This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix A to be commutative is that A be a normal matrix.

QUEENS COLLEGE

REMARKS ON REGULARITY OF METHODS OF SUMMATION

G. E. FORSYTHE AND A. C. SCHAEFFER

A doubly infinite matrix[1] (a_{mn}) $(m, n=1, 2, \cdots)$ is said to be *regular*, if for every sequence $x=\{x_n\}$ with limit x' the corresponding sums $y_m=\sum_{n=1}^{\infty}a_{mn}x_n$ exist for $m=1, 2, \cdots$, and if $\lim_{m\to\infty} y_m=x'$. An apparently more inclusive definition of regularity is that for each sequence x with limit x' the sums defining y_m shall exist for all $m\geqq m_0(x)$ and $\lim_{m\to\infty} y_m=x'$. Tamarkin[2] has shown that (a_{mn}) is regular in the latter sense if and only if there exists an m_1 *independent of* x such that the matrix (a_{mn}) $(m\geqq m_1,\ n\geqq 1)$ is regular in the former sense. Using point set theory in the Banach space (c), he proves a theorem[3] from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

THEOREM 1. *Suppose the doubly infinite matrix* (a_{mn}) *has the property that for each sequence* $x=\{x_n\}$ *with limit* 0 *there exists an* $m_0=m_0(x)$ *such that for all* $m\geqq m_0(x)$, $u_m=\lim\sup_{k\to\infty}\left|\sum_{n=1}^{k}a_{mn}x_n\right|<\infty$. *Then there exists an* m_1 *such that* $\sum_{n=1}^{\infty}|a_{mn}|<\infty$ *for all* $m\geqq m_1$.

If in addition $\lim_{m\to\infty} u_m=0$ for each sequence x with limit 0, it will follow[4] that there exists an N such that $\sum_{n=1}^{\infty}|a_{mn}|\leqq N<\infty$, for all $m\geqq m_1$.

To prove Theorem 1, suppose there were an infinite sequence $m_1<m_2<\cdots$ such that $\sum_{n=1}^{\infty}|a_{mn}|=\infty$ for $m\in\{m_\nu\}$. Let $x_1, \cdots, x_{k_1}$ be chosen with unit moduli and with amplitudes such that

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[1] In this note a_{mn}, x_n and x' denote finite complex numbers.

[2] J. D. Tamarkin, *On the notion of regularity of methods of summation of infinite series*, this Bulletin, vol. 41 (1935), pp. 241–243.

[3] J. D. Tamarkin, loc. cit., p. 242, lines 1–6.

[4] See, for example, I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, vol. 151 (1921), pp. 79–111; p. 85, Theorem 4.

$$\sum_{n=1}^{k_1} a_{m_1 n} x_n = \sum_{n=1}^{k_1} | a_{m_1 n} x_n | > 1.$$

Let $x_{k_1+1}, \cdots, x_{k_2}$ be chosen with moduli 1/2 and with amplitudes such that

$$\left| \sum_{n=k_1+1}^{k_2} a_{m_2 n} x_n \right| > 2 + \sum_{n=1}^{k_1} | a_{m_2 n} x_n |.$$

Let $x_{k_2+1}, \cdots, x_{k_3}$ be chosen with moduli 1/3 and with amplitudes such that

$$\left| \sum_{n=k_2+1}^{k_3} a_{m_1 n} x_n \right| > 3 + \sum_{n=1}^{k_2} | a_{m_1 n} x_n |.$$

Writing $y_m(k) = \sum_{n=1}^{k} a_{mn} x_n$, the sequence $\{x_n\}$ and integers $k_1 < k_2 < \cdots$ are thus chosen successively so that $|y_{m_1}(k_1)| > 1$, $|y_{m_2}(k_2)| > 2$; $|y_{m_1}(k_3)| > 3$, $|y_{m_2}(k_4)| > 4$, $|y_{m_3}(k_5)| > 5$; $|y_{m_1}(k_6)| > 6$, $\cdots$; while $|x_n| = 1/r$, for $k_{r-1} < n \leqq k_r$. This is a sort of alternating or "sweeping-out" process. So defined, $\{x_n\}$ is a sequence with limit 0, but $\limsup_{k\to\infty} |\sum_{n=1}^{k} a_{mn} x_n| = \infty$, for $m \in \{m_\nu\}$. This contradiction completes the proof of Theorem 1.

The matrix (a_{mn}) is said to be *null-preserving*, if for every sequence $x = \{x_n\}$ with limit 0 the corresponding sums defining y_m exist for $m = 1, 2, \cdots$ and if $\lim_{m\to\infty} y_m = 0$. An apparently more inclusive definition of null-preserving is that for each sequence x with limit 0 we have $u_m = \limsup_{k\to\infty} |\sum_{n=1}^{k} a_{mn} x_n| < \infty$ for all $m \geqq m_0(x)$ and $\lim_{m\to\infty} u_m = 0$. We remark that it is a consequence of Theorem 1 that (a_{mn}) is null-preserving in the latter sense if and only if there exists an m_1 such that the matrix (a_{mn}) $(m \geqq m_1, n \geqq 1)$ is null-preserving in the former sense.[5]

To consider a problem which is related to the above in the method of proof, let each element of a matrix (a_{mn}) be either $+1$ or -1. For $0 \leqq t \leqq 1$ and $n = 1, 2, \cdots$ let $\{\phi_n(t)\}$ be the Rademacher orthogonal functions,[6] and let $y_{mk}(t) = \sum_{n=1}^{k} a_{mn} \phi_n(t)$. Then it is well known[7] that for almost all t, for all $m = 1, 2, \cdots$ and for all $\epsilon > 0$, $\lim_{k\to\infty} k^{-1/2-\epsilon} y_{mk}(t) = 0$. It is clear that for a particular fixed m there is a t such that $\lim_{k\to\infty} k^{-1} y_{mk}(t) = 1$. The problem is to show that there is a t such that

[5] For conditions that (a_{mn}) be null-preserving, see T. Kojima, *On generalized Toeplitz's theorems on limit and their applications*, Tôhoku Mathematical Journal, vol. 12 (1917), pp. 291–326; p. 300.

[6] A. Zygmund, *Trigonometrical Series*, Warsaw, 1935, p. 5.

[7] For references to this and more precise results, see A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, Berlin, 1933, pp. 60–61.

simultaneously for all $m=1, 2, \cdots$, $\limsup_{k\to\infty} k^{-1}y_{mk}(t)=1$. That there exists such a t can be shown by using the alternating process of Theorem 1.

Theorem 2 follows immediately from a theorem of Banach.[8]

THEOREM 2. *If E_m is a linear manifold satisfying Baire's condition*[9] *in a Banach space E ($m=1, 2, \cdots$) and if $\lim_{m\to\infty} E_m=E$, then there exists an m_1 such that $E_{m_1}=E$.*

Theorem 2 furnishes a Banach space analogue and a proof of Theorem 1 which is related to Tamarkin's proof. To see this, let E be the Banach space (c_0) of sequences $x=\{x_n\}$ convergent to 0, with $\|x\|=\max_n |x_n|$, and with addition and multiplication by a (complex) scalar defined as usual. Let (a_{mn}) be as in Theorem 1. Let E_m be the subset of E for which $\limsup_{k\to\infty}\left|\sum_{n=1}^{k} a_{rn}x_n\right| < \infty$ for all $r\geqq m$. The hypotheses of Theorem 2 are satisfied, and from its conclusion it may be proved directly for an arbitrary $m\geqq m_1$ that $\sum_{n=1}^{\infty} |a_{mn}| < \infty$.

STANFORD UNIVERSITY

[8] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 22, Theorem 2.

[9] See S. Banach, loc. cit., p. 17. By considering a Hamel base for E, G. W. Mackey has remarked to the authors that Theorem 2 is false if the words "satisfying Baire's condition" are omitted.