BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

FIXED-POINTS OF FINITE GROUP ACTIONS ON CONTRACTIBLE COMPLEXES

BY ROBERT OLIVER[1]

Communicated by Glen E. Bredon, October 1, 1974

This is a summary of results obtained while attempting to classify the finite complexes which can be fixed-point sets of cellular actions of a given group on finite contractible CW complexes. Here, by a cellular action is meant one where the action of any group element takes the interior of any cell to the interior of some other cell, and takes a cell to itself only via the identity map. For groups of prime power order, the question has already been answered by P. A. Smith [4] and Lowell Jones [2]: if $|G| = p^n$, a finite complex can be a fixed-point set if and only if it is $\mathbf{Z}_p$-acyclic.

The main tools for answering the question for groups not of prime power order are certain functions defined below, which serve as bookkeeping devices for controlling the fixed-point structure of a space with group action. Let $\mathcal{G}^1$ denote the class of finite groups G with a normal subgroup $P \triangleleft G$ of prime power order such that G/P is cyclic. A *resolving function* for a finite group G is defined to be a function φ: $\mathcal{S}(G) \longrightarrow \mathbf{Z}$ (where $\mathcal{S}(G)$ is the set of subgroups) such that:

(1) φ is constant on conjugacy classes of subgroups.

(2) $[N(H): H] \mid \varphi(H)$ for all $H \subseteq G$.

(3) $\Sigma_{K \supseteq H} \varphi(K) = 0$ for all $H \in \mathcal{G}^1$.

Now define a *G-resolution* of a finite complex F to be any n-dimensional $(n-1)$-connected complex X $(n \geqslant 2)$ such that G acts on X with fixed-point set F, and such that $H_n(X)$ is a projective $\mathbf{Z}[G]$-module. To any G-resolution X there corresponds a unique resolving function φ satisfying

$$\chi(X^H) = 1 + \sum_{K \supseteq H} \varphi(K) \quad \text{for all } 0 \neq H \subseteq G,$$

$$\sum_{H \subseteq G} \varphi(H) = 0.$$

AMS (MOS) subject classifications (1970). Primary 55C35; Secondary 57E25.

[1] The work described here was done while the author was holding an NSF Graduate Fellowship.

Conversely, it has been shown that for any G not of prime power order, given an integral resolving function φ and a finite complex F with $\chi(F) = \varphi(G) + 1$, there is a G-resolution X of F which realizes φ. Letting $m(G)$ denote a generator of $\{\varphi(G)\colon \varphi \text{ is a resolving function for } G\}$, this proves

THEOREM 1. *If G is not of prime power order, then a finite complex F has a G-resolution if and only if $\chi(F) \equiv 1 \pmod{m(G)}$.*

For any n-dimensional G-resolution X of F, an obstruction is defined lying in the projective class group $\widetilde{K}_0(\mathbf{Z}[G])$: $\gamma_G(F, X) = (-1)^n [H_n(X)]$. A subgroup

$$\mathcal{B}(G) = \{\gamma_G(pt, X)\colon X \text{ is a } G\text{-resolution of a point}\}$$

is defined. If X_1 and X_2 are two G-resolutions of F, then $\gamma_G(F, X_1) - \gamma_G(F, X_2) \in \mathcal{B}(G)$, and so there is a well-defined obstruction $\gamma_G(F) \in \widetilde{K}_0(\mathbf{Z}[G])/\mathcal{B}(G)$, which is zero if and only if F is the fixed-point set of an action of G on some finite contractible complex. It has been shown that $\gamma_G(F)$ depends only on $\chi(F)$, and so there is an integer n_G such that:

THEOREM 2. *For any group G not of prime power order, a finite complex F is the fixed-point set of an action of G on some finite contractible complex if and only if $\chi(X) \equiv 1 \pmod{n_G}$.*

For the calculation of $m(G)$, the following notation will be used. For q prime, let $\mathcal{G}^q$ denote the class of all finite groups G with a normal subgroup $H \in \mathcal{G}^1$ of q-power index. Set $\mathcal{G} = \bigcup_q \mathcal{G}^q$. Then

THEOREM 3. *If $G \in \mathcal{G}^1$, then $m(G) = 0$. If $G \notin \mathcal{G}^1$, then $m(G)$ is a product of distinct primes (or 1), and $q \mid m(G)$ if and only if $G \in \mathcal{G}^q$. In particular, $m(G) = 1$ if and only if $G \notin \mathcal{G}$.*

Attempts to calculate n_G completely have so far been unsuccessful. The best which has been done is to show that $m(G) \mid n_G \mid m(G)^2$. In particular, $n_G = 1$ if and only if $G \notin \mathcal{G}$.

Of particular interest is the case of smooth fixed-point free actions on disks. These can be obtained from cellular fixed-point free actions on finite contractible complexes using the methods of [1] or [3]. The above results immediately yield:

THEOREM 4. *A finite group G has a smooth fixed-point free action on a (sufficiently high dimensional) disk if and only if $G \notin \mathcal{G}$. In particular, any* *nonsolvable group has such an action. A finite abelian group has such an action if and only if it has three or more noncyclic Sylow subgroups.*

(Theorem 4 was proved, for solvable groups, in the author's thesis [3] by different methods.)

REFERENCES

1. E. E. Floyd and R. W. Richardson, *An action of a finite group on an n-cell without stationary points*, Bull. Amer. Math. Soc. **65** (1959), 73–76. MR **20** #7276.

2. Lowell Jones, *The converse to the fixed-point theorem of P. A. Smith.* I, Ann. of Math. (2) **94** (1971), 52–68. MR **45** #4427.

3. R. Oliver, *Smooth fixed-point free actions of compact Lie groups on disks*, Thesis, Princeton University, Princeton, N. J., 1974.

4. P. A. Smith, *Fixed-points of periodic transformations*, Amer. Math. Soc. Colloq. Publ., vol. 27, Amer. Math. Soc. Providence, R. I., 1942, pp. 350–373.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

MATEMATISK INSTITUT, ÅRHUS UNIVERSITET, ÅRHUS, DENMARK