ON TOPOLOGIES FOR FUNCTION SPACES

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Given topological spaces X, T, and Y and a function h from $X \times T$ to Y which is continuous in x for each fixed t, there is associated with h a function h^* from T to $F = Y^X$, the space whose elements are the continuous functions from X to Y. The function h^* is defined as follows: $h^*(t) = h_t$, where $h_t(x) = h(x, t)$ for every x in X. The correspondence between h and h^* is obviously one-to-one.

Although the continuity of any particular h depends only on the given topological spaces X, T, and Y, the topology of the function space F is involved in the continuity of h^*. It would be desirable to so topologize F that the functions h^* which are continuous are precisely those which correspond to continuous functions h. It has been known for a long time that this is possible if X satisfies certain conditions, chief among which is the condition of *local compactness* (Theorem 1). This condition is often felt to be too restrictive (since it practically excludes the possibility of X itself being a function space), and several years ago, in a letter, Hurewicz proposed to me the problem of defining such a topology for F when X is not locally compact. At that time I showed by an example (essentially Theorem 3) that this is not generally possible. Recently I discovered that, by restricting the range of T in a very reasonable way, one of the standard topologies for F has the desired property even for spaces X which are not locally compact (Theorem 2). In this last result the condition of local compactness is replaced by the first countability axiom and this appeals to me as a less troublesome condition.

It should be pointed out that the problem is motivated by the special case in which T is the unit interval. When T is the unit interval, h is a homotopy and h^* is a path in the function space; in the topology of deformations, equivalence of the concepts of "homotopy" and of "function-space path" is usually required.

Among the various possible topologies for F there is one, which I shall call the compact-open[1] (co.o.) topology, which seems to be the most natural. For any two sets, A in X and W in Y, let $M(A, W)$ denote the set of mappings $f \in F$ for which $f(A) \subset W$. The co.o. topology is defined by selecting as a sub-basis for the open sets of F the

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[1] Terminology followed in this note is generally that of Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942.

sets $M(A, W)$ where A ranges over the compact subsets of X and W ranges over the open subsets of Y.

THEOREM 1. *If X is regular and locally compact, Y an arbitrary topological space, and if F has the co.o. topology, then continuity of h is equivalent to continuity of h^* for any topological space T.*

THEOREM 2. *If X is a space which satisfies the first countability axiom, Y an arbitrary topological space, and if F has the co.o. topology, then continuity of h is equivalent to continuity of h^* for any T which satisfies the first countability axiom.*

THEOREM 3. *If X is separable metrizable and Y is the real line, then in order that it be possible to so topologize F that continuity of h and of h^* are equivalent, it is necessary and sufficient that X be locally compact.*

LEMMA 1. *If F has the co.o. topology, then continuity of h implies continuity of h^* under no restrictions on the topological spaces X, T, and Y.*

PROOF. Let W be an open set in Y and A a compact set in X and let t_0 be a point in $h^{*-1}(M(A, W))$. Then $A \times t_0 \subset h^{-1}(W)$. Since $h^{-1}(W)$ is open it is the union of open sets $U_\alpha \times V_\alpha$. Since A is compact, $A \times t_0$ is contained in a finite union $\bigcup_{i=1}^n U_i \times V_i$ with each V_i a neighborhood of t_0. Then $\bigcap_{i=1}^n V_i$ is an open neighborhood of t_0 and is contained in $h^{*-1}(M(A, W))$.

PROOF OF THEOREM 1. In view of the lemma it is sufficient to prove that continuity of h^* implies continuity of h. Let W be an open set in Y and let (x_0, t_0) be a point in $h^{-1}(W)$. Since $h^*(t_0)$ is continuous in x there exists an open neighborhood U of x_0 such that $h^*(t_0) \in M(U, W)$. Because of the conditions on X there is an open neighborhood R of x_0 such that $\overline{R}$ is compact and contained in U. Since $M(\overline{R}, W)$ is open and contains $h^*(t_0)$ there is an open neighborhood V of t_0 such that $h^*(V) \subset M(\overline{R}, W) \subset M(R, W)$. Thus $R \times V$ is an open neighborhood of (x_0, t_0) which is contained in $h^{-1}(W)$.

PROOF OF THEOREM 2. As before we have to prove that continuity of h^* implies continuity of h. Let W be an open set in Y and suppose that $h^{-1}(W)$ is not open. Then there is a point (x_0, t_0) in $h^{-1}(W)$ which is also in the closure of the complement of $h^{-1}(W)$. Let $\{G_n\}$ be a base for the open sets of $X \times T$ at the point (x_0, t_0) and choose, for each integer n, a point (x_n, t_n) in the intersection of $\bigcap_{i \leq n} G_i$ and the complement of $h^{-1}(W)$. Since $h^*(t_0)$ is continuous in x there exists an open neighborhood U of x_0 such that $h^*(t_0) \in M(U, W)$. Let

$A = U \cap \bigcup_{n=0}^{\infty} x_n$. Since A is compact, $M(A, W)$ is open and since h^* is continuous and $t_0 \in h^{*-1}(M(A, W))$, there is a neighborhood V of t_0 such that $h^*(V) \subset M(A, W)$. There is an integer N such that $x_n \in U$ and $t_n \in V$ whenever $n > N$. Hence $h(x_n, t_n) \in W$ for every n greater than N. This contradiction with the choice of the points (x_n, t_n) proves that $h^{-1}(W)$ is open. Thus h is continuous.

LEMMA 2. *Let X be a separable metrizable space, let Y be the real line, and suppose that the topology of F is such that continuity of h for $T = [0, 1]$ implies the continuity of h^*. Let $W = (a, b)$ be a finite open interval in Y and let A be a closed subset of X which is not compact. Then the set $M(A, W)$ has no interior points.*

PROOF. Since A is not compact there is a sequence $\{x_n\}$ in A such that $\bigcup_{n=1}^{\infty} x_n$ is closed in X. Given any element $h^*(0)$ of the set $M(A, W)$ let us define

$$h_t(x_n) = \min \{1 + b, h_0(x_n) + nt\}.$$

Since the function h is defined over the closed set $X \times [0] \cup (\bigcup_{n=1}^{\infty} x_n) \times [0, 1]$ it may be extended continuously over the normal space $X \times [0, 1]$. If $t > 0$ there is an integer n such that $a + nt > 1 + b$; hence $h^*(t)$ is in the complement of $M(A, W)$ for every positive t. By hypothesis the topology of F is such that h^* is continuous. Hence $h^*(0)$ belongs to the closure of the complement of $M(A, W)$.

LEMMA 3. *If the topology for F is such that continuity of h^* always implies continuity of h then, given a point x_0 in X, an open set W in Y, and an element f_0 in $M(x_0, W)$, there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F.*

PROOF. Define $\phi(x, f) = f(x)$ for every $(x, f) \in X \times F$. Since $\phi^*(f) = f$, ϕ^* is continuous and hence ϕ is also continuous. Since $\phi^{-1}(W)$ is therefore open there must be a neighborhood R of x_0 and a neighborhood V of f_0 such that $\phi(R, V) \subset W$. Thus $f_0 \in V \subset M(R, W)$ and hence $M(R, W)$ is a neighborhood of f_0 in F.

PROOF OF THEOREM 3. Let W be the finite open interval (a, b) and suppose that the topology of F is such that continuity of h and of h^* are equivalent for every T. From Lemma 3 it follows that, given any point x_0 in X and any element f_0 in $M(x_0, W)$, there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F. Since X is regular there is a neighborhood U of x_0 whose closure is contained in R, so that $M(\overline{U}, W)$ is also a neighborhood of f_0. Since f_0 is an interior point of $M(\overline{U}, W)$ it follows from Lemma 2 that $\overline{U}$ is compact. Thus X must be locally compact. This proves the necessity of the condition; sufficiency is a consequence of Theorem 1.

COROLLARY. *If Y is the real line and X is separable metrizable but not locally compact, then F does not satisfy the first countability axiom in the co.o. topology.*

PROOF. Let W be the finite open interval (a, b). If F satisfied the first countability axiom then Theorem 2 would apply to yield the continuity of the function ϕ defined above. If x_0 is a point at which X is not locally compact and f_0 any element in $M(x_0, W)$, then it follows from the proof of Lemma 3 that there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F. Let U be a neighborhood of x_0 whose closure is contained in R, so that f_0 is an interior point of $M(\overline{U}, W)$. Since $\overline{U}$ is not compact this is not in agreement with Lemmas 1 and 2. This contradiction shows that F does not satisfy the first countability axiom in the co.o. topology.

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