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LIMIT THEORY FOR THE SAMPLE COVARIANCE AND CORRELATION FUNCTIONS OF MOVING AVERAGES

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Let $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ be a moving average process where the Z_t's are iid and have regularly varying tail probabilities with index $\alpha > 0$. The limit distribution of the sample covariance function is derived in the case that the process has a finite variance but an infinite fourth moment. Furthermore, in the infinite variance case $(0 < \alpha < 2)$, the sample correlation function is shown to converge in distribution to the ratio of two independent stable random variables with indices α and $\alpha/2$, respectively. This result immediately gives the limit distribution for the least squares estimates of the parameters in an autoregressive process.

1. Introduction. We consider the discrete time moving average process

$$X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}, \tag{1.1}$$

where $\{Z_t, \ -\infty < t < \infty\}$ is an independent and identically distributed (iid) sequence of random variables with regularly varying tail probabilities. More specifically, we assume

$$P(|Z_k| > x) = x^{-\alpha} L(x) \tag{1.2}$$

with $\alpha > 0$ and $L(x)$ a slowly varying function at ∞ and,

$$\frac{P(Z_k > x)}{P(|Z_k| > x)} \to p \quad \text{and} \quad \frac{P(Z_k < -x)}{P(|Z_k| > x)} \to q \tag{1.3}$$

as $x \to \infty$, $0 \le p \le 1$ and $q = 1 - p$. Under these assumptions on the noise sequence, the series defined in (1.1) exists (cf. Cline, 1983) provided

$$\sum_{j=-\infty}^{\infty} |c_j|^{\delta} < \infty \quad \text{for some } 0 < \delta < \alpha,\ \delta \le 1. \tag{1.4}$$

Note that any stationary ARMA process driven by the $\{Z_t\}$ sequence, has such a representation.

There has been increasing interest in modelling certain time series phenomena by an ARMA process with heavy tailed noise variables. For example, certain

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signal processes appear to be modelled better when the signal and/or noise has a heavy-tailed distribution rather than a Gaussian distribution. Mertz (1965) and Stuck and Kleiner (1974) have demonstrated this for telephone signals, as has Evans (1969) for signals with ELF noise and Rybin (1978) for strong narrow band signals. Fama (1965) has similarly modelled stock market prices. Reeves (1969) has considered air turbulence and Safiullin and Chabdarov (1978) have investigated radio navigation with processes involving non-Gaussian noise. The ARMA model is usually the basis for such processes.

In Davis and Resnick (1985), the weak limit behavior of the sample covariance function for the $\{X_t\}$ sequence was derived in the $0 < \alpha < 2$ case. It then followed immediately that the sample correlation function $\hat{\rho}(h) = \sum_{t=1}^{n-h} X_t X_{t+h} / \sum_{t=1}^{n} X_t^2$, $h > 0$, converges in probability to the analogue of the correlation function defined by $\rho(h) = \sum_{j=-\infty}^{\infty} c_j c_{j+h} / \sum_{j=-\infty}^{\infty} c_j^2$. A more refined result for the sample correlation function from an AR(p) process with errors satisfying (1.2) and (1.3) was given by Kanter and Steiger (1974) and Hannan and Kanter (1977). They proved that for any $\delta > \alpha$,

$$n^{1/\delta}(\hat{\rho}(h) - \rho(h)) \to_P 0$$

with a similar result holding for the least squares estimates of the parameters in the AR(p) model. Yohai and Maronna (1977) also considered AR(p) processes and showed that $n^{1/2}(\hat{\rho}(h) - \rho(h))$ is bounded in probability provided the Z_t's are symmetrically distributed and $E \log^+|Z_t| < \infty$. We provide a much more precise description of the limiting behavior of $\hat{\rho}(h)$ for infinite order moving averages which includes as a special case the AR(p) process considered by the above authors. Of course if the Z_t's have a finite variance then $n^{1/2}(\hat{\rho}(h) - \rho(h))$ is asymptotically normal under mild restrictions on the coefficients $\{c_j\}$ (cf. Anderson, 1971, page 489).

In Section 2, the limit distribution of the sample covariance function is derived for the case $2 \le \alpha < 4$. In the special case, $2 < \alpha < 4$, the process has a finite variance but an infinite fourth moment. It turns out that, as in the $0 < \alpha < 2$ case, the limit behavior of the sample covariance function is determined by the partial sums $\sum_{t=1}^{n} Z_t^2$. We also consider in Section 2 the situation when Z_t^2 belongs to the normal domain of attraction with an infinite variance.

The weak limit of the sample correlation function in the infinite variance case ($0 < \alpha < 2$) is considered in Section 4. It is shown that there exists a slowly varying function at ∞, $\tilde{L}(\cdot)$, such that $n^{1/\alpha}\tilde{L}(n)(\hat{\rho}(h) - \rho(h))$ converges in distribution to the ratio of two independent stable random variables with indices α and $\alpha/2$, respectively. If the tail distribution of $|Z_1|$ is asymptotically equivalent to a Pareto (as is the case when the Z_i's have a stable distribution), then we may take $\tilde{L}(n) = (\log n)^{-1/\alpha}$. Whereas the asymptotic properties of the sample covariance function are governed by the partial sums $\sum_{t=1}^{n} Z_t^2$, the weak limit behavior of the sample correlation function is determined by the vector of partial sums $(\sum_{t=1}^{n} Z_t^2, \sum_{t=1}^{n} Z_t Z_{t+1}, \dots, \sum_{t=1}^{n} Z_t Z_{t+h})$. In Section 3, we show that this sequence of vector-valued random variables converges in distribution to a vector $(S_0, S_1, \dots, S_h)$ of independent nonnormal stable random variables. This result is proved using point process techniques and ideas from extreme value theory. The limit random variables for the sample correlation function $\hat{\rho}(h)$ can then be identified as

$$Y_h = \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h))S_j/S_0.$$

In the classical case ($\mathrm{var}(Z_t) < \infty$) the same result is true where the S_j's $j \geq 1$ are iid $N(0,1)$ rv's and $S_0 \equiv 1$ and this provides an easy way to compute asymptotic covariances of the $\hat{\rho}(h)$'s. Further discussion on this point is contained in Section 4.

The limit results derived for the sample correlation function enable $\hat{\rho}(h)$ to be used for model identification and estimation of parameters in the class of ARMA models. In particular, limit distributions for method-of-moments type estimators of the parameters in an ARMA process can be derived (some examples are considered in Section 5). These estimators will be weakly consistent regardless of the value of α. On the other hand, if more detailed information about the distribution of the residuals is known (for example the value of α in (1.2)) then there may be better estimation methods such as minimizing the α-dispersion (Stuck, 1978, and Cline, 1983) or minimizing absolute deviations (Bloomfield and Steiger, 1983). In the absence of knowledge about α, one may fall back on the following iterative procedure: (a) obtain preliminary estimates of the parameters using the sample correlation function; (b) estimate α based on the resulting estimated residuals from (a) (cf. Hall (1982), DuMouchel, 1983); (c) update the estimated parameters by minimizing the α-dispersion between the predicted and observed values.

2. Sample covariance function. The aim of this section is to derive the weak limit of the sample covariance function for the process $\{X_t\}$ satisfying (1.1) with $2 \leq \alpha < 4$. Assume

$$X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j} \quad \text{with} \quad \sum_{j=-\infty}^{\infty} |c_j| < \infty, \tag{2.1}$$

where the Z_t satisfies (1.2) and (1.3). Put $a_n = \inf\{x\colon P(|Z_1| > x) \leq n^{-1}\}$ and define the sample covariance function by

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n} X_t X_{t+h}, \qquad h \geq 0.$$

The following proposition is the key step in evaluating the weak limit behavior of $\hat{\gamma}(h)$.

PROPOSITION 2.1. *If $2 \leq \alpha < 4$ and $EZ_t = 0$, then for every positive integer h,*

$$a_n^{-2}\left(n\hat{\gamma}(h) - \sum_{t=1}^{n} \sum_{i=-\infty}^{\infty} c_i c_{i+h} Z_{t-i}^2\right) \to_P 0. \tag{2.2}$$

PROOF. We have

$$\begin{aligned}
a_n^{-2}&\left(\sum_{t=1}^{n} X_t X_{t+h} - \sum_{t=1}^{n}\sum_{i=-\infty}^{\infty} c_i c_{i+h} Z_{t-i}^2\right)\\
&= a_n^{-2}\left(\sum_{t=1}^{n}\sum_{i\neq j} c_i c_{j+h} Z_{t-i} Z_{t-j}\right)\\
&= a_n^{-2}\sum_{t=1}^{n}\sum_{i\neq j} c_i c_{j+h}\left(Z_{t-i}1_{[|Z_{t-i}|\le a_n]} - \mu_n\right)\left(Z_{t-j}1_{[|Z_{t-j}|\le a_n]} - \mu_n\right)\\
&\quad + a_n^{-2}\mu_n\sum_{t=1}^{n}\sum_{i\neq j} c_i c_{j+h}\left(Z_{t-i}1_{[|Z_{t-i}|\le a_n]} + Z_{t-j}1_{[|Z_{t-j}|\le a_n]}\right)\\
&\quad + a_n^{-2}\sum_{t=1}^{n}\sum_{i\neq j} c_i c_{j+h} Z_{t-i} Z_{t-j} 1_{[|Z_{t-i}|> a_n \text{ or } |Z_{t-j}|> a_n]}\\
&\quad - n a_n^{-2}\mu_n^2\sum_{i\neq j} c_i c_{j+h}\\
&= A + B + C + D,
\end{aligned}$$

where $\mu_n = EZ_1 1_{[|Z_1|\le a_n]}$. We shall show that A, B, $C \to_P 0$ and $D \to 0$.

Define

$$Z_{t,n} = Z_t 1_{[|Z_t|\le a_n]} - \mu_n$$

and we have

$$\operatorname{var}(A) = a_n^{-4}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{i\neq j}\sum_{k\neq l} c_i c_{j+h} c_k c_{l+h} E\left(Z_{t-i,n} Z_{t-j,n} Z_{s-k,n} Z_{s-l,n}\right).$$

Since $\{Z_{t,n}, -\infty < t < \infty\}$ is for each n an iid sequence of zero mean random variables, the above expectation is zero unless $\{t-i, t-j\} = \{s-k, s-l\}$. When this is the case, the expectation is of the form

$$\begin{aligned}
EZ_{1,n}^2 Z_{2,n}^2 &= EZ_{1,n}^2 EZ_{2,n}^2\\
&\le \left(EZ_1^2 1_{[|Z_1|\le a_n]}\right)^2 = \sigma_n^4,
\end{aligned}$$

where $\sigma_n^2 = EZ_1^2 1_{[|Z_1|\le a_n]}$. Hence

$$\begin{aligned}
\operatorname{var}(A) &\le a_n^{-4}\sigma_n^4\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{i\neq j}\left(|c_i||c_{j+h}||c_{s-t+i}||c_{s-t+j+h}|\right.\\
&\qquad\qquad \left. + |c_i||c_{j+h}||c_{s-t+j}||c_{s-t+i+h}|\right)\\
&\le \sigma_n^4 a_n^{-4}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\left(\sum_i |c_i c_{i+s-t}|\right)^2 + \left(\sum_i |c_i c_{i+h+s-t}|\right)\left(\sum_j |c_{j+h} c_{j+s-t}|\right)\right)\\
&\le \sigma_n^4 a_n^{-4} n\sum_{|t|<n}\left(\left(\sum_i |c_i c_{i+t}|\right)^2 + \left(\sum_i |c_i c_{i+h+t}|\right)\left(\sum_j |c_{j+h} c_{t+j}|\right)\right)\\
&\le 2\left(\sum_i |c_i|\right)^4 n a_n^{-4}\sigma_n^4.
\end{aligned}$$

For $2 < \alpha < 4$, σ_n^2 has a finite limit and in the $\alpha = 2$ case it is slowly varying by Karamata's theorem (Feller, 1971). So in either case σ_n^4 is slowly varying. Moreover, a_n is regularly varying with index $1/\alpha$, which together with the slow variation of σ_n^4, implies $na_n^{-4}\sigma_n^4 \to 0$ as $n \to \infty$. Thus, $\text{var}(A) \to 0$ as desired.

As for the term B, we have

$$E|B| \le 2na_n^{-2}|\mu_n|\left(\sum_i |c_i|\right)^2 E|Z_1|1_{[|Z_1| \le a_n]}$$
$$\le 2\left(\sum_i |c_i|\right)^2 E|Z_1|na_n^{-2}|\mu_n|.$$

Since $EZ_t = 0$ by assumption,

$$|\mu_n| = |EZ_1 1_{[|Z_1| > a_n]}| \le E|Z_1|1_{[|Z_1| > a_n]} \sim \frac{\alpha - 1}{\alpha}\frac{a_n}{n}$$

by Karamata's theorem. Hence $na_n^{-2}|\mu_n| \to 0$ as $n \to \infty$.

Next

$$E|C| \le na_n^{-2}\left(\sum_i |c_i|\right)^2 E|Z_1Z_2|1_{[|Z_1| > a_n \text{ or } |Z_2| > a_n]}$$
$$\le 2na_n^{-2}\left(\sum_i |c_i|\right)^2 E|Z_2|E|Z_1|1_{[|Z_1| > a_n]}$$
$$\to 0$$

by Karamata's theorem as for B. Finally, $D = O(na_n^{-2}\mu_n^2) \to 0$ since for B we have already proved $na_n^{-2}|\mu_n| \to 0$ and this completes the proof. □

For $\alpha > 2$, define

$$\gamma(h) = \text{cov}(X_t, X_{t+h})$$
$$= \sigma^2\left(\sum_{j=-\infty}^{\infty} c_j c_{j+h}\right),$$

where $\sigma^2 = \text{var}(Z_t)$. The next theorem gives the main result of this section. Here and in what follows, convergence in distribution is denoted by " $\Rightarrow$."

THEOREM 2.2. *Suppose $\{X_t\}$ is given by* (2.1) *where $\{Z_t\}$ satisfies* (1.2) *and* (1.3) *with $2 \le \alpha < 4$. If $EZ_t = 0$, then for any positive integer l*

$$(2.3)\quad \left(na_n^{-2}(\hat\gamma(h) - b_{h,n}), 0 \le h \le l\right) \Rightarrow S\left(\sum_j c_j^2, \sum_j c_j c_{j+1}, \dots, \sum_j c_j c_{j+l}\right),$$

where S is a stable random variable with index $\alpha/2$ and $b_{h,n} = \sum_{i=-\infty}^{\infty} c_i c_{i+h} EZ_1^2 1_{[|Z_1| \le a_n]}, 0 \le h \le l$. Moreover, if $2 < \alpha < 4$, then

$$(2.4)\quad \left(na_n^{-2}(\hat\gamma(h) - \gamma(h)), 0 \le h \le l\right) \Rightarrow \left(S - \frac{\alpha}{\alpha - 2}\right)(\gamma(0), \dots, \gamma(l))/\sigma^2.$$

PROOF. By Theorem 4.1 in Davis and Resnick (1984),

$$a_n^{-2}\sum_{t=1}^{n}\sum_{i=-\infty}^{\infty} c_i c_{i+h}\left(Z_{t-i}^2-\sigma_n^2\right) \Rightarrow \sum_{i=-\infty}^{\infty} c_i c_{i+h} S \quad \text{for all } h \ge 0,$$

where $\sigma_n^2 = EZ_1^2 1_{[|Z_1|\le a_n]}$ and S is a stable random variable with index $\alpha/2$. From the proof of this same theorem, we have for any positive integer l

$$a_n^{-2}\left(\sum_{t=1}^{n}\sum_i c_i^2\left(Z_{t-i}^2-\sigma_n^2\right), \sum_{t=1}^{n}\sum_i c_i c_{i+1}\left(Z_{t-i}^2-\sigma_n^2\right),\dots, \sum_{t=1}^{n}\sum_i c_i c_{i+l}\left(Z_{t-i}^2-\sigma_n^2\right)\right)$$

$$\Rightarrow S\left(\sum_j c_j^2, \sum_j c_j c_{j+1},\dots,\sum_j c_j c_{j+l}\right).$$

This combined with Proposition 2.1 proves (2.3).

If $\alpha > 2$, then $\sigma_n^2 \to \sigma^2$ and by Karamata's theorem,

$$n\sigma^2 a_n^{-2} - n\sigma_n^2 a_n^{-2} = n a_n^{-2} EZ_1^2 1_{[|Z_1|> a_n]} \to \frac{\alpha}{\alpha-2},$$

so that by the convergence of types result, (2.4) holds. □

COROLLARY. *The same limit law is attained in Theorem* 2.2 *if* $\hat{\gamma}(h)$ *is replaced by a mean corrected version*

$$\tilde{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_t-\overline{X})(X_{t+h}-\overline{X}), \quad \textit{where } \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The proof of this corollary is analogous to that of the corollary following Theorem 4.2 in Davis and Resnick (1985) and is therefore omitted. Also note that the corollary remains true if $EZ_t \neq 0$ by considering the process $X_t - EX_t = \sum_{j=-\infty}^{\infty} c_j(Z_{t-j} - EZ_{t-j})$.

Corresponding to the case $\alpha = 4$ we have the following result.

PROPOSITION 2.3. *Suppose* $\{X_t\}$ *is defined by* (2.1) *with* $EZ_t = 0$ *and*

$$EZ_1^4 1_{[|Z_1|\le t]} = L(t)$$

is slowly varying with $\lim_{t\to\infty} L(t) = \infty$. *Define* α_n *by*

$$\frac{nL\left(\alpha_n^{1/2}\right)}{\alpha_n^2} \to 1,$$

so that α_n *is regularly varying with index* $\frac{1}{2}$. *If* $a_n = \alpha_n^{1/2}$ *then in* $\mathbb{R}^{l+1}$

(2.5) $$\left(n a_n^{-2}(\hat{\gamma}(h)-\gamma(h)), 0 \le h \le l\right) \Rightarrow N\cdot(\gamma(0),\dots,\gamma(l))/\sigma^2,$$

where N *is a* $N(0,1)$ *random variable.*

REMARKS. (1) Define $L_1(x) = L(x^{1/2})$ so that L_1 is also slowly varying (de Haan, 1970, page 21). Then α_n must satisfy $nL_1(\alpha_n)/\alpha_n^2 \to 1$. Set $U_1(x) = x^2/L_1(x)$ so that U_1 is regularly varying with index 2 and α_n satisfies $U_1(\alpha_n) \sim n$, and this shows α_n may be taken as the asymptotic inverse of U_1 at the point n (cf. Seneta, 1976, page 21).

(2) For the classical result assuming $EZ_1^4 < \infty$, see Anderson (1971, page 478).

PROOF. We begin by showing the analogue of Proposition 2.1. The difference

$$a_n^{-2}\left(n\hat{\gamma}(h) - \sum_{t=1}^{n}\sum_{i=-\infty}^{\infty}(c_i c_{i+h})Z_{t-i}^2\right)$$

is again decomposed into the pieces $A + B + C + D$.

We have $\operatorname{var}(A) = O(na_n^{-4})$. Since $L(t) \to \infty$ we have $\alpha_n/\sqrt{n} \to \infty$ and hence

$$na_n^{-4} = n\alpha_n^{-2} \to 0,$$

as desired. For B we have

$$E|B| \leq (\text{const})na_n^{-2}E|Z_1|1_{[|Z_1| > a_n]}.$$

Since $L(t) + \int_0^t z^4 P[|Z_1| \varepsilon\, dz]$ we have

$$\begin{aligned} E|Z_1|1_{[|Z_1| > a_n]} &= \int_{a_n}^{\infty} zP[|Z_1| \varepsilon\, dz] = \int_{a_n}^{\infty} t^{-3}L(dt) \\ &= 3\int_{a_n}^{\infty} L(s)s^{-4}\, ds - L(a_n)a_n^{-3} \\ &= a_n^{-3}L(a_n)\left\{\int_1^{\infty} 3(L(a_n s)/L(a_n))s^{-4}\, ds - 1\right\}, \end{aligned}$$

so that

$$na_n^{-2}E|Z_1|1_{[|Z_1| > a_n]} = nL(a_n)a_n^{-5}\left\{\int_1^{\infty} 3(L(a_n s)/L(a_n))s^{-4}\, ds - 1\right\}.$$

However, since $nL(a_n)a_n^{-4} \to 1$, the above term is asymptotic to

$$a_n^{-1}\left\{\int_1^{\infty} 3(L(a_n s)/L(a_n))s^{-4}\, ds - 1\right\},$$

which goes to zero since $a_n \to \infty$ and the expression within the braces goes to zero by Karamata's theorem. The term $E|C|$ is handled in the same way and D is of smaller order than $E|B|$ so the analogue of Proposition 2.1 is proved. □

Before continuing with the proof we need the following result.

PROPOSITION 2.4. *Suppose* $\{X_t\}$ *satisfies* (2.1) *with* $EZ_1 = 0$ *and* $U(t) = EZ_1^2 1_{[|Z_1| \leq t]}$ *slowly varying. Define* g_n *by*

$$ng_n^{-2}EZ_1^2 1_{[|Z_1| \leq g_n]} \to 1.$$

Then

$$g_n^{-1}\sum_{t=1}^{n} X_t \Rightarrow \left(\sum_{j=-\infty}^{\infty} c_j\right)N,$$

where N *is* $N(0,1)$.

PROOF. A proof can be fashioned after the method used in Davis and Resnick (1985) to prove Theorem 4.1. We have Z_1 in the domain of attraction of the normal so that $g_n^{-1}\sum_1^n Z_i \Rightarrow N$. Furthermore for $m \geq 1$

$$\mathbf{Y}_n = \left\{g_n^{-1}\sum_{t=1}^{n} Z_{t-j}, |j| \leq m\right\} \Rightarrow (N, N, \dots, N)$$

in $\mathbb{R}^{2m+1}$ and therefore by the continuous mapping theorem

$$(c_{-m}, \dots, c_m) \cdot \mathbf{Y}_n \Rightarrow \left(\sum_{|j| \leq m} c_j\right) N.$$

It remains to show

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[|g_n^{-1}\sum_{t=1}^{n} X_t - (c_{-m}, \dots, c_m) \cdot \mathbf{Y}_n| > \delta\right] = 0 \tag{2.6}$$

for any $\delta > 0$ as well as

$$\left(\sum_{|j|\leq m} c_j\right) N \Rightarrow \left(\sum_{j=-\infty}^{\infty} c_j\right) N, \qquad m \to \infty. \tag{2.7}$$

The validity of (2.7) is obvious.

We have that

$$\begin{aligned} g_n^{-1}\sum_{t=1}^{n} X_t - (c_{-m}, \dots, c_m) \cdot \mathbf{Y}_n &= g_n^{-1}\sum_{t=1}^{n}\sum_{|j|>m} c_j Z_{t-j} \\ &= g_n^{-1}\sum_{t=1}^{n}\sum_{|j|>m} c_j\left(Z_{t-j}1_{[|Z_{t-j}|\leq g_n]} - EZ_1 1_{[|Z_1|\leq g_n]}\right) \\ &\quad + g_n^{-1} n\left(\sum_{|j|>m} c_j\right) EZ_1 1_{[|Z_1|\leq g_n]} \\ &\quad + g_n^{-1}\sum_{t=1}^{n}\left(\sum_{|j|>m} c_j\right) Z_{t-j}1_{[|Z_{t-j}|>g_n]} \\ &= \alpha + \beta + \gamma. \end{aligned}$$

Now

$$\lim_{m\to\infty}\limsup_{n\to\infty} P[|\alpha| > \delta] = 0$$

by an argument identical to one used in the proof of Theorem 4.1 of Davis and Resnick (1985). (We use the fact that $ng_n^{-2}EZ_1^2 1_{[|Z_1|\leq g_n]} \to 1$.) For the other two terms we calculate

$$\begin{aligned} |EZ_1 1_{[|Z_1|\leq g_n]}| &= |EZ_1 1_{[|Z_1|>g_n]}| \\ &\leq E|Z_1| 1_{[|Z_1|>g_n]} = \int_{g_n}^{\infty} tP[|Z_1| \varepsilon\, dt] = \int_{g_n}^{\infty} t^{-1} U(dt) \\ &= \int_{g_n}^{\infty} s^{-2} U(s)\, ds - g_n^{-1} U(g_n), \end{aligned}$$

and so applying Karamata's theorem (recall U is slowly varying) we get

$$\lim_{n\to\infty} \frac{g_n E Z_1 1_{[|Z_1|\le g_n]}}{U(g_n)} = 0.$$

Thus

$$\begin{aligned}|\beta| &\le g_n^{-1} n |EZ_1 1_{[|Z_1|\le g_n]}| \sum_{|j|>m} |c_j| \\ &= \frac{nU(g_n)}{g_n^2} \frac{g_n|EZ_1 1_{[|Z_1|\le g_n]}|}{U(g_n)} \sum_{|j|>m} |c_j| \\ &\sim g_n |EZ_1 1_{[|Z_1|\le g_n]}| \sum_{|j|>m} |c_j| / U(g_n) \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Likewise

$$\begin{aligned}\lim_{m\to\infty} \limsup_{n\to\infty} P[|\gamma|>\delta] &\le \lim_{m\to\infty} \limsup_{n\to\infty} \delta^{-1} g_n^{-1} n |EZ_1 1_{[|Z_1|>g_n]}| \sum_{|j|>m} |c_j| \\ &= 0,\end{aligned}$$

as desired for the verification of (2.6). □

CONTINUATION OF THE PROOF OF PROPOSITION 2.3. From Proposition 2.4 we have (recall $\sigma^2 = EZ_1^2 = \operatorname{var}(Z_1)$)

$$\alpha_n^{-1} \sum_{t=1}^{n} \sum_{i=-\infty}^{\infty} c_i c_{i+h} \left(Z_{t-i}^2 - \sigma^2\right) \Rightarrow \left(\sum_{i=-\infty}^{\infty} c_i c_{i+h}\right) N$$

and hence, from the analogue of Proposition 2.1,

$$n a_n^{-2} (\hat{\gamma}(h) - \gamma(h)) \Rightarrow \left(\sum c_i c_{i+h}\right) N.$$

The assertion of Proposition 2.3 easily follows.

REMARK. The same limit law holds if $\hat{\gamma}(h)$ is replaced by the mean corrected version:

$$\tilde{\gamma}(h) = \frac{1}{n} \sum_{1}^{n} (X_t - \overline{X})(X_{t+h} - \overline{X}).$$

3. Sample covariance function of $\{Z_t\}$. Assume $\{Z_t\}$ is iid and satisfies (1.2) and (1.3) with $0 < \alpha < 2$. As before define

$$a_n = \inf\{x : P(|Z_1| > x) \le n^{-1}\}. \tag{3.1}$$

Applying Theorem 4.2 in Davis and Resnick (1985) to the Z_t sequence (i.e., take $c_j = 0,\ j \ne 0$ and $c_0 = 1$), we obtain

$$a_n^{-2} \sum_{t=1}^{n} Z_t Z_{t+h} \Rightarrow S \cdot 0 = 0 \quad \text{for all } h > 0$$

and

$$a_n^{-2} \sum_{t=1}^{n} Z_t^2 \Rightarrow S,$$

where S is a positive stable random variable with index $\alpha/2$. In this section, we give a different normalization for the partial sums $\Sigma_{t=1}^{n} Z_t Z_{t+h}$, $h > 0$ in order to get a nondegenerate weak limit. Not surprisingly, these partial sums (i.e., sample covariances) at different lags turn out to be asymptotically independent. This will be the main building block for deriving the limit distribution of the sample correlation function of the X_t process in the next section.

Throughout this section we shall assume $E|Z_1|^\alpha = \infty$. It then follows from Theorem 3.3(iv) in Cline (1983), that the product $Z_0 Z_1$ belongs to the α-domain of attraction. That is, $Z_0 Z_1$ satisfies

$$\frac{P(|Z_0 Z_1| > tx)}{P(|Z_0 Z_1| > t)} \to x^{-\alpha} \quad \text{as } t \to \infty,\ x > 0, \tag{3.2}$$

and

$$\frac{P(Z_0 Z_1 > t)}{P(|Z_0 Z_1| > t)} \to p^2 + (1-p)^2 \quad \text{as } t \to \infty, \tag{3.3}$$

where p is given in (1.3).

Define

$$\tilde{a}_n = \inf\{x\colon P(|Z_0 Z_1| > x) \le n^{-1}\}. \tag{3.4}$$

We first show that

$$\tilde{a}_n / a_n \to \infty. \tag{3.5}$$

Observe that for a fixed positive number M,

$$\frac{P(|Z_0 Z_1| > t)}{P(|Z_0| > t)} \ge \frac{P(|Z_0| > t/|Z_1|, |Z_1| \le M)}{P(|Z_0| > t)}$$

$$= \int_0^M \frac{P(|Z_0| > t/y)}{P(|Z_0| > t)} P(|Z_1| \varepsilon\, dy).$$

We then have, by Fatou's lemma and (1.2),

$$\liminf_{t \to \infty} \frac{P(|Z_0 Z_1| > t)}{P(|Z_0| > t)} \ge \int_0^M y^\alpha P(|Z_1| \varepsilon\, dy)$$

and upon letting $M \to \infty$, the lower bound converges to $E|Z_1|^\alpha = \infty$. It now is easy to check that (3.5) must hold.

The joint asymptotic behavior of the partial sums $(\Sigma_{t=1}^{n} Z_t^2, \Sigma_{t=1}^{n} Z_t Z_{t+1}, \ldots, \Sigma_{t=1}^{n} Z_t Z_{t+h})$ is handled using point processes techniques. For background on point processes, see Kallenberg (1976). Set $\mathbf{Y}_t = (Z_t, Z_t Z_{t+1}, \ldots, Z_t Z_{t+h})$ for $t = 0, \pm 1, \pm 2, \ldots$ and define $\mathbf{a}_n^{-1} \mathbf{Y}_t = (a_n^{-1} Z_t, \tilde{a}_n^{-1} Z_t Z_{t+1}, \ldots, \tilde{a}_n^{-1} Z_t Z_{t+h})$. The relevant sequence of point processes for this problem is given by

$$I_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}\mathbf{Y}_t},$$

which is defined on the state space $E = \mathbb{R}^{h+1} \setminus \{(0,0,\dots,0)\}$, where ε_x is the measure assigning unit mass to the point x and zero elsewhere. In defining a point process on E, we shall use the convention that if a point falls outside the state space it does not contribute to the sum. $\mathscr{E}$ will denote the usual product σ-algebra on E modified so that the compact subsets of E are those compact sets in $\mathbb{R}^{n+1}$ which are bounded away from $(0,0,\dots,0)$.

It will be shown that the sequence $\{I_n\}$ converges in distribution to a Poisson process defined as follows: Let

$$\sum_{k=1}^{\infty} \varepsilon_{j_k^{(1)}}, \sum_{k=1}^{\infty} \varepsilon_{j_k^{(2)}}, \dots, \sum_{k=1}^{\infty} \varepsilon_{j_k^{(h)}}$$

be h iid Poisson processes on $\mathbb{R} \setminus \{0\}$ with intensity measure given by

$$\tilde{\lambda}(dx) = \alpha\tilde{p}x^{-\alpha-1}1_{(0,\infty)}(x)\,dx + \alpha\tilde{q}(-x)^{-\alpha-1}1_{(-\infty,0)}(x)\,dx,$$

where $\tilde{p} = p^2 + (1-p)^2$ and $\tilde{q} = 1 - \tilde{p}$. Further let $\sum_{k=1}^{\infty} \varepsilon_{j_k^{(0)}}$ also be a Poisson process on $\mathbb{R} \setminus \{0\}$ independent of the h Poisson processes above with intensity $\lambda(dx) = \alpha p x^{-\alpha-1}1_{(0,\infty)}(x)\,dx + \alpha q(-x)^{-\alpha-1}1_{(-\infty,0)}(x)\,dx$. The limit point process is then

$$I = \sum_{k=1}^{\infty} \sum_{i=0}^{h} \varepsilon_{j_k^{(i)} \cdot \mathbf{e}_i},$$

where $\mathbf{e}_i \in \mathbb{R}^{h+1}$ is the basis element with ith component equal to one and the rest zero. In other words, the points of I are located on the coordinate axes, the points $\{j_k^{(i)}, k = 1,2,\dots\}$ lying on the axis determined by $\mathbf{e}_i$.

In order to establish $I_n \Rightarrow I$ it is convenient to first specify a class of sets (as in Section 2 of Davis and Resnick, 1985) which generate $\mathscr{E}$. Let S be the collection of all sets B of the form

$$B = (b_0, c_0] \times (b_1, c_1] \times \cdots \times (b_h, c_h],$$

which are bounded away from $(0,0,\dots,0)$ and $b_i < c_i$, $b_i \neq 0$, $c_i \neq 0$ for $i = 0,1,\dots,h$. It is clear that S is a DC-semiring (cf. Kallenberg, 1976, page 3). Moreover, since $B \in S$ is bounded away from zero, either

(C1) $$B \cap \{y\mathbf{e}_i \colon y \in \mathbb{R}\} = \phi \quad \text{for } i = 0,\dots,h,$$

or

(C2) $$B \cap \{y\mathbf{e}_i \colon y \in \mathbb{R}\} = \begin{cases} (b_j, c_j], & i = j, \\ \phi, & i \neq j. \end{cases}$$

That is, B has either empty intersection with all of the coordinate axes or intersects exactly one in an interval. Note that in (C2), $b_i < 0 < c_i$ for $i \neq j$ and $0 \notin (b_j, c_j]$. Further properties of these sets are developed in following proposition.

PROPOSITION 3.1.

(i) $nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B) \to 0$ *if* $B \in S$ *satisfies* C1.

(ii) $nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B) \to \lambda(b_0, c_0]$ *if* $B \in S$ *satisfies* C2 *with* $j = 0$,
$\to \tilde{\lambda}(b_j, c_j]$ *if* $B \in S$ *satisfies* C2 *with* $j \neq 0$.

(iii) $nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_1, \mathbf{a}_n^{-1}\mathbf{Y}_t \in B_2) \to 0$ *if* B_1 *and* $B_2 \in S$ *and* $1 < t \leq 1 + h$.

(iv) $n^2P(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_1, \mathbf{a}_n^{-1}\mathbf{Y}_t \in B_2) \leq C$ *for all* n *and* $t > 1 + h$ *where* C *is a constant depending only on the sets* B_1 *and* B_2 *in* S.

PROOF. (i) Setting $x^* = |b_0| \wedge |c_0| > 0$ and $y^* = |b_1| \wedge |c_1| > 0$, we have

$$\begin{aligned} nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B) &\leq nP(|Z_1| > a_n x^*, |Z_1 Z_2| > \tilde{a}_n y^*) \\ &\leq nP(|Z_1| > a_n M) \\ &\quad + nP(|Z_1| > a_n x^*, |Z_1 Z_2| > \tilde{a}_n y^*, |Z_1| \leq a_n M). \end{aligned}$$

From (1.2) and (3.1) we have $nP(|Z_1| > a_n M) \to M^{-\alpha}$ as $n \to \infty$, which can be made arbitrarily small by choosing M large. The second term is bounded by

$$\begin{aligned} nP\left(|Z_1| > a_n x^*, |Z_2| > \frac{\tilde{a}_n}{a_n}\frac{y^*}{M}\right) &\leq nP(|Z_1| > a_n x^*)P\left(|Z_2| > \frac{\tilde{a}_n}{a_n}\frac{y^*}{M}\right) \\ &\to (x^*)^{-\alpha} \cdot 0 = 0 \end{aligned}$$

since $\tilde{a}_n/a_n \to \infty$ by (3.5).

(ii) Suppose $j = 0$. Then, with $x^* = |b_0| \wedge |c_0|$, $y^* = \min_{1 \leq i \leq h}(|b_i| \wedge |c_i|) > 0$ and using an elementary bound, we have

$$|nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B) - nP(a_n b_0 < Z_1 \leq a_n c_0)| \leq nhP(|Z_1| > a_n x^*, |Z_1 Z_2| > \tilde{a}_n y^*),$$

which goes to zero as $n \to \infty$ by the proof in (i). Moreover, it follows from (1.2) and (1.3) that $nP(a_n b_0 < Z_1 \leq a_n c_0) \to \lambda(b_0, c_0]$. The argument for the case $j \neq 0$ is handled in the same manner and is omitted.

(iii) If either B_1 or B_2 satisfies C1, then we are done by (i). So suppose B_1 and B_2 satisfy C2 with $B_1 \cap \mathbf{e}_j = (b_j^{(1)}, c_j^{(1)}] \neq \phi$, $B_2 \cap \mathbf{e}_{j'} = (b_{j'}^{(2)}, c_{j'}^{(2)}] \neq \phi$. Then if $j \neq 0$ and $j' \neq 0$,

$$(3.6)\quad nP(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_1, \mathbf{a}_n^{-1}Y_t \in B_2) \leq nP(|Z_1 Z_{1+j}| > \tilde{a}_n x^*, |Z_t Z_{t+j}| > \tilde{a}_n y^*),$$

where $x^* = |b_j^{(1)}| \wedge |c_j^{(1)}|$ and $y^* = |b_{j'}^{(2)}| \wedge |c_{j'}^{(2)}|$. Now if $t \neq 1 + j$ and $t + j' \neq 1 + j$, then by independence

$$\begin{aligned} &nP(|Z_1 Z_{1+j}| > \tilde{a}_n x^*, |Z_t Z_{t+j'}| > \tilde{a}_n y^*) \\ &\quad = nP(|Z_1 Z_{1+j}|| > \tilde{a}_n x^*)P(|Z_t Z_{t+j'}| > \tilde{a}_n y^*) \\ &\quad \to 0. \end{aligned}$$

On the other hand, if $t = 1 + j$ or $t + j' = 1 + j$, then we have the bound

$$\begin{aligned} &nP(|Z_1 Z_2| > \tilde{a}_n x^*, |Z_2 Z_3| > \tilde{a}_n y^*) \\ &\quad \leq nP(|Z_2| > a_n M) + nP(|Z_1 Z_2| > \tilde{a}_n x^*, |Z_2 Z_3| > \tilde{a}_n y^*, |Z_2| \leq a_n M) \\ &\quad \leq nP(|Z_2| > a_n M) + nP(|Z_1 Z_2| > \tilde{a}_n x^*)P\left(|Z_3| > \frac{\tilde{a}_n}{a_n M} y^*\right) \\ &\quad \to M^{-\alpha} \quad \text{as } n \to \infty, \end{aligned}$$

where we have used (3.5) in he second term. Since M is arbitrary the left side of (3.6) must have a zero limit. The other cases $j = 0$ or $j' = 0$ are done in a similar way.

(iv) This follows easily from (i) and (ii) since for $t > 1 + h$ the vectors $\mathbf{Y}_1$ and $\mathbf{Y}_t$ are independent. □

PROPOSITION 3.2. *Let* $\{Z_t\}$ *be* iid *satisfying* (1.2) *and* (1.3) *with* $0 < \alpha < 2$ *and suppose* $E|Z_1|^\alpha = \infty$. *If* a_n *and* $\tilde{a}_n$ *are given by* (3.1) *and* (3.4) *we have*

$$I_n \Rightarrow I$$

in the sense of convergence of point processes on the space E (cf. *Kallenberg*, 1976).

PROOF. Since the point process I is simple, it suffices to show by Theorem 4.7 in Kallenberg (1976) that

$$EI_n(B) \to EI(B) < \infty \quad \text{for all } B \in S \tag{3.7}$$

and

$$P(I_n(R) = 0) \to P(I(R) = 0) \tag{3.8}$$

for all sets R which are a finite union of disjoint sets in S.

Clearly (3.7) is automatic from (i) and (ii) of Proposition 3.1 because I has all of its points on the coordinate axes. Now suppose $R = \cup_{j=1}^m B_j$ is a union of disjoint sets in S. For a fixed positive integer k, define $I^*_{[n/k]}(R) = \sum_{t=1}^{[n/k]} \varepsilon_{\mathbf{a}_n^{-1}\mathbf{Y}_t}(R)$ where $[x]$ is the greatest integer $\le x$. Using a Bonferroni-type inequality, stationarity, and the disjointness of the sets B_j, we have

$$\begin{aligned}\sum_{j=1}^m [n/k] P\big(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_j\big) - \sum_{i=1}^m \sum_{j=1}^m \sum_{t=2}^{[n/k]} [n/k] P\big(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_i, \mathbf{a}_n^{-1}\mathbf{Y}_t \in B_j\big) \\ \le P\big(I^*_{[n/k]}(R) > 0\big) \le \sum_{j=1}^m [n/k] P\big(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_j\big).\end{aligned}$$

It follows from above that

$$\sum_{j=1}^m [n/k] P\big(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_j\big) = EI^*_{[n/k]}(R) \to k^{-1}EI(R)$$

as $n \to \infty$. Applying Proposition 3.1(iii) and (iv), we also have

$$\limsup_{n\to\infty} \sum_{t=2}^{[n/k]} [n/k] P\big(\mathbf{a}_n^{-1}\mathbf{Y}_1 \in B_i, \mathbf{a}_n^{-1}\mathbf{Y}_t \in B_j\big) = o(1/k) \quad \text{as } k \to \infty$$

for $i, j = 1, \dots, m$, so that

$$\begin{aligned}1 - k^{-1}EI(R) &\le \liminf_{n\to\infty} P\big(I^*_{[n/k]}(R) = 0\big) \\ &\le \limsup_{n\to\infty} P\big(I^*_{[n/k]}(R) = 0\big) \le 1 - k^{-1}EI(R) + o(1/k).\end{aligned} \tag{3.9}$$

Since the vector-valued process $\mathbf{Y}_t$ is h-dependent, a standard argument (cf. Leadbetter, Lindgren, and Rootzén, 1983, Chapters 3 and 5) gives

$$P^k\big(I^*_{[n/k]}(R) = 0\big) - P\big(I_n(R) = 0\big) \to 0 \quad \text{as } n \to \infty \tag{3.10}$$

for every positive integer k. Taking the kth power of (3.9) and using (3.10), we obtain

$$\begin{aligned}\big(1 - k^{-1}EI(R)\big)^k &\le \liminf_{n\to\infty} P\big(I_n(R) = 0\big) \le \limsup_{n\to\infty} P\big(I_n(R) = 0\big)\\ &\le \big(1 - k^{-1}EI(R) + o(1/k)\big)^k.\end{aligned}$$

Now letting $k \to \infty$, we have $P(I_n(R) = 0) \to e^{-EI(R)}$. But I is a Poisson process so that $e^{-EI(R)} = P(I(R) = 0)$ which verifies (3.8) as desired. □

THEOREM 3.3. *Let $\{Z_t\}$ be iid satisfying* (1.2) *and* (1.3) *with $0 < \alpha < 2$ and $E|Z_1|^\alpha = \infty$. Then, if a_n and $\tilde{a}_n$ are given by* (3.1) *and* (3.4),

$$\left(a_n^{-2}\sum_{t=1}^n Z_t^2, \tilde{a}_n^{-1}\sum_{t=1}^n (Z_tZ_{t+1} - \mu_n), \dots, \tilde{a}_n^{-1}\sum_{t=1}^n (Z_tZ_{t+h} - \mu_n)\right) \Rightarrow (S_0, S_1, \dots, S_h),$$

where $\mu_n = EZ_1Z_21_{[|Z_1Z_2| \le \tilde{a}_n]}$ and $S_0, S_1, \dots, S_h$ are independent stable random variables; S_0 is positive with index $\alpha/2$ and $S_1, S_2, \dots, S_h$ are identically distributed with index α.

PROOF. Adapting the argument used in Section 2 of Resnick (1986) and in Section 4 of Davis and Resnick (1985) (see also Resnick and Greenwood, 1978) it is easy to show, for any $0 < \delta < 1$,

$$\begin{aligned}\Bigg(a_n^{-2}\sum_{t=1}^n Z_t^2 1_{[|Z_t| > a_n\delta]}, \tilde{a}_n^{-1}\sum_{t=1}^n \Big(&Z_tZ_{t+i}1_{[|Z_tZ_{t+i}| > \tilde{a}_n\delta]}\\ &- EZ_1Z_2 1_{[\tilde{a}_n\delta < |Z_1Z_2| \le \tilde{a}_n]}\Big), 1 \le i \le h\Bigg)\\ \Rightarrow (S_0^\delta, S_1^\delta, \dots, S_h^\delta),&\end{aligned}$$

where

$$S_0^\delta = \sum_{k=1}^\infty \big(j_k^{(0)}\big)^2 1_{[|j_k^{(0)}| > \delta]}$$

and

$$S_i^\delta = \sum_{k=1}^\infty j_k^{(i)} 1_{[|j_k^{(i)}| > \delta]} - \int_{|s| \in (\delta, 1]} s\tilde{\lambda}(ds)$$

for $i = 1, 2, \dots, h$. Clearly, $S_0^\delta, S_1^\delta, \dots, S_h^\delta$ are independent since the points $\{j_k^{(0)}\}, \{j_k^{(1)}\}, \dots, \{j_k^{(h)}\}$ are independent. The Itô representation implies $S_i^\delta \Rightarrow S_i$ as $\delta \to 0$, $i = 0, 1, \dots, h$ (cf. Resnick, 1986) where the vector $(S_0, S_1, \dots, S_h)$ is as described in the statement of the theorem. In view of Billingsley (1968, Theorem 4.2), the proof is complete once we show

$$(3.11)\qquad \lim_{\delta\to 0}\limsup_{n\to\infty} E\left(a_n^{-2}\sum_{t=1}^{n} Z_t^2 1_{[|Z_t|\le a_n\delta]}\right) = 0$$

and

$$(3.12)\quad \lim_{\delta\to 0}\limsup_{n\to\infty} \operatorname{var}\left(\tilde{a}_n^{-1}\sum_{t=1}^{n} Z_t Z_{t+i} 1_{[|Z_t Z_{t+i}|\le \tilde{a}_n\delta]}\right) = 0, \qquad i = 1,\dots,h.$$

The expectation in (3.11) is equal to $n/a_n^2\, EZ_1^2 1_{[|Z_1|\le a_n\delta]}$ which has the desired limit by Karamata's theorem (Feller, 1971, page 283). Since the process $\{Z_t Z_{t+i}, t = 0, \pm 1, \pm 2, \dots\}$ is i-dependent, (3.12) holds by the comment on the top of page 266, Davis (1983). □

REMARKS. (1) If the distribution of Z_t is symmetric then so is the distribution of $Z_t Z_{t+1}$, in which case $\mu_n = 0$.

(2) For $0 < \alpha < 1$, the theorem remains valid without centering the terms $Z_t Z_{t+i}$ by μ_n.

(3) In the case $1 < \alpha < 2$, $EZ_1Z_2 = (EZ_1)^2$ exists and from Karamata's theorem, $n\tilde{a}_n^{-2}(E(Z_1Z_2) - \mu_n) = n\tilde{a}_n^{-2}E(Z_1Z_2 1_{[|Z_1Z_2| > \tilde{a}_n]}) \to$ const. Thus, by the convergence of types result, Theorem 3.3 is also valid if μ_n is replaced by $\mu^2 = (EZ_1)^2$.

4. Sample correlation function of $\{X_t\}$. As before let $\{Z_t\}$ be iid satisfying (1.2) and (1.3) with $0 < \alpha < 2$, $E|Z_t|^\alpha = \infty$, and define

$$(4.1)\qquad X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j},$$

where

$$(4.2)\qquad \sum_{j=-\infty}^{\infty} |c_j|^\delta |j| < \infty \quad \text{with} \begin{cases} \delta = 1, & \text{if } \alpha > 1 \\ 0 < \delta < \alpha & \text{if } \alpha \le 1. \end{cases}$$

We shall first concentrate on the unadjusted sample correlation function defined by

$$(4.3)\qquad \hat{\rho}(h) = \frac{C(h)}{C(0)}, \qquad h \ge 0,$$

where

$$(4.4)\qquad C(h) = \sum_{t=1}^{n} X_t X_{t+h}.$$

The sum in (4.4) is terminated at n rather than $n - h$ for notational simplicity in the following arguments. All of the results in this section, however, remain valid if the upper limit is $n - h$. Put $\rho(h) = \sum_j c_j c_{j+h}/\sum_j c_j^2$, which in the case that $\operatorname{var}(Z_t) < \infty$, is equal to $\operatorname{corr}(X_t, X_{t+h})$. In Davis and Resnick (1985, Theorem 4.2) it was shown under condition (1.4) that $\hat{\rho}(h) \to_P \rho(h)$. Here, we consider the limit distribution of $\hat{\rho}(h)$, suitably normalized. We begin with the following proposition which is similar to Lemma 8.4.3 in Anderson (1971).

PROPOSITION 4.1. *Assume* (4.1), (4.2), *and* $E|Z_1|^\alpha = \infty$. *Then for every positive integer* h,

$$\tilde{a}_n^{-1}a_n^2\left(\hat{\rho}(h) - \rho(h) - [C(0)]^{-1}\sum_{t=1}^{n}\sum_{\substack{i,j\\ i\neq j}} c_i\big(c_{j+h} - c_j\rho(h)\big)Z_{t-i}Z_{t-j}\right) \tag{4.5}$$
$$\to_P 0,$$

where a_n *and* $\tilde{a}_n$ *are given by* (3.1) *and* (3.4), *respectively.*

PROOF. We have

$$\begin{aligned}
\hat{\rho}(h) - \rho(h) &= [C(0)]^{-1}(C(h) - \rho(h)C(0))\\
&= [C(0)]^{-1}\sum_{t=1}^{n}\left(\sum_i\sum_j c_ic_jZ_{t-i}Z_{t+h-j} - \rho(h)\sum_i\sum_j c_ic_jZ_{t-i}Z_{t-j}\right)\\
&= [C(0)]^{-1}\sum_{t=1}^{n}\sum_i\sum_j c_i\big(c_{j+h} - c_j\rho(h)\big)Z_{t-i}Z_{t-j},
\end{aligned}$$

so that the difference in (4.5) is equal to

$$\begin{aligned}
&\tilde{a}_n^{-1}a_n^2[C(0)]^{-1}\sum_{t=1}^{n}\sum_i\big(c_ic_{i+h}^2 - c_i^2\rho(h)\big)Z_{t-i}^2\\
&\quad= \tilde{a}_n^{-1}a_n^2[C(0)]^{-1}\sum_i\left(\big(c_ic_{i+h} - c_i^2\rho(h)\big)\sum_{t=1}^{n}Z_{t-i}^2\right)\\
&\quad= \tilde{a}_n^{-1}a_n^2[C(0)]^{-1}\sum_i\big(c_ic_{i+h} - c_i^2\rho(h)\big)\left(\sum_{t=1}^{n}Z_t^2 + U_{n,i}\right),
\end{aligned}$$

where $U_{n,i} = \sum_{t=1-i}^{n-i}Z_t^2 - \sum_{t=1}^{n}Z_t^2$ is the sum of at most $2i$ random variables. Since $a_n^{-2}C(0)$ converges in distribution (Theorem 4.2 in Davis and Resnick, 1985) and $\sum_i(c_ic_{i+h} - c_i^2\rho(h)) = 0$ it suffices to show

$$\limsup_{n\to\infty} E\left|\sum_i\big(c_ic_{i+h} - c_i^2\rho(h)\big)U_{n,i}\right|^{\delta/2} < \infty, \tag{4.6}$$

δ defined in (4.2). Because $\delta < \alpha$, $E|Z_1|^\delta < \infty$, so that by the triangle inequality and assumption (4.2), we have

$$\begin{aligned}
&E\left|\sum_i\big(c_ic_{i+h} - c_i^2\rho(h)\big)U_{n,i}\right|^{\delta/2}\\
&\quad\le \sum_i\big(|c_ic_{i+h}|^{\delta/2} + |c_i|^{\delta}|\rho(h)|^{\delta/2}\big)E|U_{n,i}|^{\delta/2}\\
&\quad\le \sum_i\big(|c_ic_{i+h}|^{\delta/2} + |c_i|^{\delta}|\rho(h)|^{\delta/2}\big)\big(2|i|E|Z_1|^{\delta}\big)
\end{aligned}$$

and by the Schwarz inequality this is bounded by

$$\leq 2E|Z_1|^{\delta}\left[\left(\sum_i |c_i|^{\delta}|i|\right)^{1/2}\left(\sum_i |c_{i+h}|^{\delta}|i|\right)^{1/2} + |\rho(h)|^{\delta/2}\left(\sum_i |c_i||i|\right)\right]$$
$$< \infty$$

by assumption (4.2). Thus (4.6) follows since the bound does not depend on n. □

PROPOSITION 4.2. *Assume* (4.1), (4.2), *and* $E|Z_1|^{\alpha} = \infty$. *Then*

$$a_n^{-2}\left(C(0) - \sum_{t=1}^{n}\sum_{i=-\infty}^{\infty} c_i Z_{t-i}^2\right) = a_n^{-2}\sum_{t=1}^{n}\sum_{\substack{i,j\\ i\neq j}} c_i c_j Z_{t-i}Z_{t-j} \to_P 0.$$

PROOF. The proof of Proposition 2.1 can be adapted to this case but a simpler argument is given here instead. Choose $0 < \delta < \alpha$ satisfying (4.2) with $\alpha < 2\delta$. The triangle inequality gives

$$E\left|a_n^{-2}\sum_{t=1}^{n}\sum_{\substack{i,j\\ i\neq j}} c_i c_j Z_{t-i}Z_{t-j}\right|^{\delta} \leq a_n^{-2\delta} n \sum_{\substack{i,j\\ i\neq j}} |c_i c_j|^{\delta} E|Z_1 Z_2|^{\delta}$$

$$\leq n a_n^{-2\delta}\left(\sum_i |c_i|^{\delta}\right)^2 \left(E|Z_1|^{\delta}\right)^2.$$

Now since a_n is regularly varying with index $1/\alpha$, $a_n^{2\delta}$ is regularly varying with index $2\delta/\alpha > 1$, and hence $n a_n^{-2\delta} \to 0$. □

Rearranging the terms in the sum (4.5), we have

$$\sum_{t=1}^{n}\sum_{\substack{i,j\\ i\neq j}} c_i\left(c_{j+h} - c_j\rho(h)\right)Z_{t-i}Z_{t-j}$$

$$= \sum_{t=1}^{n}\sum_i\sum_{j\neq 0} c_i\left(c_{i-j+h} - c_{i-j}\rho(h)\right)Z_{t-i}Z_{t-i+j} \tag{4.7}$$

$$= \sum_{j\neq 0}\sum_{t=1}^{n}\sum_i \psi_{i,j}Z_{t-i}Z_{t-i+j},$$

where $\psi_{i,j} = c_i(c_{i-j+h} - c_{i-j}\rho(h))$, $i = 0, \pm 1, \pm 2, \dots$, $j = \pm 1, \pm 2, \dots$.

PROPOSITION 4.3. *Assume* (4.1), (4.2) *and* $E|Z_1|^{\alpha} = \infty$. *As* $n \to \infty$ *we have*

(i)
$$\tilde{a}_n^{-1}\left(\sum_{t=1}^{n}\left(\sum_i \psi_{i,j}Z_{t-i}Z_{t-i+j} + \sum_i \psi_{i,-j}Z_{t-i}Z_{t-i-j}\right) - \sum_i(\psi_{i,j} + \psi_{i,-j})\sum_{t=1}^{n} Z_t Z_{t+j}\right) \to_P 0$$

for each $j > 0$ *and*

(ii)
$$a_n^{-2}\left(\sum_{t=1}^{n}\sum_{i} c_i^2 Z_{t-i}^2 - \sum_i c_i^2 \sum_{t=1}^{n} Z_t^2\right) \to_P 0,$$

and therefore $a_n^{-2}\,(C(0) - \sum_i c_i^2 \sum_{t=1}^n Z_t^2) \to_P 0$.

PROOF. (i) Interchanging the order of summation and regrouping terms, the difference in (i) becomes

$$\tilde{a}_n^{-1}\sum_i \psi_{i,j}\left(\sum_{t=1-i}^{n-i} Z_t Z_{t+j} - \sum_{t=1}^{n} Z_t Z_{t+j}\right)$$
$$+\tilde{a}_n^{-1}\sum_i \psi_{i,-j}\left(\sum_{t=1-i-j}^{n-i-j} Z_t Z_{t+j} - \sum_{t=1}^{n} Z_t Z_{t+j}\right)$$
$$= \tilde{a}_n^{-1}\sum_i \psi_{i,j} V_{n,i} + \tilde{a}_n^{-1}\sum_i \psi_{i,-j} W_{n,i},$$

where

$$V_{n,i} = \sum_{t=1-i}^{n-i} Z_t Z_{t+j} - \sum_{t=1}^{n} Z_t Z_{t+j}$$

and

$$W_{n,i} = \sum_{t=1-i-j}^{n-i-j} Z_t Z_{t+j} - \sum_{t=1}^{n} Z_t Z_{t+j}.$$

However with δ as chosen in (4.2)

$$\limsup_n E\left|\sum_i \psi_{i,j} V_{n,i}\right|^{\delta} \le \limsup_n \sum_i |\psi_{i,j}|^{\delta} E|V_{n,i}|^{\delta}$$
$$\le 2\sum_i |\psi_{i,j}|^{\delta} |i| E|Z_1|^{\delta}|Z_2|^{\delta} < \infty,$$

whence $\tilde{a}_n^{-1}\sum_i \psi_{i,j} V_{n,i} \to_P 0$. The same argument also gives $\tilde{a}_n^{-1}\sum_i \psi_{i,-j} W_{n,i} \to_P 0$, which proves (i).

(ii) The above argument also works in this case but with δ replaced by $\delta/2$. The last statement follows from Proposition 4.2. □

THEOREM 4.4. *Suppose* $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ *where* $\{c_j\}$ *satisfies* (4.2) *and* $\{Z_t\}$ *satisfies* (1.2) *and* (1.3), *and* $E|Z_1|^{\alpha} = \infty$, $0 < \alpha < 2$. *If* a_n *and* $\tilde{a}_n$ *are given by* (3.1) *and* (3.4), *then for any positive integer* l,

(4.8) $\quad (\tilde{a}_n^{-1} a_n^2(\hat{\rho}(h) - \rho(h) - d_{h,n}/C(0)), 1 \le h \le l) \Rightarrow (Y_1, Y_2, \dots, Y_l)$

in $\mathbb{R}^l$, *where*

$$d_{h,n} = \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h)) \sum_i c_i^2 EZ_1Z_2 1_{[|Z_1Z_2| \le \tilde{a}_n]},$$

$$Y_h = \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h)) S_j/S_0,$$

and $S_0, S_1, S_2, \ldots$ *are independent stable random variables as described in* Theorem 3.3 (*i.e.*, S_0 *is positive with index* $\alpha/2$ *and* $S_1, S_2, \ldots$ *are identically distributed with index* α). *In addition, if either*

(i) $0 < \alpha < 1$, *or*
(ii) $\alpha = 1$ *and the distribution of* Z_t *is symmetric, or*
(iii) $1 < \alpha < 2$ *and* $EZ_1 = 0$,

then (4.8) *holds with* $d_{h,n} = 0$, $h = 1, \ldots, l$, *and a location change in the* S_j*'s*, $j \ge 1$.

Observe that since both a_n and $\tilde{a}_n$ are regularly varying with index $1/\alpha$, the normalization $a_n^2/\tilde{a}_n$ is also regularly varying with index $1/\alpha$. That is, $a_n^2/\tilde{a}_n = n^{1/\alpha}\tilde{L}(n)$ for some slowly varying function $\tilde{L}$.

PROOF. From Proposition 4.3, Theorem 3.3, and the continuous mapping theorem, we have for any fixed positive integer m,

$$(4.9)\qquad \left(a_n^{-2}C(0), \tilde{a}_n^{-1} \sum_{0<|j|\le m} \sum_{t=1}^{n} \left(\sum_i \psi_{i,j}(Z_{t-i}Z_{t-i+j} - \mu_n) \right) \right) \Rightarrow \left(\sum_i c_i^2 S_0, \sum_{j=1}^{m} \sum_i (\psi_{i,j} + \psi_{i,-j}) S_j \right),$$

where $\mu_n = EZ_1Z_2 1_{[|Z_1Z_2| \le \tilde{a}_n]}$. The dependence of $\psi_{i,j}$ on h is temporarily suppressed. The plan of the proof is to first show that (4.9) remains valid with m replaced by ∞ and then make use of Propositions 4.1–4.3 to derive the weak limit of $\tilde{a}_n^{-1}a_n^2(\hat{\rho}(h) - \rho(h) - d_{h,n}/C(0))$.

To establish the limit in (4.9) with m replaced by ∞ it suffices to show (cf. Billingsley, 1968, Theorem 4.2) that

$$(4.10)\qquad \lim_{m\to\infty} \limsup_{n\to\infty} P\left(\tilde{a}_n^{-1} \left| \sum_{|j|>m} \sum_{t=1}^{n} \sum_i \psi_{i,j}(Z_{t-i}Z_{t-i+j} - \mu_n) \right| > \gamma \right) = 0$$

for every $\gamma > 0$ and

$$(4.11)\qquad \sum_{j=1}^{m} \sum_i (\psi_{i,j} + \psi_{i,-j}) S_j \Rightarrow \sum_{j=1}^{\infty} \sum_i (\psi_{i,j} + \psi_{i,-j}) S_j.$$

The limit in (4.11) can be checked using characteristic functions since

$\sum_{j=1}^{\infty}|\sum_{i}(\psi_{i,j}+\psi_{i,-j})|^{\alpha}<\infty$. As for (4.10), we have the bound

$$P\left(\tilde{a}_n^{-1}\sum_{|j|>m}\sum_{t=1}^{n}\sum_{i}\psi_{i,j}(Z_{t-i}Z_{t-i+j}-\mu_n)| > \gamma\right)$$

$$\le P\left(\tilde{a}_n^{-1}\left|\sum_{|j|>m}\sum_{t=1}^{n}\sum_{i}\psi_{i,j}\left(Z_{t-i}Z_{t-i+j}1_{[|Z_{t-i}Z_{t-i+j}|\le \tilde{a}_n]}-\mu_n\right)\right| > \gamma/2\right)$$

$$+P\left(\tilde{a}_n^{-1}\left|\sum_{|j|>m}\sum_{t=1}^{n}\sum_{i}\psi_{i,j}Z_{t-i}Z_{t-i+j}1_{[|Z_{t-i}Z_{t-i+j}|>\tilde{a}_n]}\right| > \gamma/2\right)$$

$$= A + B.$$

Applying Chebyshev's inequality to A gives, after some simplification (see the proof of Proposition 2.1),

$$A \le 4\gamma^{-2}\tilde{a}_n^{-2}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{i}\sum_{|j|>m}\sum_{|j'|>m}|\psi_{i,j}|\big(|\psi_{s-t+i,j'}|+|\psi_{s-t+i-j,j'}|$$

$$+|\psi_{s-t+i+j',j'}|+|\psi_{s-t+i-j+j',j'}|\big)\sigma_n^2,$$

where $\sigma_n^2 = E|Z_1Z_2|^2 1_{[|Z_1Z_2|\le \tilde{a}_n]}$. A change of variables in the summation gives the bound

$$A \le 4\gamma^{-2}\tilde{a}_n^{-2}n\sum_{t=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\sum_{|j|>m}\sum_{|j'|>m}|\psi_{i,j}|\big(|\psi_{t+i,j'}|$$

$$+|\psi_{t+i-j,j'}|+|\psi_{t+i+j',j'}|+|\psi_{t+i-j+j',j'}|\big)\sigma_n^2,$$

and since $\sum_{t=-\infty}^{\infty}|\psi_{t+k,j'}| = \sum_{t=-\infty}^{\infty}|\psi_{t,j'}|$ for all integers k,

$$A \le 4\gamma^{-2}\tilde{a}_n^{-2}n4\left(\sum_{i=-\infty}^{\infty}\sum_{|j|>m}|\psi_{i,j}|\right)^2\sigma_n^2.$$

The absolute summability of the c_j's ensures that all of the above sums involving $|\psi_{i,j}|$ are finite and in particular $\lim_{m\to\infty}\sum_{|j|>m}\sum_{i=-\infty}^{\infty}|\psi_{i,j}| = 0$. Thus by Karamata's theorem ($\tilde{a}_n^{-2}n\sigma_n^2 \to \alpha/(2-\alpha)$), we have $\lim_{m\to\infty}\limsup_{n\to\infty}A = 0$.

With δ as given in (4.2)

$$B \le 2^{\delta}\gamma^{-\delta}\tilde{a}_n^{-\delta}n\sum_{|j|>m}\sum_{i}|\psi_{i,j}|^{\delta}E|Z_1Z_2|^{\delta}1_{[|Z_1Z_2|>\tilde{a}_n]}$$

and again by Karamata's theorem, $n\tilde{a}_n^{-\delta}E|Z_1Z_2|^{\delta}1_{[|Z_1Z_2|>\tilde{a}_n]} \to \alpha/(\alpha-\delta)$ so that $\lim_{m\to\infty}\limsup_{n\to\infty}B = 0$, which establishes (4.9) with m replaced by ∞.

Now from Proposition 4.1 and (4.7), we have

$$\tilde{a}_n^{-1}a_n^2(\hat{\rho}(h)-\rho(h)) = \tilde{a}_n^{-1}a_n^2(C(0))^{-1}\sum_{j\ne 0}\sum_{t=1}^{n}\sum_{i}\psi_{i,j}Z_{t-i}Z_{t-i+j}+o_p(1).$$

Since

$$\sum_{i}(\psi_{i,j}+\psi_{i,-j})/\sum_{i}c_i^2 = \rho(h+j)+\rho(h-j)-2\rho(j)\rho(h),$$

we then have

$$\tilde{a}_n^{-1} a_n^2(\hat{\rho}(h) - \rho(h) - d_{h,n}/C(0)) = \tilde{a}_n^{-1} a_n^2 (C(0))^{-1} \sum_{j \neq 0} \sum_{t=1}^{n} \sum_i \psi_{i,j}(Z_{t-i}Z_{t-i+j} - \mu_n) + o_p(1).$$

It follows by applying the continuous mapping theorem to (4.9) that

$$\tilde{a}_n^{-1} a_n^2(\hat{\rho}(h) - \rho(h) - d_{h,n}/C(0)) \Rightarrow \left(\sum_{j=1}^{\infty} \sum_i (\psi_{i,j} + \psi_{i,-j}) S_j / \sum_i c_i^2 S_0 \right) = Y_h.$$

The proof of the joint convergence in (4.8) is essentially the same as the above argument. The only difference is that the vector in (4.9) is extended to an $l + 1$ – dimensional vector where the $(h + 1)$th component is given by

$$\tilde{a}_n^{-1} \sum_{0<|j|<m} \sum_{t=1}^{n} \sum_i \psi_{i,j}^{(h)}(Z_{t-i}Z_{t-i+j} - \mu_n), \qquad h = 1, 2, \dots l.$$

Finally, the last statement of the theorem is an immediate consequence of Remarks 1–3 in Section 3. □

In the following two results, we consider the limit laws of the mean corrected version of the sample correlation function defined by

$$\tilde{\rho}(h) = \sum_{t=1}^{n} (X_t - \overline{X})(X_{t+h} - \overline{X}) \Big/ \sum_{t=1}^{n} (X_t - \overline{X})^2,$$

where $\overline{X} = \sum_{t=1}^{n} X_t/n$.

COROLLARY 1. *Suppose* $1 < \alpha < 2$. *Then for any positive integer* l,

$$(\tilde{a}_n^{-1} a_n^2(\tilde{\rho}(h) - \rho(h)), 1 \le h \le l) \Rightarrow (Y_1, Y_2, \dots, Y_l).$$

PROOF. Since the function $\tilde{\rho}(h)$ is location invariant, we may assume without loss of generality that $EZ_t = 0$ (otherwise consider the process $X_t - EX_t = \sum_{j=-\infty}^{\infty} c_j(Z_{t-j} - EZ_{t-j})$). In view of Theorem 4.4, it suffices to show $\tilde{\rho}(h) - \hat{\rho}(h) = o_p(\tilde{a}_n a_n^{-2})$. Using the identity $\sum_{t=1}^{n} X_t^2 - \sum_{t=1}^{n}(X_t - \overline{X})^2 = n\overline{X}^2$, we have

$$\tilde{\rho}(h) - \hat{\rho}(h) = \left(\hat{\rho}(h) n\overline{X}^2 - \overline{X} \sum_{t=1}^{n} X_{t+h} \right) \Big/ \sum_{t=1}^{n} (X_t - \overline{X})^2. \tag{4.12}$$

In Section 4 of Davis and Resnick (1985), it was shown that $\sum_{t=1}^{n}(X_t - \overline{X})^2 = O_p(a_n^2)$, $\sum_{t=1}^{n} X_t = O_p(a_n) = o_p(\tilde{a}_n)$, and $\hat{\rho}(h) \to_P \rho(h)$. Since $\overline{X} \to EX_1 = 0$ and $\sum_{t=1}^{n} X_{t+h}/n \to EX_1 = 0$ a.s. by the ergodic theorem, this implies $\tilde{\rho}(h) - \hat{\rho}(h) = o_p(\tilde{a}_n a_n^{-2})$ as desired. □

In the $0 < \alpha < 1$ case, the sample mean plays a dominant role in determining the limit distribution of $\tilde{\rho}(h)$. In order to describe this result, it is necessary to first define two random variables. Let $\{j_k: k = 1, 2, \dots\}$ be the points of a Poisson process on $\mathbb{R} \setminus \{0\}$ with intensity $\lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)\,dx + \alpha q(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)\,dx$, where p and q are given in (1.2). Now if $0 < \alpha < 1$, then $\sum_{k=1}^{\infty} |j_k| < \infty$ a.s. so that the random variables $S = \sum_{k=1}^{\infty} j_k$ and $S_0 = \sum_{k=1}^{\infty} j_k^2$ are well-defined. In particular, S and S_0 each have a stable distribution with index α and $\alpha/2$, respectively.

COROLLARY 2. *Suppose* $0 < \alpha < 1$. *Then for any positive integer* l

$$(n(\tilde{\rho}(h) - \rho(h)), 1 \le h \le l) \Rightarrow ((\rho(h) - 1), 1 \le h \le l)\Big(\sum_i c_i\Big)^2 S^2 \Big/ \Big(\sum_i c_i^2 S_0\Big).$$

REMARK. Some properties of the distribution function of S^2/S_0 are studied in Logan et al. (1973). See also Cline (1983).

PROOF. Let $\{j_k\}$ be the points of a Poisson process as described above. Using an argument similar to that given in Section 4 of Davis and Resnick (1985) (see also Resnick, 1986, Section 4) it is easy to show

$$\begin{aligned}(4.13)\quad \left(a_n^{-1} \sum_{t=1}^{n} X_t, a_n^{-2} \sum_{t=1}^{n} (X_t - \bar{X})^2\right) &\Rightarrow \left(\Big(\sum_i c_i\Big)\Big(\sum_{k=1}^{\infty} j_k\Big), \Big(\sum_i c_i^2\Big)\Big(\sum_{k=1}^{\infty} j_k^2\Big)\right) \\ &= \left(\Big(\sum_i c_i\Big) S, \Big(\sum_i c_i^2\Big) S_0\right).\end{aligned}$$

Now rearranging the identity in (4.12), we have

$$\begin{aligned}(4.14)\quad n(\tilde{\rho}(h) - \rho(h)) &= n(\hat{\rho}(h) - \rho(h)) \\ &\quad + \frac{n(\hat{\rho}(h) - 1) n\bar{X}^2}{\sum_{t=1}^{n} (X_t - \bar{X})^2} + \frac{n\bar{X}\big(\sum_{j=1}^{h} (X_j - X_{n+j})\big)}{\sum_{t=1}^{n} (X_t - \bar{X})^2}.\end{aligned}$$

By Theorem 4.4 the first term is $O_p(\tilde{a}_n a_n^{-2} n) = o_p(1)$ since $\alpha < 1$. The third term in (4.14) is also negligible because $n\bar{X} = O_p(a_n)$, $(\sum_{t=1}^{n} (X_t - \bar{X})^2)^{-1} = O_p(a_n^{-2})$, and $\sum_{j=1}^{h} (X_j - X_{n+j}) = O_p(1)$ so that the product of the three terms is $O_p(a_n^{-1}) = o_p(1)$. As for the middle term,

$$\frac{n(\hat{\rho}(h) - 1) n\bar{X}^2}{\sum_{t=1}^{n} (X_t - \bar{X})^2} \Rightarrow \frac{(\rho(h) - 1)(\sum_i c_i S)^2}{(\sum_i c_i^2) S_0}$$

follows from (4.13) and the weak consistency of $\hat{\rho}(h)$. Finally the joint convergence in the statement of the corollary is clear. □

We close this section with a comparison of the standard result for the correlation function in the finite variance case and Theorem 4.4. Assuming that Z_t has a finite variance and a zero mean, Theorem 8.4.6 of Anderson (1971) gives

$$n^{1/2}(\hat{\rho}(1) - \rho(1), \hat{\rho}(2) - \rho(2), \dots, \hat{\rho}(l) - \rho(l)) \Rightarrow (V_1, V_2, \dots, V_l),$$

where the limit vector has a multivariate normal distribution with mean zero and covariance matrix given by Bartlett's formula

$$r_{gh} = \sum_{j=-\infty}^{\infty} \big(\rho(g+j)\rho(h+j) + \rho(g-j)\rho(h+j) - 2\rho(j)\rho(g)\rho(h+j) - 2\rho(j)\rho(h)\rho(g+j) + 2\rho^2(j)\rho(g)\rho(h)\big).$$

However, by checking covariances the components in the limit vector may be written as

$$V_h = \sum_{j=1}^{\infty} \big(\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h)\big)S_j, \qquad h = 1, 2, \dots, l, \tag{4.15}$$

where $\{S_j\}$ is a sequence of iid $N(0,1)$ random variables. This corresponds to the numerator portion of the limit in Theorem 4.4 with $\alpha = 2$. In fact, S_j may be identified as the weak limit of $\sigma^{-2}n^{-1/2}\sum_{t=1}^{n} Z_t Z_{t+j}$, $j = 1, 2, \dots$. Moreover, in the finite variance case, the sample variance

$$n^{-1}\sum_{t=1}^{n} X_t^2 \to_P \sum_{j=-\infty}^{\infty} c_j^2 \operatorname{var}(Z_1) > 0,$$

whereas $a_n^{-2}\sum_{t=1}^{n} X_t^2 \Rightarrow \sum_{j=-\infty}^{\infty} c_j^2 S_0$ in the $0 < \alpha < 2$ case. This phenomenon accounts for the division by S_0 in Theorem 4.4 and not in (4.15).

5. Examples. In this section, we consider applications of Theorem 4.4 to some time series models. Throughout this section, assume the hypotheses of Theorem 4.4 are met and, for simplicity, suppose the distribution of Z_t is symmetric and that the distribution of $|Z_t|$ is asymptotically equivalent to a Pareto. It then follows that

$$(n/\log n)^{1/\alpha}(\hat{\rho}(h) - \rho(h)) \Rightarrow \sum_{j=1}^{\infty} \big(\rho(h+j) + \rho(j-h) - 2\rho(j)\rho(h)\big)S_j/S_0 \tag{5.1}$$

and $S_1, S_2, \dots$ is now an iid sequence of symmetric α-stable random variables, independent of the positive $\alpha/2$-stable random variable S_0.

The numerator of the limit in (5.1) is also a symmetric α-stable random variable with characteristic function given by

$$\exp\left\{-\sum_{j=1}^{\infty} |\rho(h+j) + \rho(j-h) - 2\rho(j)\rho(h)|^{\alpha}|t|^{\alpha}\right\}. \tag{5.2}$$

Extending the notion of variance for a Gaussian random variable, Stuck (1978) defined the dispersion of a random variable with characteristic function (5.2) by

$$\text{disp} = \sum_{j=1}^{\infty} |\rho(h+j) + \rho(j-h) - 2\rho(j)\rho(h)|^{\alpha}. \tag{5.3}$$

(See also Cline, 1983.) The limit in (5.1) is then equal in distribution to

$(\text{disp})^{1/\alpha}S_1/S_0$. Notice that upon setting $\alpha = 2$ in (5.3), we get this asymptotic variance of $\hat{\rho}(h)$ in the traditional finite second moment setting.

5.1. MA(q). Suppose $\{X_t\}$ is the finite moving average

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

Then, since $\rho(h) = 0$ for $|h| > q$, we have for $h > q$

$$(n/\log n)^{1/\alpha}(\hat{\rho}(h) - \rho(h)) \Rightarrow \left(1 + 2\sum_{j=1}^{q} |\rho(j)|^\alpha\right)^{1/\alpha} S_1/S_0.$$

5.2. Estimation of θ in a MA(1). For the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, $\rho(1) = \theta/(1+\theta^2)$. A method-of-moments-type estimator for θ is found by solving the latter equation for θ. Choosing the solution with the constraint $|\theta| \le 1$ (cf. Fuller, 1976) gives

$$\hat{\theta} = \begin{cases} \left(1 - (1 - 4\hat{\rho}^2)^{1/2}\right)/(2\hat{\rho}), & \text{if } 0 \le |\hat{\rho}| \le 0.5, \\ -1, & \hat{\rho} \le -0.5, \\ 1, & \hat{\rho} > 0.5, \end{cases}$$

where $\hat{\rho} = \hat{\rho}(1)$. Letting $g(\rho)$ denote the inverse of the function $\theta/(1+\theta^2)$ with $|\theta| \le 1$, we have by the mean value theorem

$$\hat{\theta} - \theta = g(\hat{\rho}) - g(\rho) = g'(\rho)(\hat{\rho} - \rho) + o_p(\hat{\rho} - \rho).$$

Hence

$$(n/\log n)^{1/\alpha}(\hat{\theta} - \theta) \Rightarrow (1 - \theta^2)^{-1}(1 + \theta^2)^2\left((1 - 2\rho^2(1))^\alpha + |\rho(1)|^\alpha\right)^{1/\alpha} S_1/S_0.$$

The dispersion of the numerator of the limit simplifies to

$$\frac{(1 + \theta^4)^\alpha + |\theta|^\alpha(1 + \theta^2)^\alpha}{(1 - \theta^2)^\alpha}.$$

By setting $\alpha = 2$, we obtain the asymptotic variance of $\hat{\theta}$ (cf. Fuller, 1976, page 343). Note that while this estimate of θ is inefficient in the finite variance case, its performance in the $0 < \alpha < 2$ case seems to be good. For example, in a simulation experiment, 100 replications of the process $Z_t - 0.4Z_{t-1}$, $t = 1, 2, \ldots, 100$, were generated where Z_t is Cauchy distributed. The mean of the $\hat{\theta}$'s was -0.40074 with a standard deviation of 0.0790. This compares favorably with the asymptotic standard deviation of $((1 - \theta^2)/n)^{1/2} = 0.0917$ for the maximum likelihood estimator of θ in a MA(1) model assuming the noise sequence is normally distributed. While comparing variances in this situation may be a bit misleading, it nevertheless gives an indication of the reasonably good performance of $\hat{\theta}$ in the $0 < \alpha < 2$ case. For some comparisons in AR(p) models see Bloomfield and Steiger (1983).

5.3. AR(1). Let $\{X_t\}$ be the AR(1) process $X_t = \phi X_{t-1} + Z_t$ where $|\phi| < 1$. In this case, $\rho(h) = \phi^{|h|}$ and estimating ϕ by $\hat{\phi} = \hat{\rho}(1)$, we have

$$(n/\log n)^{1/\alpha}(\hat{\phi} - \phi) \Rightarrow \left(\sum_{j=1}^{\infty} \left(\phi^{1+j} + \phi^{j-1} - 2\phi^j\phi\right)^\alpha \right)^{1/\alpha} S_1/S_0$$

$$= \frac{1 - \phi^2}{(1 - \phi^\alpha)^{1/\alpha}} S_1/S_0.$$

5.4. Yule–Walker estimates. The Yule–Walker matrix equation for the AR(p) model $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$, assuming $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p \neq 0$, $|z| \leq 1$, is

$$R\boldsymbol{\phi} = \boldsymbol{\rho}, \tag{5.4}$$

where R is the $p \times p$ matrix $[\rho(i-j)]_{i,j=1}^p$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_p)'$, and $\boldsymbol{\rho} = (\rho(1), \dots, \rho(p))'$. The Yule–Walker estimate of $\boldsymbol{\phi}$ is then defined as the solution of (5.4) with R and $\boldsymbol{\rho}$ replaced by $\hat{R} = [\hat{\rho}(i-j)]_{i,j=1}^p$ and $\hat{\boldsymbol{\rho}} = (\hat{\rho}(1), \dots, \hat{\rho}(p))'$, respectively. As in Yohai and Maronna (1977), for $\mathbf{z} \in \mathbb{R}^p$ define $\psi(\mathbf{z}) = R(\mathbf{z})^{-1}\mathbf{z}$ where $R(\mathbf{z}) = [z_{|i-j|}]_{i,j=1}^p$ and $z_0 \equiv 1$. Since $\hat{R} \to_p R$ and R is nonsingular, this implies $\psi(\hat{\boldsymbol{\rho}})$ is well defined for large n. The mean value theorem then gives

$$\hat{\boldsymbol{\phi}} - \boldsymbol{\phi} = D(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}) + o_p(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}),$$

where D is the $p \times p$ matrix of partial derivatives of ψ evaluated at $\boldsymbol{\rho}$. Consequently,

$$(n/\log n)^{1/\alpha}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \Rightarrow D\mathbf{Y},$$

where $\mathbf{Y} = (Y_1, Y_2, \dots, Y_p)'$ with $Y_h = \sum_{j=1}^{\infty}(\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h))S_j/S_0$, $h = 1, \dots, p$.

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