into prominence by the imaginations of great intellects, were fraught with difficulty. There seemed to be an absence, partial or complete, of the law and order that investigators had been accustomed to find in the wide realm of continuous quantity. The country as explored was found to be full of pitfalls for the unwary. Many a lesson concerning the danger of hasty generalisation had to be learnt and taken to heart. Many a false step had to be retraced. Many a road which a first reconnaissance had shown to be straight for a short distance, was found on further exploration, to suddenly change its direction and to break up into a number of paths which wandered in a fitful manner in country of increasing natural difficulty. There were few vanishing points in the perspective. Few, also, and insignificant were the peaks from which a general view could be gathered of any considerable portion of the country. The surveying instruments were inadequate to cope with the physical character of the land. The province of the Theory of Numbers was forbidding. Many a man returned empty-handed and baffled from the pursuit, or else was drawn into the vortex of a kind of Maelström and had his heart crushed out of him. But early in the last century the dawn of a brighter day was breaking. A combination of great intellects—Legendre, Gauss, Eisenstein, Stephen Smith, &c.—succeeded in adapting some of the existing instruments of research in continuous quantity to effective use in discontinuous quantity. These adaptations are of so difficult and ingenious a nature that they are to-day, at the commencement of a new century, the wonder and, I may add, the delight of beholders. True it is that the beholders are few. To attain to the point of vantage is an arduous task demanding alike devotion and courage. I am reminded, to take a geographical analogy, of the Hamilton Falls, near Hamilton Inlet, in Labrador. I have been informed that to obtain a view of this wonderful natural feature demands so much time and intrepidity, and necessitates so many collateral arrangements, that a few years ago only nine white men had feasted their eyes on falls which are finer than those of Niagara. The labours of the mathematicians named have resulted in the formation of a large body of doctrine in the Theory of Numbers. Much that, to the superficial observer, appears to lie on the threshold of the subject is found to be deeply set in it and to be only capable of attack after problems at first sight much more complicated have been solved. The mirage that distorted the scenery and obscured the perspective

> has been to some extent dissipated; certain vanishing points have been ascertained; certain elevated spots giving extensive views have been either found or constructed. The point I wish to urge is, that these specialists in the Theory of Numbers were successful for the reason that they were not specialists at all in any narrow meaning of the word. Success was only possible because of the wide learning of the investigator; because of his accurate knowledge of the instruments that had been made effective in other branches; and because he had grasped the underlying principles which caused those instruments to be effective in particular cases.

MacMahon then moves on to combinatorics, the subject that he dominated during his lifetime [**11**, pp. 891–892].

> The combinatorial analysis may be described as occupying an extensive region between the algebras of discontinuous and continuous quantity. It is to a certain extent a science of enumeration, of measurement by means of integers, as opposed to measurement of quantities which vary by infinitesimal increments. It is also concerned with arrangements in which differences of quality and relative position in one, two, or three dimensions, are factors. Its chief problem is the formation of connecting roads between the sciences of discontinuous and continuous quantity. To enable, on the one hand, the treatment of quantities which vary *per saltum*, either in magnitude or position, by the methods of the science of continuously varying quantity and position, and on the other hand to reduce problems of continuity to the resources available for the management of discontinuity. These two roads of research should be regarded as penetrating deeply into the domains which they connect.
>
> In the early days of the revival of mathematical learning in Europe the subject of 'combinations' cannot be said to have rested upon a scientific basis. It was brought forward in the shape of a number of isolated questions of arrangement, which were solved by mere counting. Their solutions did not further the general progress, but were merely valuable in connection with the special problems. Life and form, however, were infused when it was recognised by De Moivre, Bernoulli, and others that it was possible to create a science of probability on the basis of enumeration and arrangement. Jacob Bernoulli, in his "Ars Conjectandi," 1713, established the fundamental principles of the Calculus of Probabilities. A systematic advance in certain questions which depend upon the partitions of numbers was only possible when Euler showed that the identity $x^a x^b = x^{a+b}$ reduced arithmetical addition to algebraical multiplication and *vice versâ*. Starting with this notion Euler developed a theory of generating functions on the expansion of which depended the formal solutions of many problems. The subsequent work of Cayley and Sylvester rested on the same idea, and gave rise to many improvements. The combinations under enumeration had all to do

> with what may be termed arrangements on a line subject to certain laws. The results were important algebraically as throwing light on the theory of Algebraic series, but another large class of problems [concerning arrays] remained untouched, and was considered as being both outside the scope and beyond the power of the method.... It will be gathered from remarks made above that I regard any department of scientific work, which seems to be narrow or isolated, as a proper subject for research. I do not believe in any branch of science, or subject of scientific work, being destitute of connection with other branches. If it appears to be so, it is especially marked out for investigation by the very unity of science. There is no necessarily pathless desert separating different regions. Now a department of pure mathematics which appeared to be somewhat in this forlorn condition a few years ago, was that which included problems of the nature of the magic square of the ancients.

MacMahon concludes with a sketch of his work extending the theory of partitions and rectangular array enumerations.

In the last twenty years a large number of researchers have greatly advanced the program for combinatorics hinted at by MacMahon. At the forefront of this work is the "MIT School". Gian-Carlo Rota is its founder and now leads it jointly with his student, Richard Stanley. Stanley, probably more than any other living mathematician, has absorbed the foundational work of MacMahon and has developed magnificent mathematical theories that have gone beyond the wildest dreams of MacMahon. I feet it would be fair to say that MacMahon's glowing account of 19th century advances in the theory of numbers could now be made of enumerative combinatorics.

In particular, when MacMahon made his allusion to magic squares he could not have envisioned the amazing proof by Stanley of the Anand-Dumir-Gupta conjecture on magic squares [**14**]. The upshot of this major breakthrough was an increasingly deep theory connecting commutative algebra and combinatorics; perhaps the most dramatic related achievement was Stanley's proof of the Upper Bound Conjecture which won him the Pólya Prize.

Fortunately for the mathematical community, Stanley has now written an excellent graduate text which introduces many of the topics beloved by MacMahon.

Chapter 1 is a very readable answer to the chapter title "What Is Enumerative Combinatorics?" Included is an elegant treatment of permutation statistics, a topic considered extensively by MacMahon. The chapter concludes with a charming account of Rota's *Twelvefold Way*.

Chapter 2, Sieve Methods, illustrates how far we have come beyond the classical "inclusion-exclusion" principle. Again MacMahon's original ideas are presented (in Example 2.2.4), and the subsequent developments are carefully presented in the text and in the numerous interesting exercises.

Chapter 3 presents an extensive treatment of the combinatorics of partially ordered sets. Again we have a very readable and deep introduction to this subject. As Stanley notes, pp. 149–50, "...it was not until 1964 that the seminal paper [**13**] of G.-C. Rota appeared that began the systematic development of posets and lattices within combinatorics."

Chapter 4, Rational Generating Functions, provides an excellent exposition of the domain now dominated by Stanley himself. Indeed the Anand-Dumir-Gupta conjecture is presented as Proposition 4.6.19, p. 232.

Historically then this is a book of major importance. It provides a widely accessible introduction to many topics in combinatorics. It presents a number of "the peaks from which a general view could be gathered of any considerable portion of the country." Furthermore, it is sure to become a standard as an introductory graduate text in combinatorics. There are a host of exercises (45 pages) and solutions (62 pages).

One might object that this book presents only one view of the multi-faceted subject of enumerative combinatorics. Stanley readily and correctly acknowledges this on p. 151.

> Among the many alternative theories to binomial posets for unifying various aspects of enumerative combinatorics and generating functions, we mention the theories of prefabs [**3**], dissects [**6**], linked sets [**4**], and species [**17**]. The most powerful of these theories is perhaps that of species, which is based on category theory. We should also mention the book of I. Goulden and D. Jackson [**5**], which gives a fairly unified treatment of a large part of enumerative combinatorics related to the counting of sequences and paths.

Presumably Volume II will fill in many further topics.

In conclusion, this is an outstanding book. Perhaps we can summarize by quoting from Rota's foreword:

> Best of all, Stanley has succeeded in dramatizing the subject, in a book that will engage from start to finish the attention of any mathematician who will open it at page one.

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One-parameter semigroups of positive operators, by W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. Edited by R. Nagel. Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986, x + 460 pp., $36.30. ISBN 3-540-16454-5

A *strongly continuous semigroup* (or (C_0) *semigroup*) is a family $T = \{T(t): t \geqslant 0\}$ of bounded linear operators on a Banach space X satisfying, for all s, $t \in [0, \infty)$ and all $f \in X$,

$$T(s+t)f = T(t)T(s)f, \qquad T(0)f = f,$$
$$T(\cdot)f\colon [0,\infty) \to X \text{ is continuous.}$$

The (*infinitesimal*) *generator* A of T is the strong derivative of T at the origin. More precisely, $f \in \mathrm{Dom}(A)$ and $Af = g$ means $\lim_{t\to 0} t^{-1}(T(t)f - f)$ exists and equals g. Informally one thinks of $T(t)$ as $\exp(tA)$, but care must be exercised in the interpretation of the exponential because in all the interesting cases the generator A is an unbounded operator.

Associated with the generator A (or more generally with a given linear operator A) is the initial value problem

$$du(t)/dt = Au(t), \qquad u(0) = f \in \mathrm{Dom}(A). \tag{1}$$

The obvious candidate for the solution is $u(t) = T(t)f$ (with the semigroup property $T(t+s)f = T(t)T(s)t$ following formally from the existence and uniqueness for (1)). Of concern is when (1) is a *well-posed problem*. This means that a solution of (1) exists, is unique, and depends continuously on the ingredients of the problem (namely f and A) in a suitable sense.

Two principal results of semigroup theory go back to E. Hille, K. Yosida, and R. Phillips, and can be stated as follows. (I) The initial value problem (1) is well posed iff A is the generator of a (C_0) semigroup T, in which case $u(\cdot) = T(\cdot)f$ is the unique solution of (1). (II) The operator A is the generator of a semigroup iff for λ real and large, $(\lambda - A)^{-1}$ exists in $\mathscr{L}(X)$ (i.e., as a bounded linear operator on X) and certain norm estimates hold.