BOOK REVIEW

Divergent series. By G. H. Hardy. Oxford University Press, 1949. 16+396 pp. $8.00.

Hardy died on December 1, 1947; during his lifetime the theory of divergent series and its applications developed into an important branch of modern analysis. It is only natural that the book bears all the marks of his own research work, but it is also a comprehensive presentation of this vastly expanded subject. The final galley proofs were read and completed by some of his younger collaborators, in particular L. S. Bosanquet.

The book consists of thirteen chapters and five additional sections. Attached to each chapter are notes and references. Chapters I and II contain a lively historical survey, particularly on Euler's, Fourier's, and Heaviside's contributions. Also some principles are discussed on which to base methods for the summation of divergent series and integrals.

Chapter III discusses general theorems concerning linear transformations of sequences and functions, with some particular examples. Let us explain a few fundamental concepts. The general linear transformation of a sequence s_n is either another sequence

$$t_m = \sum_{n=0}^{\infty} c_{m,n}s_n, \qquad m = 0, 1, 2, \cdots,$$

or a function $t(x) = \sum c_n(x)s_n$, where x is a continuous parameter. A linear transform of a function $s(y)$ is $t(x) = \int_0^\infty c(x, y)s(y)dy$; $(c_{m,n})$ is the matrix of the transform t_m, $c(x, y)$ is the kernel of the transform $t(x)$.

In Chapter IV special methods are discussed (Nörlund, Euler, Abel, and others). Chapter V is concerned with Hölder, Cesàro, and Riesz means. Hölder and Cesàro means of first order, denoted by $(H, 1)$ and $(C, 1)$, are the same: $h'_n = (1/(n+1)) \sum_0^n s_\nu$; by iteration: $h_n^{(2)} = (1/(n+1)) \sum_0^n h'_\nu$, and $h_n^{(r)} = (1/(n+1)) \sum_0^n h_\nu^{(r-1)}$. The Cesàro means are defined by taking $s'_n = \sum_0^n s_\nu$, $s_n^{(r)} = \sum_0^n s_\nu^{(r-1)}$, $c_n^{(r)} = s_n^{(r)}/A_n^{(r)}$, where $A_n^{(r)}$ is $s_n^{(r)}$ when all $s_n = 1$.

The next two chapters are devoted to Tauberian theorems for Cesàro and for Abel means. Tauber proved in 1897 that Abel summability and $\lim na_n = 0$ imply convergence of $\sum a_n$; this elementary result was the starting point of a long chain of investigations to establish inverse theorems of summability dealing with the question:

Under what conditions for the terms a_n does summability of a series $\sum a_n$ imply its convergence? A significant result of Hardy and Landau is that if $\sum a_n$ is summable (C, k) for some k and if $na_n > -M$ (M constant for all n) then $\sum a_n$ is convergent. Many generalizations concerning Cesàro and Riesz's summability are discussed. The methods employed led to the concept of slowly oscillating sequences and functions and to a connection with Fourier transforms and closure theorems. Chapter VII concludes with the "high indices" Theorem of Hardy and Littlewood.

Chapter VIII is concerned with the methods of Euler and Borel and their generalizations, with application to analytic continuation and to summation of certain asymptotic series. Chapter IX analyzes Tauberian Problems for Borel and Euler summability. The principal Tauberian Theorem is that Borel summability and $a_n n^{1/2} = O(1)$ imply convergence of $\sum a_n$.

The main subject of Chapter X is to discuss the convergence and summability properties of the Cauchy product of two series. Chapter XI discusses Hausdorff means and related transforms. In Chapter XII, N. Wiener's theory of general Tauberian problems is presented; here the tool is the theory of Fourier transforms. It includes applications to the theory of primes, and to special summability methods, such as Borel summability and Riemann summability.

Chapter XIII discusses the Euler-Maclaurin sum formula. The first of the appendices discusses the evaluation of certain definite integrals by means of divergent series; appendices II and III give applications to Fourier series. In particular appendix II analyzes the Fourier kernel of certain methods of summation such as Cesàro's method and de la Vallée Poussin's method. In appendix III, Riemann and Abel summability are applied to Fourier series. Appendix IV deals with Lambert and Ingham summability and their role in the analytic theory of numbers. Appendix V discusses theorems concerning a method related to Abel summability.

The large amount of material evidently did not permit more detail concerning applications to trigonometric series, or a discussion of the subject of orthogonal series and of multiple series. It seems worthwhile to write another book on these subjects.

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