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LARGE-SAMPLE PROPERTIES OF PARAMETER ESTIMATES FOR STRONGLY DEPENDENT STATIONARY GAUSSIAN TIME SERIES

BY ROBERT FOX[1] AND MURAD S. TAQQU[2,3]

Cornell University

A strongly dependent Gaussian sequence has a spectral density $f(x, \theta)$ satisfying $f(x, \theta) \sim |x|^{-\alpha(\theta)} L_\theta(x)$ as $x \to 0$, where $0 < \alpha(\theta) < 1$ and $L_\theta(x)$ varies slowly at 0. Here θ is a vector of unknown parameters. An estimator for θ is proposed and shown to be consistent and asymptotically normal under appropriate conditions. These conditions are satisfied by fractional Gaussian noise and fractional ARMA, two examples of strongly dependent sequences.

1. Introduction. Let X_j, $j \geq 1$, be a stationary Gaussian sequence with mean μ and spectral density $\sigma^2 f(x, \theta)$, $-\pi \leq x \leq \pi$, where $\mu, \sigma^2 > 0$ and the vector $\theta \in E \subset R^p$ are unknown parameters. Denote the covariance by $\sigma^2 r_k(\theta)$, so that

$$E(X_j - \mu)(X_{j+k} - \mu) = \sigma^2 r_k(\theta) = \sigma^2 \int_{-\pi}^{\pi} e^{ikx} f(x, \theta)\, dx.$$

(We are not assuming that σ^2 is the variance of X_j.) Let $R_N(\theta)$ be the $N \times N$ matrix with j, kth entry $r_{j-k}(\theta)$. Thus $\sigma^2 R_N(\theta)$ is the covariance matrix of $X_1, \ldots, X_N$. Our object is to estimate θ and σ^2 based on the observations $X_1, \ldots, X_N$.

We are interested in *strongly dependent* sequences X_j, that is, in sequences $f(x, \theta) \sim |x|^{-\alpha(\theta)} L_\theta(x)$ as $x \to 0$, where $0 < \alpha(\theta) < 1$ and $L_\theta(x)$ varies slowly at 0. These sequences have covariances that decrease too slowly to permit the normalized partial sums

$$S_{[Nt]} = \frac{\sum_{j=1}^{[NT]} X_j}{\sqrt{N}}, \qquad t \geq 0$$

to converge weakly to Brownian motion. Because of this fact, strongly dependent sequences play an important role in the theory of self-similar stochastic processes. Two examples, fractional Gaussian noise and fractional ARMA's, are described at the end of this section. We will estimate simultaneously all the unknown

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[1] Presently at Rochester Institute of Technology, Department of Mathematics, Rochester, NY 14623.

[2] Presently at Boston University, Department of Mathematics, Boston, MA 02215.

[3] Research supported by the National Science Foundation grant ECS-80-15585 at Cornell University.

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parameters symbolized by the vector θ, and not just the exponent $\alpha(\theta)$ in isolation.

A number of approaches to parameter estimation for strongly dependent sequences have been considered in the literature. These include the R/S technique, periodogram estimation, and maximum likelihood estimation. Theoretical properties of the R/S estimates have been investigated by Mandelbrot (1975) and Mandelbrot and Taqqu (1979). Periodogram estimation has been considered by Mohr (1981), Graf (1983), and Geweke and Porter-Hudak (1983). Hipel and McLeod (1978) have discussed computational considerations involved in the application of maximum likelihood estimation. See also Todini and O'Connell (1979).

The study of maximum likelihood estimation for strongly dependent sequences is a special case of the problem of maximum likelihood estimation for dependent observations. Sweeting (1980) has given conditions under which the maximum likelihood estimator is consistent and asymptotically normally distributed. Basawa and Prakasa Rao (1980) and Basawa and Scott (1983) survey theorems and examples in this area. In order to apply these results it would be necessary to study the second derivatives of $R_N^{-1}(\theta)$. To avoid this difficulty we will follow the approach suggested by Whittle (1951). This involves maximizing

$$\frac{1}{\sigma}\exp\left\{-\frac{\mathbf{Z}'A_N(\theta)\mathbf{Z}}{2N\sigma^2}\right\}. \tag{1.1}$$

Here $\mathbf{Z} = (X_1 - \bar{X}_N, \dots, X_N - \bar{X}_N)'$, $\bar{X}_N = (1/N)\sum_{j=1}^N X_j$, and $A_N(\theta)$ is the $N \times N$ matrix with entries $[A_N(\theta)]_{jk} = a_{j-k}(\theta)$, where

$$a_j(\theta) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} e^{ijx}[f(x,\theta)]^{-1}\,dx. \tag{1.2}$$

For the approximation of the inverse of $(R(\theta))_{i\ge 0, j\ge 0}$ by $(A_N(\theta))_{i\ge 0, j\ge 0}$, see Bleher (1981) and also Beran and Kuensch (1985). Notice, that, by Parseval's relation, the doubly infinite matrix $A(\theta)$ with entries $a_{j-k}(\theta)$, $-\infty < j, k < \infty$, is the inverse of the doubly infinite matrix $R(\theta)$ with entries $r_{j-k}(\theta)$, $-\infty < j, k < \infty$.

Thus we define estimators $\bar{\theta}_N$ and $\bar{\sigma}_N^2$ to be those values of θ and σ^2 which maximize $(1/\sigma)\exp\{-(1/2N\sigma^2)\mathbf{Z}'A_N(\theta)\mathbf{Z}\}$. This is equivalent to choosing $\bar{\theta}_N$ to minimize

$$\sigma_N^2(\theta) = \frac{\mathbf{Z}'A_N(\theta)\mathbf{Z}}{N} \tag{1.3}$$

and then setting $\bar{\sigma}_N^2 = \sigma_N^2(\bar{\theta}_N)$. It will be convenient to use the fact that

$$\sigma_N^2(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x,\theta)]^{-1} I_N(x)\,dx, \tag{1.4}$$

where

$$I_N(x) = \frac{\left|\sum_{j=1}^N e^{ijx}(X_j - \bar{X}_N)\right|^2}{2\pi N}. \tag{1.5}$$

Walker (1964) showed that $\bar{\theta}_N$ is consistent and asymptotically normal in many cases in which the sequence $\{X_j\}$ is weakly dependent (and not necessarily Gaussian). Hannan (1973) improved these results and was able to use them to prove consistency and asymptotic normality of the maximum likelihood estimator. Dunsmuir and Hannan (1976) gave extensions to the case of vector-valued observations.

We show that for strongly dependent Gaussian sequences $\{X_j\}$ the estimator $\bar{\theta}_N$ is consistent and asymptotically normal. The conditions under which this result holds are given in Section 2. Our results apply to fractional Gaussian noise and fractional ARMA's.

Fractional Gaussian noise was introduced by Mandelbrot and Van Ness (1968) and has been widely used to model strongly dependent geophysical phenomena. It is a stationary Gaussian sequence with mean 0 and covariance

$$r_k = EX_jX_{j+k} = \frac{C}{2}\{|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\},$$

where H is a parameter satisfying $\frac{1}{2} < H < 1$ and $C > 0$. This covariance satisfies

$$r_k \sim CH(2H-1)k^{2H-2} \quad \text{as } k \to \infty.$$

The spectral density $f(x, H)$ of fractional Gaussian noise is given by

$$f(x,H) = CF(H)f_0(x,H), \tag{1.6}$$

where

$$f_0(x,H) = (1-\cos x)\sum_{k=-\infty}^{\infty}|x+2k\pi|^{-1-2H}, \qquad -\pi \le x \le \pi, \tag{1.7}$$

and

$$F(H) = \left\{\int_{-\infty}^{\infty}(1-\cos x)|x|^{-1-2H}\,dx\right\}^{-1} \tag{1.8}$$

[see Sinai (1976)]. As $x \to 0$ we have

$$f(x,H) \sim \frac{CF(H)}{2}|x|^{1-2H}.$$

Fractional Gaussian noise is the unique Gaussian sequence with the property that $S_{mN} = \sum_{j=1}^{mN} X_j$ has the same distribution as $m^H S_N$ for all $m, N \ge 1$. Further properties are discussed in Mandelbrot and Taqqu (1979).

Another example to which our results apply is fractional ARMA. To define it, let $g(x,\xi) = \sum_{j=0}^{p}\xi_j x^j$ and $h(x,\phi) = \sum_{j=0}^{q}\phi_j x^j$, where $\xi = (\xi_0, \dots, \xi_r)$ and $\phi = (\phi_0, \dots, \phi_q)$. Suppose that $g(x,\xi)$ and $h(x,\phi)$ have no zeros on the unit circle and no zeros in common. For $0 < d < \frac{1}{2}$, define the spectral density

$$f(x,d,\xi,\phi) = C|e^{ix}-1|^{-2d}\left|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\right|^2, \qquad -\pi \le x \le \pi. \tag{1.9}$$

A Gaussian sequence with mean 0 and spectral density $f(x, d, \xi, \phi)$ is called a fractional ARMA process. Heuristically, it is the sequence which, when differenced d times, yields an ARMA process with spectral density

$$C\left|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\right|^2.$$

Granger and Joyeux (1980) and Hosking (1981) have proposed the use of fractional ARMA's to model strongly dependent phenomena, since

$$f(x,d,\xi,\phi) \sim C\left|\frac{g(1,\xi)}{h(1,\phi)}\right|^2 |x|^{-2d} \quad \text{as } x \to 0.$$

2. Statements of the theorems. Let X_j, $j \geq 1$ be a stationary Gaussian sequence with mean μ and spectral density $\sigma^2 f(x,\theta)$, where $\mu, \sigma^2 > 0$ and $\theta \in E$ are unknown parameters. The set $E \subset R^p$ is assumed to be compact. Let σ_0^2 and θ_0 be the true values of the parameters. We assume that θ_0 is in the interior of E.

If θ and θ' are distinct elements of E, we suppose that the set $\{x\colon f(x,\theta) \neq f(x,\theta')\}$ has positive Lebesgue measure, so that different θ's correspond to different dependence structures. Assume the functions $f(x,\theta)$ are normalized so that

$$\int_{-\pi}^{\pi} \log f(x,\theta)\,dx = 0, \qquad \theta \in E.$$

Let $f^{-1}(x,\theta) = 1/f(x,\theta)$.

Remark. The condition $\int_{-\pi}^{\pi}\log f(x,\theta)\,dx > -\infty$ guarantees that the sequence $\{X_j\}$ admits a backward expansion

$$X_j = \sigma \sum_{k=0}^{\infty} b_k(\theta)\varepsilon_{j-k},$$

where ε_j, $j \geq 1$, are independent standard normal random variables. The first coefficient $b_0(\theta)$ is the one-step prediction standard deviation of the sequence $Y_j = X_j/\sigma$. It is given by

$$b_0(\theta) = 2\pi \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(x,\theta)\,dx\right\}.$$

[See Hannan (1970), page 137.] If $\int_{-\pi}^{\pi}\log f(x,\theta)\,dx = 0$ for all θ, it follows, that $b_0(\theta) = 2\pi$ and so $\{Y_j\}$ has one-step prediction standard deviation independent of θ.

We will refer to the following conditions.

Conditions A. We say that $f(x,\theta)$ satisfies conditions A.1–A.6 if there exists $0 < \alpha(\theta) < 1$ such that for each $\delta > 0$

(A.1) $g(\theta) = \int_{-\pi}^{\pi}\log f(x,\theta)\,dx$ can be differentiated twice under the integral sign.

(A.2) $f(x, \theta)$ is continuous at all (x, θ), $x \neq 0$, $f^{-1}(x, \theta)$ is continuous at all (x, θ), and

$$f(x, \theta) = O(|x|^{-\alpha(\theta)-\delta}) \quad \text{as } x \to 0.$$

(A.3) $\partial/\partial\theta_j f^{-1}(x, \theta)$ and $\partial^2/\partial\theta_j \partial\theta_k f^{-1}(x, \theta)$ are continuous at all (x, θ),

$$\frac{\partial}{\partial\theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-\delta}) \quad \text{as } x \to 0, \qquad 1 \le j \le p,$$

and

$$\frac{\partial^2}{\partial\theta_j \partial\theta_k} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-\delta}) \quad \text{as } x \to 0, \qquad 1 \le j, k \le p.$$

(A.4) $\partial/\partial x f(x, \theta)$ is continuous at all (x, θ), $x \neq 0$, and

$$\frac{\partial}{\partial x} f(x, \theta) = O(|x|^{-\alpha(\theta)-1-\delta}) \quad \text{as } x \to 0.$$

(A.5) $\partial^2/\partial x \, \partial\theta_j f^{-1}(x, \theta)$ is continuous at all (x, θ), $x \neq 0$, and

$$\frac{\partial^2}{\partial x \, \partial\theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-1-\delta}) \quad \text{as } x \to 0, \qquad 1 \le j \le p.$$

(A.6) $\partial^3/\partial^2 x \, \partial\theta_j f^{-1}(x, \theta)$ is continuous at all (x, θ), $x \neq 0$, and

$$\frac{\partial^3}{\partial^2 x \, \partial\theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-2-\delta}) \quad \text{as } x \to 0, \qquad 1 \le j \le p.$$

REMARK. The constants that appear in the $O(\cdot)$ conditions may depend on the parameters θ and δ. If $L(x)$ varies slowly as $x \to 0$, then $L(x) = O(|x|^{-\delta})$ as $x \to 0$ for every $\delta > 0$. [See Feller (1971), page 277]. Thus (A.1)–(A.6) will be satisfied if the indicated continuity holds, $f(x, \theta)$ varies regularly as $x \to 0$ with exponent $-\alpha(\theta)$, $\partial/\partial x/ f(x, \theta)$ with exponent $-\alpha(\theta) - 1$, $\partial/\partial\theta_j$, $f^{-1}(x, \theta)$, and $\partial^2/\partial\theta_j \partial\theta_k f^{-1}(x, \theta)$ with exponent $\alpha(\theta)$, $\partial^2/\partial x \, \partial\theta_j f^{-1}(x, \theta)$ with exponent $\alpha(\theta) - 1$, and $\partial^3/\partial^2 x \, \partial\theta_j f^{-1}(x, \theta)$ with exponent $\alpha(\theta) - 2$.

DEFINITION OF THE ESTIMATOR. Consider the quadratic form $\sigma_N^2(\theta)$ given in (1.3). Let $\bar{\theta}_N$ be a value of θ which minimizes $\sigma_N^2(\theta)$. Put $\bar{\sigma}_N^2 = \sigma_N^2(\bar{\theta}_N)$. The following theorem establishes the strong consistency of the estimators $\bar{\theta}_N$ and $\bar{\sigma}_N^2$.

THEOREM 1. *If $f(x, \theta)$ satisfies conditions* (A.2) *and* (A.4), *then with probability* 1

$$\lim_{N\to\infty} \bar{\theta}_N = \theta_0 \quad \textit{and} \quad \lim_{N\to\infty} \bar{\sigma}_N^2 = \sigma_0^2.$$

To state the next theorem, let $W(\theta)$ be the $p \times p$ matrix with j, kth entry

$$w_{jk}(\theta) = \int_{-\pi}^{\pi} f(x,\theta) \frac{\partial^2}{\partial\theta_j\,\partial\theta_k} f^{-1}(x,\theta)\,dx.$$

THEOREM 2. *If conditions* (A.1)–(A.6) *are satisfied then the random vector* $\sqrt{N}(\bar{\theta}_N - \theta_0)$ *tends in distribution to a normal random vector with mean* 0 *and covariance matrix* $4\pi W^{-1}(\theta_0)$.

Theorems 1 and 2 are proven in Section 3.

REMARK 1. As in Hannan (1973), it can be shown that the results hold if $\sigma_N^2(\theta)$ in (1.4) is replaced by

$$\sum_k f^{-1}\left(\frac{2\pi k}{N}, \theta\right) I_N\left(\frac{2\pi k}{N}\right),$$

where $-N/2 < k \leq [N/2]$. This last expression may be useful for computational purposes.

REMARK 2. $\bar{\theta}_N - \bar{\theta}_0$ is asymptotically of the order of $1/\sqrt{N}$. Geweke and Porter-Hudak (1983) have obtained asymptotic results for an estimator resulting from a regression based on the periodogram estimates. This estimator converges to the true value of the parameter at a slower speed than $1/\sqrt{N}$.

REMARK 3. The sample mean $\bar{X}_N$ converges to $\mu = EX_j$ at a slower speed than $\bar{\theta}_N$ converges to θ_0 because (up to a slowly varying function in the normalization) $N^{1/2-\alpha/2}(\bar{X}_N - \mu)$ converges to a normal distribution [see Taqqu (1975)].

APPLICATIONS. Theorems 1 and 2 can be applied to fractional Gaussian noise and to fractional ARMA. In order to apply them to fractional Gaussian noise, restrict the parameter H to a compact subset of $(\frac{1}{2}, 1)$ and choose the normalization constant $CF(H)$ in (1.6) as

$$\text{(2.1)}\quad CF(H) = \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left\{(1-\cos x)\sum_{k=-\infty}^{\infty} |x + 2k\pi|^{-1-2H}\right\} dx\right],$$

so that

$$\int_{-\pi}^{\pi} \log f(x,H)\,dx = 0.$$

Similarly, Theorems 1 and 2 can be applied to a fractional ARMA process by restricting the parameter (d, ξ, ϕ) to a compact set and choosing C in (1.9) as

$$\text{(2.2)}\quad C = C(d,\xi,\phi) = \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left\{|e^{ix}-1|^{-2d}\left|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\right|^2\right\} dx\right].$$

THEOREM 3. *The conclusions of Theorems* 1 *and* 2 *hold if* $X_j - \mu$ *is fractional Gaussian noise with* $\frac{1}{2} < H < 1$ *or a fractional ARMA process with* $0 < d < \frac{1}{2}$.

Theorem 3 is proven in Section 4.

REMARK. We have supposed that the mean μ of the sequence X_j is unknown. If it is known, merely replace the periodogram $I_N(x)$ in (1.5) by

$$\tilde{I}_N(x) = \frac{\left|\sum_{j=1}^{N} e^{ijx}(X_j - \mu)\right|^2}{2\pi N}.$$

COROLLARY 1. *When* $\mu = EX_j$ *is known, Theorems* 1, 2, *and* 3 *hold if* $I_N(x)$ *is replaced by* $\tilde{I}_N(x)$.

3. Proofs of Theorems 1 and 2. Retain the assumptions and definitions made in Section 2 prior to the statement of Theorem 1. Introduce $r_k(\theta) = \int_{-\pi}^{\pi} e^{ikx} f(x, \theta)\,dx$, so that $E(X_j - \mu)(X_{j+k} - \mu) = \sigma_0^2 r_k(\theta_0)$. Adopt the convention that functions defined in $[-\pi, \pi]$ are extended to $[-2\pi, 2\pi]$ in such a way as to have period 2π.

LEMMA 1. *Let* $g(x, \theta)$ *be a continuous function on* $[-\pi, \pi] \times E$. *If* (A.2) *and* (A.4) *hold, then with probability* 1

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} g(x, \theta) I_N(x)\,dx = \sigma_0^2 \int_{-\pi}^{\pi} g(x, \theta) f(x, \theta_0)\,dx$$

uniformly in θ.

PROOF. Note that $I_N(x)$ has Fourier coefficients

$$\int_{-\pi}^{\pi} e^{ikx} I_N(x)\,dx = \begin{cases} W(k, N), & |k| < N, \\ 0, & |k| > N, \end{cases}$$

where

$$\begin{aligned} W(k, N) &= \frac{1}{N} \sum_{j=1}^{N-k} (X_j - \overline{X}_N)(X_{j+k} - \overline{X}_N) \\ &= \frac{1}{N} \sum_{j=1}^{N-k} \{X_j - \mu - (\overline{X}_N - \mu)\}\{X_{j+k} - \mu - (\overline{X}_N - \mu)\} \\ &= \frac{\sum_{j=1}^{N-k}(X_j - \mu)(X_{j+k} - \mu)}{N} + \frac{N-k}{N}(\overline{X}_N - \mu)^2 \\ &\quad - (\overline{X}_N - \mu)\frac{\sum_{j=1}^{N-k}(X_j - \mu)}{N} - (\overline{X}_N - \mu)\frac{\sum_{j=k+1}^{N}(X_j - \mu)}{N}. \end{aligned}$$

The sequence $\{X_j\}$ is ergodic since it is Gaussian with spectral density $f(x, \theta_0)$

that satisfies $\int_{-\pi}^{\pi} f(x, \theta_0)\, dx > -\infty$. Therefore $(\bar{X}_N - \mu)$ tends to 0 as $N \to \infty$, as do the last three terms on the right-hand side. Hence

$$\lim_{N\to\infty} W(k, N) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N-k} (X_j - \mu)(X_{j+k} - \mu)$$
$$= \sigma_0^2 r_k(\theta_0).$$

This means that the proof of Lemma 1 can be carried out exactly as that of Lemma 1 of Hannan (1973). □

PROOF OF THEOREM 1. The proof uses Lemma 1 and the fact that $\int_{-\pi}^{\pi} \log f(x, \theta)\, dx = 0$ for all θ. Proceed as in the proof of Theorem 1 of Hannan (1973). □

To establish Theorem 2, we use the following four lemmas.

LEMMA 2. *Let $\{b_N\}$ be a sequence of constants tending to ∞. Let $\partial/\partial\theta\, \sigma_N^2(\theta)$ be the random vector with jth component equal to $\partial/\partial\theta_j\, \sigma_N^2(\theta)$. If* (A.2)–(A.4) *hold and Y is a random vector such that $b_N\, \partial/\partial\theta\, \sigma_N^2(\theta_0)$ tends to Y in distribution as $N \to \infty$, then $b_N(\bar{\theta}_N - \theta_0)$ tends to $(-2\pi/\sigma_0^2)W^{-1}(\theta_0)Y$ in distribution as $N \to \infty$.*

PROOF. Let $\partial^2/\partial\theta^2\, \sigma_N^2(\theta)$ be the $p \times p$ random matrix with j, kth entry $\partial^2/\partial\theta_j\, \partial\theta_k\, \sigma_N^2(\theta)$. According to the mean value theorem

$$\frac{\partial}{\partial\theta}\sigma_N^2(\bar{\theta}_N) = \frac{\partial}{\partial\theta}\sigma_N^2(\theta_0) + \left[\frac{\partial^2}{\partial\theta^2}\sigma_N^2(\theta_N^*)\right](\bar{\theta}_N - \theta_0),$$

where $|\theta_N^* - \theta_0| < |\bar{\theta}_N - \theta_0|$. Since θ_0 is in the interior of E, Theorem 1 implies that $\bar{\theta}_N$ is in the interior of E for large N. Since $\bar{\theta}_N$ minimizes $\sigma_N^2(\theta)$, it follows that $\partial/\partial\theta\, \sigma_N^2(\bar{\theta}_N) = 0$ for large N. Thus for large N

$$\frac{\partial}{\partial\theta}\sigma_N^2(\theta_0) = \left[-\frac{\partial^2}{\partial\theta^2}\sigma_N^2(\theta_N^*)\right](\bar{\theta}_N - \theta_0).$$

Because

$$\frac{\partial^2}{\partial\theta_j\, \partial\theta_k}\sigma_N^2(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial\theta_j\, \partial\theta_k} f^{-1}(x, \theta) I_N(x)\, dx,$$

it follows from Lemma 1 and Theorem 1 that with probability 1

$$\frac{\partial^2}{\partial\theta_j\, \partial\theta_k}\sigma_N^2(\theta_N^*) \to \frac{\sigma_0^2}{2\pi} w_{jk}(\theta_0).$$

Therefore

$$-b_N \frac{\sigma_0^2}{2\pi} W(\theta_0)(\bar{\theta}_N - \bar{\theta}_0)$$

tends in distribution to Y, completing the proof of Lemma 2. □

Lemma 3. *If* (A.1), (A.2), *and* (A.3) *hold then*

$$w_{jk}(\theta) = \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta_j} f^{-1}(x, \theta) \right) \left(\frac{\partial}{\partial \theta_k} f^{-1}(x, \theta) \right) f^2(x, \theta)\, dx.$$

Proof. If the right-hand side is denoted J, then by (A.1),

$$0 = \int_{-\pi}^{\pi} \frac{\partial^2}{\partial \theta_j\, \partial \theta_k} \log f(x, \theta)\, dx = -J + \int_{-\pi}^{\pi} f^{-1}(x, \theta) \frac{\partial^2}{\partial \theta_j\, \partial \theta_k} f(x, \theta)\, dx.$$

Hence

$$\begin{aligned} w_{jk}(\theta) &= \int_{-\pi}^{\pi} f(x, \theta) \frac{\partial^2}{\partial \theta_j\, \partial \theta_k} f^{-1}(x, \theta)\, dx \\ &= 2J - \int_{-\pi}^{\pi} f^{-1}(x, \theta) \frac{\partial^2}{\partial \theta_j\, \partial \theta_k} f(x, \theta)\, dx \\ &= J. \quad \square \end{aligned}$$

Lemma 4. *If conditions* (A.2) *and* (A.4) *hold, then for every* $\delta > 0$

$$r_k(\theta) = O(k^{\alpha(\theta) - 1 + \delta}) \quad \text{as } k \to \infty.$$

Proof. Fix θ and put $f(x) = f(x, \theta)$. Since f is periodic,

$$\begin{aligned} 2|r_k(\theta)| &= \left| \int_{-\pi}^{\pi} e^{ikx} \left[f(x) - f\left(x + \frac{\pi}{k}\right) \right] dx \right| \\ &\le \int_{-\pi}^{\pi} \left| f(x) - f\left(x + \frac{\pi}{k}\right) \right| dx \\ &= \int_{-\pi}^{-2\pi/k} + \int_{-2\pi/k}^{\pi/k} + \int_{\pi/k}^{\pi}. \end{aligned}$$

Conditions (A.2) and (A.4) imply that there is a constant $C = C(\theta, \delta)$ such that

$$f(x) \le C|x|^{-\alpha(\theta) - \delta}$$

and

$$\left| \frac{\partial}{\partial x} f(x) \right| \le C|x|^{-\alpha(\theta) - 1 - \delta}$$

for x bounded away from $\pm 2\pi$, say $|x| \le 2\pi - 1$. (Since f is periodic, f need not be continuous at 2π.) By the mean value theorem

$$\begin{aligned} \int_{\pi}^{2\pi/k} \left| f(x) - f\left(x + \frac{\pi}{k}\right) \right| dx &\le C \frac{\pi}{k} \int_{-\pi}^{-2\pi/k} \left| x + \frac{\pi}{k} \right|^{-\alpha(\theta) - 1 - \delta} dx \\ &= C \frac{\pi}{k} \int_{-\pi + \pi/k}^{-\pi/k} |x|^{-\alpha(\theta) - 1 - \delta}\, dx = O(k^{\alpha(\theta) - 1 + \delta}) \end{aligned}$$

as $k \to \infty$. A similar argument shows that

$$\int_{\pi/k}^{\pi} \left| f(x) - f\left(x + \frac{\pi}{k}\right) \right| dx = O(k^{\alpha(\theta)-1+\delta}).$$

We also have

$$\begin{aligned}\int_{-2\pi/k}^{\pi/k} \left| f(x) - f\left(x + \frac{\pi}{k}\right) \right| dx &\le \int_{-2\pi/k}^{\pi/k} f(x)\, dx + \int_{-2\pi/k}^{\pi/k} f\left(x + \frac{\pi}{k}\right) dx \\ &\le C \int_{-2\pi/k}^{\pi/k} |x|^{-\alpha(\theta)-\delta}\, dx \\ &\quad + C \int_{-2\pi/k}^{\pi/k} \left| x + \frac{\pi}{k} \right|^{-\alpha(\theta)-\delta} dx \\ &= 2C \int_{-2\pi/k}^{\pi/k} |x|^{-\alpha(\theta)-\delta}\, dx = O(k^{\alpha(\theta)-1+\delta}).\end{aligned}$$

This completes the proof of Lemma 4. □

Lemma 5. *If conditions* (A.3), (A.5), *and* (A.6) *hold, then for every* $\delta > 0$ *and every* $1 \le j \le p$,

$$\int_{-\pi}^{\pi} e^{ikx} \left[\frac{\partial}{\partial \theta_j} f^{-1}(x, \theta) \right] dx = O(k^{-\alpha(\theta)-1+\delta}) \quad \text{as } k \to \infty.$$

Proof. Since $\partial/\partial\theta_j f^{-1}(x, \theta)$ is symmetric, integration by parts yields

$$\begin{aligned} e_k(\theta) &:= \int_{-\pi}^{\pi} e^{ikx} \frac{\partial}{\partial \theta_j} f^{-1}(x, \theta)\, dx \\ &= -\frac{1}{ik} \int_{-\pi}^{\pi} e^{ikx} \frac{\partial^2}{\partial x\, \partial \theta_j} f^{-1}(x, \theta)\, dx. \end{aligned}$$

The argument in Lemma 4 can now be applied since

$$\frac{\partial^2}{\partial x\, \partial \theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-1-\delta}) \quad \text{with } 0 < -(\alpha(\theta) - 1) < 1.$$

We thus get

$$\begin{aligned} e_k(\theta) &= \frac{1}{k} O(k^{-(\alpha(\theta)-1)-1+\delta}) \\ &= O(k^{-\alpha(\theta)-1+\delta}) \quad \text{as } k \to \infty. \quad \square \end{aligned}$$

The proof of Theorem 2 uses the following result which is a consequence of Theorem 4 of Fox and Taqqu (1983).

Proposition 1. *Let* $f(x)$ *and* $g(x)$ *be symmetric real-valued functions whose sets of discontinuities have Lebesgue measure* 0. *Suppose that there exist* $\alpha < 1$

and $\beta < 1$ *such that* $\alpha + \beta < \frac{1}{2}$ *and such that for each* $\delta > 0$

$$f(x) = O\big(|x|^{-\alpha-\delta}\big) \quad \text{as } x \to 0$$

and

$$g(x) = O\big(|x|^{-\beta-\delta}\big) \quad \text{as } x \to 0.$$

If $\{X_j\}$ *is a stationary, mean* 0, *Gaussian sequence with spectral density* $f(x)$, *then*

$$\sqrt{N}\left\{\int_{-\pi}^{\pi} I_N(x)g(x)\,dx - E\int_{-\pi}^{\pi} I_N(x)g(x)\,dx\right\}$$

tends in distribution to a normal random variable with mean 0 *and variance*

$$4\pi\int_{-\pi}^{\pi}[f(x)g(x)]^2\,dx.$$

REMARK. Proposition 1 is established by showing that the cumulants of order greater than two tend to zero as $N \to \infty$. Non-Gaussian limits may occur if the $\{X_j\}$ are non-Gaussian [see Fox and Taqqu (1985)].

PROOF OF THEOREM 2. Let α_0 denote $\alpha(\theta_0)$. Define $m_N = E\,\partial/\partial\theta\,\sigma_N^2(\theta_0)$ and let $m_{N,j} = E\,\partial/\partial\theta_j\,\sigma_N^2(\theta_0)$ be the jth coordinate of m_N. Let $c_1, \dots, c_p$ be fixed constants and consider the random variable

$$\begin{aligned} Y_N &= \sum_{j=1}^{p} c_j\left[\frac{\partial}{\partial\theta_j}\sigma_N^2(\theta_0) - m_{N,j}\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{j=1}^{p} c_j\frac{\partial}{\partial\theta_j} f^{-1}(x,\theta_0)\right] I_N(x)\,dx - \sum_{j=1}^{p} c_j m_{N,j}. \end{aligned}$$

Under condition (A.3) the function in brackets is $O(|x|^{\alpha_0-\delta})$ as $x \to 0$ for every $\delta > 0$. Apply Proposition 1 with

$$\begin{aligned} \alpha &= \alpha_0, \\ \beta &= -\alpha_0, \\ f(x) &= \sigma_0^2 f(x,\theta_0), \end{aligned}$$

and

$$g(x) = \sum_{j=1}^{p} c_j\frac{\partial}{\partial\theta_j} f^{-1}(x,\theta_0),$$

and conclude that $\sqrt{N}\,Y_N$ tends in distribution as $N \to \infty$ to a normal random variable with mean 0 and variance s^2 given by

$$\begin{aligned} s^2 &= \frac{\sigma_0^4}{\pi}\int_{-\pi}^{\pi} f^2(x,\theta_0)\left[\sum_{j=1}^{p} c_j\frac{\partial}{\partial\theta_j} f^{-1}(x,\theta_0)\right]^2 dx \\ &= \sum_{j=1}^{p}\sum_{k=1}^{p} c_j c_k\frac{\sigma_0^4}{\pi}\int_{-\pi}^{\pi} f^2(x,\theta_0)\left(\frac{\partial}{\partial\theta_j} f^{-1}(x,\theta_0)\right)\left(\frac{\partial}{\partial\theta_k} f^{-1}(x,\theta_0)\right) dx. \end{aligned}$$

An application of Lemma 3 yields

$$s^2 = \sum_{j=1}^{p} \sum_{k=1}^{p} c_j c_k \frac{\sigma_0^4}{\pi} w_{ij}(\theta_0).$$

Since $c_1, \dots, c_p$ were arbitrary, we have shown that $\sqrt{N}(\partial/\partial\theta\, \sigma_N^2(\theta_0) - m_N)$ tends in distribution to a normal random vector with mean 0 and covariance matrix $\sigma_0^4/\pi W(\theta_0)$. Therefore Theorem 2 will follow from Lemma 2 if we show that under the conditions of Theorem 2

$$\lim_{N\to\infty} \sqrt{N} m_{N,l} = 0, \qquad l = 1, \dots, p.$$

To prove this, define

$$\mu_{N,l} = E\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta_l} f^{-1}(x, \theta_0) \tilde{I}_N(x)\, dx \right\},$$

where

$$\tilde{I}_N(x) = \frac{\left|\sum_{j=1}^{N} e^{ijx}(X_j - \mu)\right|^2}{2\pi N}.$$

It follows from Lemma 8.1 of Fox and Taqqu (1983) that $\lim_{N\to\infty}\sqrt{N}(m_{N,l} - \mu_{N,l}) = 0$. Thus it suffices to show

$$\lim_{N\to\infty} \sqrt{N} \mu_{N,l} = 0, \qquad l = 1, \dots, p. \tag{3.1}$$

We have

$$\mu_{N,l} = \frac{1}{(2\pi)^2 N} \sum_{j=1}^{N} \sum_{k=1}^{N} e_{j-k}(\theta_0)(X_j - \mu)(X_k - \mu),$$

where

$$e_k = e_k(\theta) = \int_{-\pi}^{\pi} e^{ikx} \frac{\partial}{\partial \theta_l} f^{-1}(x, \theta)\, dx.$$

Set also $r_k = r_k(\theta_0)$. Then

$$\mu_{N,l} = \frac{\sigma_0^2}{(2\pi)^2 N} \sum_{j=1}^{N} \sum_{k=1}^{N} e_{j-k} r_{j-k}.$$

Note that $e_k r_k$ is the kth Fourier coefficient of the convolution

$$h(x) = \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta_l} f^{-1}(y, \theta_0) \right) f(y - x, \theta_0)\, dy.$$

Note also that

$$\begin{aligned} h(0) &= \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta_l} f^{-1}(y, \theta_0) \right) f(y, \theta_0)\, dy \\ &= -\int_{-\pi}^{\pi} \frac{\partial}{\partial \theta_l} \log f(y, \theta_0)\, dy = 0, \end{aligned}$$

where we have used (A.1).

To prove (3.1), observe that by Lemmas 4 and 5, there is a $0 < \delta < \frac{1}{4}$ such that as $k \to \infty$,

$$e_k r_k = O(k^{-2+2\delta}). \tag{3.2}$$

Observe also that $e_k r_k$ is the kth Fourier coefficient of h, so that

$$\sum_{k=-\infty}^{\infty} e_k r_k = h(0) = 0. \tag{3.3}$$

We have

$$\begin{aligned}\frac{(2\pi)^2}{\sigma_0^2}\sqrt{N}\mu_{N,l} &= \sqrt{N}\,\frac{\sum_{j=1}^{N}\sum_{j=1}^{N} e_{j-k}r_{j-k}}{N}\\ &= \sqrt{N}\sum_{|k|<N} e_k r_k\left(1-\frac{k}{N}\right)\\ &= N^{1/2}\sum_{|k|<N} e_k r_k - N^{-1/2}\sum_{|k|<N} k e_k r_k.\end{aligned}$$

Because of (3.3), the first term equals $-N^{1/2}\sum_{|k|\geq N} e_k r_k$, which is $O(N^{-1/2+2\delta})$ by (3.2). The second term is also $O(N^{-1/2+2\delta})$ by (3.2). These terms tend to zero as $N \to \infty$, establishing (3.1). This completes the proof of Theorem 2. □

REMARK. Conditions (A.4), (A.5), and (A.6) were used in the proofs of Lemmas 4 and 5 to show that $r_k = O(k^{\alpha(\theta)-1+\delta})$ and $e_k = O(k^{-\alpha(\theta)-1+\delta})$ as $k \to \infty$. In specific cases, a Tauberian theorem may be applied to $f(x,\theta)$ and $\partial/\partial\theta_j f^{-1}(x,\theta)$ to yield such estimates on r_k and e_k.

4. Proof of Theorem 3. In order to verify that conditions A are satisfied for fractional Gaussian noise and fractional ARMA's, it is convenient to check the following conditions which are stronger than conditions A.

CONDITIONS B. We say that $f(x,\theta)$ satisfies conditions B.1–B.4 if there is a continuous function $0 < \alpha(\theta) < 1$ and constants $C(\delta)$, $C_0(\delta)$ such that for each $\delta > 0$

(B.1) $f(x,\theta)$ is continuous at all (x,θ), $x \neq 0$ and

$$f(x,\theta) \geq C_0(\delta)|x|^{-\alpha(\theta)+\delta}.$$

(B.2) $f(x,\theta) \leqq C(\delta)|x|^{-\alpha(\theta)-\delta}$.

(B.3) $\partial/\partial\theta_j f(x,\theta)$ and $\partial^2/\partial\theta_j\,\partial\theta_k f(x,\theta)$ are continuous at all (x,θ), $x \neq 0$,

$$\left|\frac{\partial}{\partial\theta_j} f(x,\theta)\right| \leq C(\delta)|x|^{-\alpha(\theta)-\delta}, \qquad 1 \leq j \leq p,$$

and

$$\left|\frac{\partial^2}{\partial\theta_j\,\partial\theta_k} f(x,\theta)\right| \leq C(\delta)|x|^{-\alpha(\theta)-\delta}, \qquad 1 \leq j, k \leq p.$$

(B.4) $\partial/\partial x\, f(x,\theta)$, $\partial^2/\partial x\,\partial\theta_j\, f(x,\theta)$, and $\partial^3/\partial x^2\,\partial\theta_j\, f(x,\theta)$ are continuous at all (x,θ), $x \neq 0$,

$$\left|\frac{\partial}{\partial x} f(x,\theta)\right| \leq C(\delta)|x|^{-\alpha(\theta)-1-\delta},$$

$$\left|\frac{\partial^2}{\partial x\,\partial\theta_j} f(x,\theta)\right| \leq C(\delta)|x|^{-\alpha(\theta)-1-\delta}, \qquad 1 \leq j \leq p,$$

and

$$\left|\frac{\partial^3}{\partial x^2\,\partial\theta_j} f(x,\theta)\right| \leq C(\delta)|x|^{-\alpha(\theta)-2-\delta}, \qquad 1 \leq j \leq p.$$

Note that conditions B do not involve the function $f^{-1}(x,\theta)$. The constants $C(\delta)$ and $C_0(\delta)$ which appear in conditions B are required to be independent of θ.

LEMMA 6. *If f satisfies conditions* B.1–B.4, *then f satisfies conditions* A.1–A.6.

PROOF. Suppose that f satisfies conditions B. It is easily seen that conditions A.2–A.6 are satisfied. For example

$$\left|\frac{\partial}{\partial\theta_j} f^{-1}(x,\theta)\right| = \frac{\left|\frac{\partial}{\partial\theta_j} f(x,\theta)\right|}{f^2(x,\theta)} \leqq \frac{C(\delta)}{C_0^2(\delta)}|x|^{\alpha(\theta)-3\delta}.$$

This implies that $\partial/\partial\theta_j\, f^{-1}(x,\theta)$ is continuous and that $\partial/\partial\theta_j\, f^{-1}(x,\theta) = O(|x|^{\alpha(\theta)-3\delta})$ as $x \to 0$.

We check that condition A.1 is satisfied. Let v_j be the jth unit vector in R^p, that is, the vector with jth component equals 1 and all other components equal 0. Then we have

$$\frac{\int_{-\pi}^{\pi}\log f(x,\theta+\varepsilon v_j)\,dx - \int_{-\pi}^{\pi}\log f(x,\theta)\,dx}{\varepsilon}$$
$$= \int_{-\pi}^{\pi}\frac{\log f(x,\theta+\varepsilon v_j) - \log f(x,\theta)}{\varepsilon}\,dx.$$

By the mean value theorem this integrand is majorized for each $x \neq 0$ by

$$\left|\frac{\partial}{\partial\theta_j}\log f(x,\theta^*(x))\right| = \frac{\left|\frac{\partial}{\partial\theta_j} f(x,\theta^*(x))\right|}{f(x,\theta^*(x))},$$

where $|\theta^*(x) - \theta| < |\varepsilon|$. Under conditions B.1 and B.3 this quotient is at most $C(\delta)/C_0(\delta)|x|^{\alpha_m-\alpha_M-2\delta}$, where

$$\alpha_m = \min_{\theta\in E}\alpha(\theta)$$

and

$$\alpha_M = \max_{\theta \in E} \alpha(\theta).$$

Since $\alpha_m - \alpha_M > -1$ we can choose δ so that $\alpha_m - \alpha_M - 2\delta > -1$, and thus

$$\int_{-\pi}^{\pi} |x|^{\alpha_m - \alpha_M - 2\delta}\, dx < \infty.$$

Hence the dominated convergence theorem implies that $\int_{-\pi}^{\pi} \log f(x, \theta)\, dx$ can be differentiated under the integral sign. A similar argument shows that a second differentiation under the integral sign can also be performed. □

PROOF OF THEOREM 3. For fractional Gaussian noise, $f(x, H) = CF(H) f_0(x, H)$, where $f_0(x, H)$ is defined in (1.7) and $CF(H)$ is defined in (2.1) as

$$CF(H) = \exp\left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_0(x, H)\, dx \right\}.$$

According to Lemma 6 it suffices to show that $f(x, H)$ satisfies conditions B.1–B.4 with $\alpha(H) = 2H - 1$. We will show that $f_0(x, H)$ satisfies conditions B with $\alpha(H) = 2H - 1$. Then Lemma 6 implies that $CF(H)$ is twice continuously differentiable, which means that $f(x, H)$ satisfied conditions B.

Note that

$$f_0(x, H) = (1 - \cos x)\left[|x|^{-1-2H} + f_1(x, H)\right],$$

where

$$f_1(x, H) = \sum_{k \neq 0} |x + 2k\pi|^{-1-2H}.$$

Since $1 - \cos x \sim |x|^2/2$ as $x \to 0$, conditions B will hold for $f_0(x, H)$ if $f_1(x, H)$ is three times continuously differentiable at all (x, H). A standard theorem on differentiation of series [Theorem 7.17 of Rudin (1964), for example] shows that this is indeed the case. Thus conditions B are satisfied for fractional Gaussian noise.

It is even simpler to verify conditions B for a fractional ARMA process because the divergent term is already factored out in that case □

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SCHOOL OF OPERATIONS RESEARCH
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853