thesis that the theorem is false for I, and establishes the desired result.

The above theorem may be stated otherwise as follows. If the sequence of horizontal functions $h(x, n)$ satisfying the conditions of the lemma approach the zero limit function monotonically, they must approach it uniformly. It follows as a corollary from this theorem that if we have a series of "interval" functions, such that $f(x, n)$ is a single-valued continuous curve in each subinterval, and if the sequence of functions approach a continuous limit function monotonically for each fixed x in I, then the approach to the limit function is uniform with respect to x. In particular, the sequence of "interval" functions may be a set of functions $f_n(x)$ each of which is continuous in I.

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NOTE ON RATIONAL PLANE CUBICS*

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1. *Introduction.* Many constructions have been devised for a rational plane cubic. One of the most interesting of them is due to Zahradnik† who noticed that the familiar construction for the cissoid of Diocles could be extended so as to generate any rational plane cubic. It is as follows: Take any conic C, a fixed point O on C, and a fixed line b. Any line l through O meets C a second time at P, and b at Q. On l lay off a segment OM equal to and in the sense PQ. The locus of the point M is a rational plane cubic R with double point at O. The tangents to R and O are the joins of O to the two points in which b meets C.

Niewenglowski‡ showed a bit later that this same construction may be applied to R, using a second fixed line dis-

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† Zahradnik, *Cissoidalcurven*, ARCHIV DER MATHEMATIK UND PHYSIK, vol. 56, p. 8.

‡ Niewenglowski, *Sur les courbes d'ordre n à point multiple d'ordre n−1*, COMPTES RENDUS, vol. 80 (1875), p. 1067.

tinct from b. A rational quartic results which has a triple point at O, the triple tangents being the joins of O to the three points in which the second fixed line meets R. Indeed, any rational plane curve of order n having a multiple point of order $n-1$ may be obtained in this way.

Evidently the point M is given by means of a parabolic substitution operating on the line l. The self-corresponding point of this substitution is the point at infinity on l. In this note we introduce homogeneous coördinates which has the effect of replacing the above metric determination of M by a purely projective one. A simple geometric characterization of the relations between the conic C, the line b, and the cubic R is obtained.

2. *Zahradnik's Formula.* Let the fixed point O be (0,0,1) Then the conic C may be written

$$ax^2+bxy+cy^2+dxz+eyz=0\ , \tag{1}$$

and the fixed line b

$$px+qy+rz=0\ . \tag{2}$$

The equation of R is, according to Zahradnik,

$$(ax^2+bxy+cy^2)\ (px+qy+rz)-(dx+ey)\ (px+qy)z=0 \tag{3}$$

a result which may be readily verified.

The tangent to C at O meets the fixed line b in a point which is a flex for R. The flex-tangent at this point is the line b. We shall not insist upon the verification of these statements at this time as their truth will appear a bit later. Moreover, one would suspect this from the construction.

In order to obtain the desired relations between C and R we solve the converse problem of finding C and b when R is given.

3. *Determination of C and b when R has Distinct Tangents at O.* If R has a double point with distinct—real or imagin-

ary—tangents, we may canonize its equation by means of a convenient choice of the triangle of reference. Let the double point be (0, 0, 1) ; $y=0$ be a line joining the double point to a real flex-point (there is at least one) ; $z=0$ be the flex-line (the line on which the three flexes of R lie) ; $x=0$ be the harmonic of $y=0$ with respect to the pair of tangents to R at O. With these assumptions the equation of R may be written

$$(R) \qquad z(x^2+\epsilon y^2)+y(3x^2-\epsilon y^2)=0\ ,$$

in which ϵ is a real number not zero.

We introduce a parameter λ into this equation by writing it in the form

$$(R_\lambda) \qquad z(x^2+\epsilon y^2)+\lambda y(3x^2-\epsilon y^2)=0\ .$$

The hessian of R_λ is

$$(H_\lambda) \qquad z(x^2+\epsilon y^2)-3\lambda y(3x^2-\epsilon y^2)=0,$$

from which we infer that it also belongs to the family. Hence the curves R_λ form a *syzygetic* family all of whose members have the same double point and tangents there. All the cubics R_λ have the same flex-points $(\sqrt{\epsilon}, \sqrt{3}, 0)$, $(-\sqrt{\epsilon}, \sqrt{3}, 0)$, (1, 0, 0) which we denote by F_1, F_2, F_3, respectively. Consequently, the members of the family have the same flex-line $z=0$. The only points of intersection of different members of the family are the flex-points and the double point. Hence a particular cubic R_{λ_0} may be individualized by specifying a point, other than the above fixed points of intersection, through which it must pass.

For R_λ Zahradnik's formula (3) becomes

$$(4) \quad z(x^2+\epsilon y^2)+\lambda y(3x^2-\epsilon y^2)\equiv(ax^2+bxy+cy^2)(px+qy+rz) \\ -(dx+ey)(px+qy)z\ .$$

There are six essential constants in the equation of a cubic with a double point at a given point, and an equal number in the equations of C and b. Hence, upon equating coefficients of like terms in (4), a finite number of solutions are obtained. They are

$$(C_\lambda^{(i)}) \qquad \frac{3\lambda}{p_i}xy+\frac{2\epsilon}{3}y^2-\frac{1}{p_i}xz+\frac{2\epsilon}{9\lambda}yz=0\ ,$$

$$(b_\lambda^{(i)}) \qquad p_ix-\frac{3\lambda}{2}y+z=0\ , \qquad (i=1,\ 2)$$

where $4\epsilon p^2=27\lambda^2$, the positive root being indicated by p_1;

$$(C_\lambda^{(i)}) \qquad x^2-\frac{\epsilon}{3}y^2-\frac{4\epsilon}{9\lambda}yz=0\ ,$$

$$(b_\lambda^{(i)}) \qquad 3\lambda y+z=0\ .$$

4. *Properties of the Solutions.* Denote the point of intersection of $b_\lambda^{(1)}$, $b_\lambda^{(2)}$, by $B_\lambda^{(3)}$, etc. Then

$$B_\lambda^{(1)}=(\sqrt{3\epsilon},\ -1,\ 3\lambda),\quad B_\lambda^{(2)}=(-\sqrt{3\epsilon},\ -1,\ 3\lambda),$$
$$B_\lambda^{(3)}=(0,\ 2,\ 3\lambda)\ .$$

As λ varies $B_\lambda^{(1)}$, $B_\lambda^{(2)}$, $B_\lambda^{(3)}$ move along $x+\sqrt{3\epsilon}\,y=0$, $x-\sqrt{3\epsilon}\,y=0$, $x=0$, respectively. The first two lines are harmonic with respect to the pair $xy=0$. The lines $b_\lambda^{(1)}$, $b_\lambda^{(2)}$, $b_\lambda^{(3)}$ are the flex-tangents at F_1, F_2, F_3, respectively. The three conics pass through the double point O. They touch by pairs in the three flex-points. In fact, $C_\lambda^{(1)}$ and $C_\lambda^{(2)}$ are tangent at $F_3=(1,\ 0,\ 0)$. Their fourth point of intersection is $(0,-1,\ 3\lambda)$. Similarly, $C_\lambda^{(1)}$ and $C_\lambda^{(3)}$ touch at $F_2=(-\sqrt{\epsilon},\ \sqrt{3},\ 0)$ and have their fourth meet at $(\sqrt{3\epsilon},\ 1,\ 6\lambda)$. The meets of $C_\lambda^{(2)}$ and $C_\lambda^{(3)}$ are found by changing the sign of the radical involving ϵ. Hence the variable point of intersection of the conics, in the above order, always lies on the lines

$$x=0\ ,\quad x-\sqrt{3\epsilon}\ y=0\ ,\quad x+\sqrt{3\epsilon}\ y=0\ ,$$

which are the same as those on which $B_\lambda^{(i)}$ move.

It is a simple matter to show that the polar of $B_\lambda^{(i)}$ with respect to $C_\lambda^{(i)}$ is the line $b_\lambda^{(i)}$ and that the tangents to $C_\lambda^{(i)}$ at

(0, 0, 1) meet R_λ in F_i. Also, that the real points of intersection of $C_\lambda^{(i)}$ and R_λ, other than O, lie on lines which pass through F_i.

The jacobian of the three conics $C_\lambda^{(i)}$ is

$$(J_\lambda) \qquad z(x^2+\epsilon y^2)+3\lambda y(3x^2-\epsilon y^2)=0\ ,$$

so that it also belongs to the syzygetic family. Let (x, y, z) be any point P on J_λ. The three polar lines of P with respect to $C_\lambda^{(i)}$ meet in a point Q whose coördinates are $(\epsilon y^2, -xy, 6\lambda xy+xz)$. This point has been called the *conjugate* of P and lies on J_λ. The conjugates of the three flex-points of the syzygetic family are $Q_1=(\sqrt{3\epsilon}, -1, 6\lambda)$, $Q_2=(-\sqrt{3\epsilon}, -1, 6\lambda)$, $Q_3=(0, 1, 3\lambda)$ from which we infer that the three points Q coincide with the variable points of intersection of the conics $C_{-\lambda}^{(i)}$ belonging to $R_{-\lambda}$. The associated cubics R_λ and $R_{-\lambda}$ meet $x=0$ in two points which form an harmonic pair with (0, 0, 1) and (0, 1, 0). They are the only cubics of the family which meet $x=0$ in these two points.

The polar conic of $R_{-\lambda}$ with respect to any point (α, β, γ) is

$$(5) \quad (\gamma-3\beta\lambda)x^2-6\alpha\lambda xy+\epsilon(\gamma+3\beta\lambda)y^2+2\alpha xz+2\beta\epsilon yz=0\ .$$

For $(\alpha, \beta, \gamma)=(\sqrt{3\epsilon}, -1, 3\lambda)$, $(+\sqrt{3\epsilon}, +1, 3\lambda)$, $(0, 2, -3\lambda)$, the equation (4) gives the three conics $C_\lambda^{(i)}$. But these points are associated with $R_{-\lambda}$ in the same manner that $B_{-\lambda}^{(i)}$ are related to R_λ, that is, they are the points $B_{-\lambda}^{(i)}$. It is a simple matter to show that $B_{-\lambda}^{(i)}$ are the poles of the flex-line $z=0$ with respect to the conics $C_\lambda^{(i)}$.

We may formulate our principal results as follows. *Any rational plane cubic R having distinct tangents at its double point can be generated, in the sense of Zahradnik, in three ways. The three fixed lines are the flex-tangents of the cubic, while the three conics are the polar conics, with respect to $R_{-\lambda}$, of the points of intersection of the three flex-tangents of $R_{-\lambda}$.*

5. *Determination of C and b when R has a Cusp at O.* Let the cusp be at (0, 0, 1) with $x=0$ as the tangent and (1, 0, 0) be the one flex. The equation of R may be written

$$x^2(\alpha y+\beta z)+\gamma y^3=0\ .$$

A real projective transformation—of no geometrical significance—may be found which will give the equation the form

$$x^2(ky+z)+y^3=0\ ,$$

where k may be any real number including zero. For this form $ky+z=0$ is, of course, the flex-tangent.

With the use of formula (3) we again find three solutions for C and b, namely,

$$(C^{(i)})\qquad 2k^2xy\mp 2k\sqrt{-k}y^2-2kxz\pm\sqrt{-k}yz=0\ ,$$

$$(b^{(i)})\qquad \pm 2k\sqrt{-k}x-2ky+z=0\ ,\qquad (i=1,2),$$

in which $i=1$ refers to the upper sign;

$$(C^{(3)})\qquad k^2x^2+ky^2+yz=0\ ,$$

$$(b^{(3)})\qquad ky+z=0\ .$$

Since k may be made to assume any real value without affecting the geometrical significance of our equations, we disregard the first two solutions as giving analytic solutions of no real interest.

As before, the fixed line $b^{(3)}$ is the flex-tangent. A simple calculation shows that $C^{(3)}$ meets the cubic R in only two real points, other than (0, 0, 1). These points lie on the line $2ky+z=0$. The two pairs of lines $y=0$, $2ky+z=0$ and $ky+z=0$, $z=0$ are harmonic. Moreover, the two tangents to $C^{(3)}$ from the flex-point are the flex-tangent and $y=0$. These properties serve to characterize $C^{(3)}$ and $b^{(3)}$. In fact, if we are given any cuspidal cubic referred to a triangle whose sides are the cuspidal tangent, the line joining the cusp to the flex-point, and a third line through the flex-point which is not the flex-tangent, we obtain the line b by drawing the flex-tangent. The conic C is determined as that conic whose tangents from the flex-point are the line joining the double point to the flex, and the flex-tangent; and which meets R in the two points cut out from R by the line through the flex harmonic with $y=0$ with respect to the pair $z=0$ and the flex-tangent. These conditions completely determine C.

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