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BOOK REVIEW

Foundations of a structural theory of set addition, by G. A. Freiman, American Mathematical Society, Translations of Mathematical Monographs, vol. 37, 1973, vii+108 pp., $15.70

This is an advanced monograph addressed primarily to experts in additive number theory. Although the only formal prerequisite is a basic knowledge of elementary number theory, some of the proofs are rather sophisticated, and while the author states that he has presented the material of the book to teacher trainees, this would be quite unthinkable in the United States.

There are three chapters entitled *Isomorphisms*, *The fundamental theorem on sums of finite sets*, and *Sums of sequences, sets of residues and point sets*. In the first chapter, which is by far the most clearly written of the three, the author introduces some basic concepts and formulates the problems to be attacked later in the book. Suppose that A and B are two sets with a binary operation, denoted in both cases by $+$. (The operations are not assumed to be associative or commutative.) Two subsets $A' \subseteq A$, $B' \subseteq B$ are called *isomorphic* if there is a bijection $a \leftrightarrow b$ from A' to B' such that $a_1+a_2=a_3+a_4$ if and only if $b_1+b_2=b_3+b_4$. For example, if $A=\boldsymbol{Z}$ and $B=\boldsymbol{Z}^2$ (where $\boldsymbol{Z}^n$ denotes the group of lattice points in n-dimensional Euclidean space $\boldsymbol{R}^n$), then the subsets

$$A' = \{0, 1, 3, 4\} \quad \text{and} \quad B' = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$$

are isomorphic. A rather grandiose statement is made to the effect that additive number theory is the study of properties invariant under isomorphism, and comparisons with Klein's *Erlanger Programm* are drawn. The reviewer finds this quite ludicrous, but it does enliven some otherwise dull moments in reading the book.

Now let K be a set of finite cardinality $|K|=k$, and $2K$ the set of all sums $a+b$, where $a, b \in K$. The author distinguishes between *direct* problems of additive number theory, where properties of $2K$ are deduced from those of K, and *inverse* problems, where properties of K are deduced from those of $2K$. The present book is primarily concerned with inverse problems, and in particular with determining the structure of K under the hypothesis that the cardinality $|2K|=T$ is small. As an example of a direct result we may take the Cauchy-Davenport theorem, which asserts that if K is a subset of $Z/(p)$ (the cyclic group of prime order p), then $T \geqq \min(2k-1, p)$.[1] The corresponding inverse theorem is that if $T=2k-1<p$, then K is an arithmetic progression.[2]

Chapter 2 begins with a strengthened form of this inverse theorem, namely that if $T<12k/5-3$ and $k<p/35$, then K is contained in an arithmetic progression of length $T-k+1$. The proof is by the method of trigonometric sums and is an excellent illustration of the ideas used in proving the fundamental theorem which forms the core of the chapter. To formulate this latter theorem, suppose that K is a subset of Z^m containing 0, and that $T<Ck$, where C is a constant $\geqq 2$. Then for k sufficiently large, there is a parallelepiped $H \subset Z^n$, where $n \leqq C-1$, and a group homomorphism $\phi: Z^n \rightarrow Z^m$ such that (1) the restriction of ϕ to H is an isomorphism in the author's sense, (2) $K \subseteq H\phi$, and (3) $|H|<ck$, where c is a constant depending only on C.

The proof of this theorem is a *tour de force*, combining the method of trigonometric sums with ideas from the geometry of numbers and probability theory. There are some special cases, however, which can be treated by more elementary methods, and where sharper conclusions can be drawn. For example, if $K \subset Z$ and $T<3k-3$, then K is contained in an arithmetic progression of length $T-k+1$ (yielding the conclusion of the fundamental theorem with $n=1$ and a sharper bound on $|H|$). When $T=3k-3$, this is no longer the case, as is shown by the example

$$K_0 = \{0, 1, \cdots, k_1 - 1, b, b+1, \cdots, b+k_2-1\},$$

where $k_1+k_2=k$ and $b+k_2 \geqq 2k$. The author proves, however, that if $k \neq 6$ and $T=3k-3$, then K is either contained in an arithmetic progression of length $2k-2$, or is affinely equivalent to K_0 for some choice of k_1, k_2, b. When $k=6$, there is the further possibility that K is affinely equivalent to the set $K_6=\{0, 1, 2, b, b+1, 2b\}$, where $b \geqq 5$. To fit this into the general scheme of the fundamental theorem, suppose for definiteness that $k_1 \geqq k_2$, and consider the set

$$\bar{K}_0 = \{(0,j) \mid 0 \leqq j < k_1\} \cup \{(1,j) \mid 0 \leqq j < k_2\}.$$

It is contained in the rectangle $H=\{(x, y) \mid x=0 \text{ or } 1, 0 \leqq y<k_1\}$. Let ϕ be the homomorphism of Z^2 onto Z defined by $(1,0)\phi=1$, $(0,1)\phi=b$. The restriction of ϕ to H is an isomorphism in the author's sense, and $\bar{K}_0\phi=K_0$. Moreover $|H|=2k_1<2k$, so the constant c of the general theorem is 2 in this case. The set K_6 can be similarly obtained as an image of the

[1] More generally, $|K_1+K_2| \geqq \min(|K_1|+|K_2|-1, p)$, where K_1, K_2 are subsets of $Z/(p)$, and K_1+K_2 is the set of all sums a_1+a_2 with $a_i \in K_i$.

[2] In fact, if $|K_1+K_2|=|K_1|+|K_2|-1<p$, then K_1 and K_2 are arithmetic progressions with a common difference. This result is due to A. G. Vosper.

triangle

$$\bar{K}_6 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\}.$$

When $T=3k-2$, there is an analogous characterization of the possible isomorphism types of K, but for larger values of T the number of cases to be considered grows rapidly, and the elementary methods become unworkable. Still, these special cases are very helpful in motivating and illustrating the fundamental theorem.

The remainder of the book is devoted to applications of this theorem. The first of them is to the theory of nonaveraging sets, i.e. sets $K \subset \boldsymbol{Z}$ with no three members in arithmetic progression. It is shown that $T/k \to \infty$ for any sequence of finite nonaveraging sets whose cardinality k tends to infinity.

Next, let $A=\{0, a_1, a_2, \cdots\}$ be an increasing sequence of nonnegative integers,

$$A(x) = |A \cap [0, x]|, \quad \text{and} \quad A_2(x) = |2A \cap [0, x]|.$$

Let $\alpha = \underline{\lim}\, A(x)/x$ and $\gamma = \lim A_2(x)/x$ be the asymptotic densities of A and $2A$ respectively, and $d(A)$ the g.c.d. of the members of A. The inequality $\gamma \geqq \min(1/d(A), 3\alpha/2)$ is derived from the elementary case of the fundamental theorem where $T<3k-3$ (described in detail above). This inequality was obtained by P. Erdös in the special case $a_1=1$.

Using the full force of the fundamental theorem, it is shown that if $\overline{\lim}\, A(x)/x<1/2d(A)$, then $\overline{\lim}\, A_2(x)/A(x) \geqq 3$, a result which was conjectured by Erdös under the stronger hypothesis that $\lim A(x)/x=0$.

Other applications include a generalization of M. Kneser's analogue of the $\alpha+\beta$ theorem for asymptotic density, further extensions of the Cauchy-Davenport theorem, and a strengthening of the Brunn-Minkowski inequality. All of these results are very impressive; however, the precise statements are too technical to give here.

Obviously this book is an important and valuable contribution to the subject of additive number theory. Among its strong points are the author's mastery of a powerful technique, the interest and beauty of the problems to which this technique is applied, and the liveliness and enthusiasm of the presentation. On the negative side there is the unnecessary clumsiness of certain proofs, and above all the enormous number of minor but annoying misprints. At the risk of reopening the cold war, the reviewer feels compelled to lodge a feeble protest against the sloppiness which mars so many otherwise superb Russian texts. (Of course a few of these slips may have been introduced by the translator, who on the whole did an excellent job, and wisely chose to remain anonymous.) G. H. Hardy once remarked that no one ever wrote five pages of mathematics without a mistake; it seems that in the area of trigonometric sums this bound can be reduced to five lines. A complete list of errata would probably fill a volume at least half the size of the monograph itself. For example, on page 59 we find the following definition of a supporting hyperplane of a convex set $D \subset \boldsymbol{R}^n$: "A hyperplane L divides the space $\boldsymbol{R}^n$ into two parts. If one of these parts does not contain D, but L contains points from D, then L is called a supporting hyperplane of D." According to this definition, every hyperplane which intersects a convex set supports it! Of course it is clear how to fix this one up, but why should we have to?[3] After all, mathematics is an art as well as a science, and such monstrosities, when they abound on every page, begin to detract from the aesthetic value of the book.

Here is another example. On pages 52–53 we read: "Let R be a convex, open domain which is symmetric with respect to the origin and has volume V, $0<V<\infty$. For each I, $1 \leqq I \leqq n$, there exists a largest number λ, say λ_I, such that λR contains I linearly independent points." In the second sentence "largest" should be replaced by "smallest", and R should be replaced by its closure. Obvious, perhaps, but most annoying all the same.[4]

The next example is of a different type. In the preface the author informs us that "the well-known significance of the concept of isomorphism between groups lies in the fact that it provides the possibility for disregarding incidental properties of each given specific group, and thus it leads to investigations of a more general kind. Furthermore it permits us to introduce an infinity of operations on the elements of a group." Here the last sentence is not really a misprint, but just nonsense.

More serious is the statement made on page 2 that any two sets of cardinality 2 are isomorphic. Since the author is a first-rate mathematician, this must be a case of Homer nodding; a counterexample is provided by the sets $A=\{0, 1\} \subset \boldsymbol{Z}$ and $B=\boldsymbol{Z}/(2)$.

Needless to say, the overwhelming majority of the mistakes are wrong subscripts, reversed inequalities, incorrect references to equations or bibliographical items[5] etc., but these do not make interesting reading in a review.

Having gotten all this off my chest, I will close on a more positive note by saluting the author on his accomplishment. He has produced some really fine mathematics and an excellent book which can be warmly recommended to devotees of number theory.

BASIL GORDON

[3] Actually, what is this definition doing at all in such an advanced treatise?

[4] The geometry of numbers seems particularly prone to distortion in this book. Thus we are told on page 61 that a convex body in $\boldsymbol{R}^m$ which contains only one lattice point has volume $\leqq 2^m$. (No mention is made of symmetry about the one lattice point.)

[5] There is poetic justice here, for on page 23 the author, through a mistake in reference numbers, attributes one of his own most striking results to Davenport!