The Annals of Statistics
1992, Vol. 20, No. 2, 971–1001

ASYMPTOTICS FOR LINEAR PROCESSES

BY PETER C. B. PHILLIPS AND VICTOR SOLO

Yale University and Macquarie University

A method of deriving asymptotics for linear processes is introduced which uses an explicit algebraic decomposition of the linear filter. The technique is closely related to Gordin's method but has some advantages over it, especially in terms of its range of application. The method offers a simple unified approach to strong laws, central limit theory and invariance principles for linear processes. Sample means and sample covariances are covered. The results accommodate both homogeneous and heterogeneous innovations as well as innovations with undefined means and variances.

1. Introduction. Since the work of McLeish (1975a, b, 1977) a popular approach to the development of asymptotics for time series has been the use of limit theorems for dependent random variables that satisfy certain mixing conditions. This approach has the advantage of allowing for heterogeneity as well as dependence; it highlights the trade-off that occurs in limit theory between moment conditions that control outlier probabilities and memory conditions that control the extent of the temporal dependence; and it conveniently accommodates nonlinear function dependence on a series' past history. The latter property has ensured that the method is especially popular in the development of asymptotics for nonlinear statistical models [e.g., Gallant (1987), Chapter 7].

In spite of these advantages, the approach does have some drawbacks. First, not all linear processes are strong mixing, for example, and it is necessary to use functions of mixing processes to accommodate even simple time series like the first order autoregression in a general theory. This is unfortunate because most of the stationary time series literature is still concerned with parametric models that fall in the linear process class. Second, the mixingale theory of McLeish is articulated in the L_2 norm and is therefore inapplicable in time series models with infinite variance errors.

The aim of the present paper is to show the versatility of an alternative approach that is especially designed for linear process. In this sense, the paper represents something of a return to more traditional methods and models such as those emphasized in major textbooks like Anderson (1971), Fuller (1976) and Hannan (1970), all of which put linear processes in a central position in the development of time series asymptotics. Our method involves little in the way of probabilistic sophistication and relies almost exclusively on limit theory for independent and identically distributed (i.i.d.) or independent and

Received December 1989; revised July 1991.
AMS 1980 *subject classifications*. Primary 60F05; secondary 60F17, 60F15.
Key words and phrases. BN decomposition, central limit theory, functional limit laws, infinite variance errors, law of iterated logarithm, linear process, stable process, strong laws.

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nonidentically distributed (i.n.i.d.) random variables (r.v.'s) and for martingale difference sequences (m.d.s.'s). The key to the approach is an algebraic decomposition of the linear filter into long-run and transitory elements that is known in the econometric literature as the Beveridge–Nelson or BN decomposition—see Beveridge and Nelson (1981). The long-run component in this decomposition yields the martingale approximation to the partial sum process of a stationary time series. In this way, the approach is related to the method pioneered by Gordin (1969) for developing central limit theorems (CLT's) for stationary processes via corresponding results for approximating martingales. A detailed treatment of that method is given in Chapter 5 of Hall and Heyde (1980).

Since our own approach relies on a purely algebraic decomposition of the linear filter, it has some advantages over the martingale approximation approach. First, it can be readily used when the time series innovations are heterogeneously distributed rather than stationary and ergodic martingale differences. Second, we may relax moment conditions and work with innovations whose first and second moments are not finite. Our method can therefore accommodate a limit theory for moving averages of r.v.'s with regularly varying tail probabilities, such as that developed in recent work by Davis and Resnick (1985a, b, 1986).

A wide spectrum of limit results is presented. We give strong laws of large numbers (SLLN's), a law of the iterated logarithm (LIL), CLT's and invariance principles (IP's), we include sample means and sample covariances of stationary and nonstationary time series and we give stable limit laws for sample moments of linear processes whose domain of attraction is not the normal distribution. Few of the results given are new and our main purpose is to exhibit a unifying theme in the treatment of linear process asymptotics. The approach should be of pedagogical interest to time series specialists.

2. Preliminaries. We start with a simple polynomial decomposition that is fundamental to our approach.

2.1 LEMMA (BN). *Let* $C(L) = \sum_0^\infty c_j L^j$. *Then*

$$C(L) = C(1) - (1 - L)\tilde{C}(L), \tag{1}$$

where $\tilde{C}(L) = \sum_0^\infty \tilde{c}_j L^j$, $\tilde{c}_j = \sum_{j+1}^\infty c_k$. *If* $p \geq 1$, *then*

$$\sum_1^\infty j^p |c_j|^p < \infty \Rightarrow \sum_0^\infty |\tilde{c}_j|^p < \infty \quad and \quad |C(1)| < \infty. \tag{2}$$

If $p < 1$, *then*

$$\sum_1^\infty j|c_j|^p < \infty \Rightarrow \sum_0^\infty |\tilde{c}_j|^p < \infty. \tag{3}$$

2.2 REMARKS. (i) For linear processes such as (13) below, the decomposition (1) yields directly the martingale approximation to the partial sum process of a stationary time series [see Hall and Heyde (1980), Chapter 5]. Because (1) is purely algebraic, it turns out to be a useful device in reducing time series asymptotics to known theorems for i.i.d., i.n.i.d. and m.d.s. sequences. The decomposition can also be applied to deduce asymptotics for higher order moments, invariance principles and stable limit laws for time series.

(ii) When $p = 2$, we have $\sum_0^\infty \tilde{c}_j^2 < \infty$ under

$$(\mathscr{S}_1) \qquad \sum_1^\infty j^2 c_j^2 < \infty.$$

An alternative condition for $\sum_0^\infty \tilde{c}_j^2 < \infty$ is

$$(\mathscr{S}_2) \qquad \sum_1^\infty j^{1/2}|c_j| < \infty.$$

Observe that $c_j = 1/j^{3/2}\ln(j+1)$, for example, satisfies $(\mathscr{S}_1)$ but fails $(\mathscr{S}_2)$, so that $(\mathscr{S}_1)$ is a useful complement to $(\mathscr{S}_2)$. Both coefficient preconditions will be used repeatedly below.

(iii) The algebraic decomposition (1) was used explicitly (but without conditions on the coefficients) by Beveridge and Nelson (1981) to decompose aggregate economic time series into permanent and transitory components. For convenience, we shall refer to (1) subsequently as the BN decomposition although it must certainly have been known and used in earlier work. For finite lag polynomials, the decomposition was used by Fuller [(1976), page 374] and by Bewley (1979). A proof of the result under $(\mathscr{S}_2)$ was given in Solo (1989).

For later development, it will be useful to have available some standard asymptotics for sequences of independent r.v.'s and martingale differences. We start with the following result of Heyde and Seneta [see Hall and Heyde (1980), page 36].

2.3 Theorem (LLN). *Let (Z_n) be a sequence of r.v.'s adapted to the filtration $(\mathscr{F}_n)$. Let Z be a dominating r.v. for which $E|Z| < \infty$ and*

$$(4) \qquad P(|Z_n| \geq x) \leq cP(|Z| \geq x)$$

for each $x \geq 0$, $n \geq 1$ and for some constant c. Then as $n \to \infty$,

$$(5) \qquad \frac{1}{n}\sum_1^n [Z_i - E(Z_i|\mathscr{F}_{i-1})] \to_p 0.$$

If $E(|Z|\ln^+|Z|) < \infty$ or if the Z_n are independent or if (Z_n) is stationary and $\mathscr{F}_n$ is the natural filtration of Z_n, then a.s. convergence applies in (5).

2.4 Remarks. (i) A sequence (Z_n) satisfying (4) is said to be *strongly uniformly integrable* (s.u.i.)—see Billingsley [(1968), page 32] and Solo (1982, 1986).

(ii) If the Z_n are identically distributed with $E|Z_0| < \infty$, then (4) is automatic and (5) holds with a.s. convergence.

For our central limit theory a useful starting point is the following result of McLeish (1974) [see also Hall and Heyde (1980)]. Suppose $(Z_i, \mathscr{F}_i)$ is an m.d.s., $S_n = \Sigma_1^n Z_i$, $U_n^2 = \Sigma_1^n Z_i^2$ and $s_n^2 = E(U_n^2) = E(S_n^2)$. Then:

2.5 THEOREM (CLT). *If* (6) *and* (7) *hold, then* $s_n^{-1} S_n \to_d N(0,1)$:

(6) $$s_n^{-2} U_n^2 \to_p 1,$$

(7) $$\max_{1 \le i \le n} |Z_{ni}| \to_p 0, \qquad Z_{ni} = s_n^{-1} Z_i.$$

The invariance principle calls for more notation. Let $[nr]$ denote the integer part of nr with $0 \le r \le 1$ and set

$$W_n(r) = s_n^{-1} S_{[nr]},$$

$$\xi_n(r) = s_n^{-1} S_i + s_n^{-1} Z_{i+1} \left(s_{i+1}^2 - s_i^2\right)^{-1} \left(r s_n^2 - s_i^2\right) \quad \text{for } s_i^2 \le r s_n^2 < s_{i+1}^2.$$

From Brown (1971) [see also Hall and Heyde (1980), page 99], we have:

2.6 THEOREM (IP). *If* (6) *and either* (7) *or* (8) *hold, then* $W_n(r)$, $\xi_n(r) \to_d W(r)$, *a standard Brownian motion on* $C[0,1]$, *where*

(8) $$\sum_1^n E\left[Z_{ni}^2 1(|Z_{ni}| > \varepsilon)\right] \to 0$$

for any $\varepsilon > 0$. *In fact, when* (6) *holds, conditions* (7) *and* (8) *are equivalent.*

2.7 ASSUMPTIONS. We work with two classes of assumptions concerning the time series innovations when these have finite means. They are letter coded as: $\mathscr{A}$ for homogeneity assumptions; and $\mathscr{B}$ for heterogeneity assumptions.

($\mathscr{A}_1$) (ε_t) is i.i.d. with zero mean and $E|\varepsilon_0| < \infty$.

($\mathscr{A}_2$) (ε_t) is i.i.d. with zero mean and $\sigma_\varepsilon^2 = E\left(\varepsilon_0^2\right) < \infty$.

($\mathscr{A}_3$) (ε_t) is i.i.d. with zero mean and finite fourth cumulant κ_4.

($\mathscr{A}_4$) (ε_t) is i.i.d. with zero mean and $E|\varepsilon_0|^p < \infty$ for some p satisfying $2 < p < \infty$.

($\mathscr{B}_1$) (ε_t) is an m.d.s. and is s.u.i. with dominating r.v. Z that satisfies $E(|Z| \ln^+ |Z|) < \infty$.

($\mathscr{B}_2$) (ε_t) is an m.d.s., is s.u.i. with $E(Z^2 \ln^+ |Z|) < \infty$ and, further, $n^{-1} \Sigma_1^n E(\varepsilon_t^2 | \mathscr{F}_{t-1}) \to_{\text{a.s.}} \sigma_\varepsilon^2$.

For the case where the innovations ε_t may have undefined means, we make the following domain of attraction assumptions (letter coded as $\mathscr{C}$). We say that ε is in the domain of attraction of a stable law with a parameter α and write $\varepsilon \in \mathscr{D}(\alpha)$ if

(9) $$P(\varepsilon > x) = c_1 x^{-\alpha} L(x)\left(1 + \alpha_1(x)\right), \qquad x > 0,\ c_1 \ge 0,$$

and

$$P(\varepsilon < -x) = c_2 x^{-\alpha} L(x)(1 + \alpha_2(x)), \qquad x > 0, c_2 \geq 0, \tag{10}$$

with $0 < \alpha < 2$, $L(x)$ a slowly varying function at ∞ and $\alpha_i(x) \to 0$ as $|x| \to \infty$. If $L(x) = 1$ in (9) and (10), then ε is in the normal domain of attraction of a stable law with parameter α and we write $\varepsilon \in \mathscr{N}\mathscr{D}(\alpha)$.

($\mathscr{C}_1$) (ε_t) is i.i.d. and $\varepsilon_t \in \mathscr{D}(\alpha)$. If $\alpha > 1$, $E(\varepsilon_t) = 0$ and if $\alpha = 1$, then $\varepsilon_t =_d -\varepsilon_t$ (i.e., ε_t is symmetrically distributed).

($\mathscr{C}_2$) (ε_t) is i.i.d. and $\varepsilon_t \in \mathscr{N}\mathscr{D}(\alpha)$. If $\alpha > 1$, $E(\varepsilon_t) = 0$ and if $\alpha = 1$, then $\varepsilon_t =_d -\varepsilon_t$.

2.8 REMARKS. (i) It follows from Theorem LLN that under ($\mathscr{A}_1$) or ($\mathscr{A}_2$) or ($\mathscr{B}_1$), we have $n^{-1}\Sigma_1^n \varepsilon_t \to_{\text{a.s.}} 0$. (ii) Similarly under ($\mathscr{B}_2$) we have both $n^{-1}\Sigma_1^n \varepsilon_t \to_{\text{a.s.}} 0$ and $n^{-1}\Sigma_1^n \varepsilon_t^2 \to_{\text{a.s.}} \sigma_\varepsilon^2$. (iii) For ($\mathscr{C}_1$) and ($\mathscr{C}_2$), we define the normalizing sequence

$$a_n = \inf\{x \colon P(|\varepsilon_0| > x) \leq n^{-1}\}.$$

Under ($\mathscr{C}_1$), we have $a_n = n^{1/\alpha} L'(n)$, where $L'(n)$ is slowly varying at infinity. Under ($\mathscr{C}_2$) we have $a_n = c n^{1/\alpha}$ for some constant c; when $\varepsilon_t =_d -\varepsilon_t$ and $c_1 = c_2 = a^\alpha$ in (9) and (10), then $c = a$. With this construction we have the following results under either ($\mathscr{C}_1$) or ($\mathscr{C}_2$) and $0 < \alpha < 2$:

$$\frac{1}{a_n}\sum_1^n \varepsilon_t \to_d U_\alpha(1), \qquad \frac{1}{a_n}\sum_1^{[nr]} \varepsilon_t \to_d U_\alpha(r), \tag{11}$$

$$\left(\frac{1}{a_n}\sum_1^{[nr]} \varepsilon_t, \frac{1}{a_n^2}\sum_1^{[nr]} \varepsilon_t^2\right) \to_d \left(U_\alpha(r), \int_0^r (dU_\alpha)^2\right). \tag{12}$$

Here $U_\alpha(r)$ is the Lévy α-stable process and $\int_0^r (dU_\alpha)^2 = [U_\alpha]_r$ is its quadratic variation process. The first result of (11) is classical [e.g., Ibragimov and Linnik (1971), Chapter 2]; the second is its functional version; and (12) is a joint functional limit law for the first and second sample moments that is proved in Resnick [(1986), pages 94–95].

3. Limit theory for linear processes.

A. *BN in direct mode and homogeneous innovations.* Suppose X_t is the linear process

$$X_t = C(L)\varepsilon_t = \sum_0^\infty c_j \varepsilon_{t-j}, \qquad C(L) = \sum_0^\infty c_j L^j, \tag{13}$$

with $0 < |C(1)| < \infty$ and

$$\sum_0^\infty c_j^2 < \infty. \tag{14}$$

Our object is to show how simply some of the classical time series asymptotics can be worked out by applying the BN decomposition (1) directly to (13). For (ε_t) we employ either $(\mathscr{A}_1)$ or $(\mathscr{A}_2)$. Applying (1) to (13), we get

$$X_t = C(1)\varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t \tag{15}$$

with

$$\tilde{\varepsilon}_t = \tilde{C}(L)\varepsilon_t = \sum_0^\infty \tilde{c}_j \varepsilon_{t-j}, \qquad \tilde{c}_j = \sum_{j+1}^\infty c_k.$$

Under $(\mathscr{A}_2)$, $E(\tilde{\varepsilon}_t^2) < \infty$ if

$$\sum_0^\infty \tilde{c}_j^2 < \infty, \tag{16}$$

which by the BN lemma holds if $(\mathscr{S}_1)$ holds. Now sum (15) to find

$$\frac{1}{n}\sum_1^n X_t = C(1)\frac{1}{n}\sum_1^n \varepsilon_t + \frac{1}{n}(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_n). \tag{17}$$

So a SLLN for X_t follows directly from a SLLN for ε_t [see Remark 2.9(i))] if only

$$n^{-1}\tilde{\varepsilon}_0 \to_{\text{a.s.}} 0 \quad \text{and} \quad n^{-1}\tilde{\varepsilon}_n \to_{\text{a.s.}} 0.$$

These hold if

$$\sum_1^\infty E(\tilde{\varepsilon}_0^2)n^{-2} < \infty \quad \text{and} \quad \sum_1^\infty E(\tilde{\varepsilon}_n^2)n^{-2} < \infty,$$

which hold if $E(\tilde{\varepsilon}_0^2) = E(\tilde{\varepsilon}_n^2) < \infty$, which holds if $(\mathscr{S}_1)$ does. Thus, we have established:

3.1 THEOREM (SLLN). *Under* $(\mathscr{A}_2)$ *and* $(\mathscr{S}_1)$, $n^{-1}\Sigma_1^n X_t \to_{a.s.} 0$.

With a little more effort and a strengthening of $(\mathscr{S}_1)$, we can relax the second moment condition in $(\mathscr{A}_2)$, giving:

3.2 THEOREM (SLLN). *Under* $(\mathscr{A}_1)$ *and* $(\mathscr{S}_3)$, $n^{-1}\Sigma_1^n X_t \to_{a.s.} 0$, *where*

$$\sum_1^\infty j|c_j| < \infty. \tag{$\mathscr{S}_3$}$$

An LIL for partial sums of X_t can be obtained in a similar way. Let $\varphi_n = \{2n \ln_2(n)\}^{1/2}$, where $\ln_2(n) = \ln(\ln(n))$. Replace (17) by the expression

$$\frac{1}{\varphi_n}\sum_1^n X_t = C(1)\frac{1}{\varphi_n}\sum_1^n \varepsilon_t + \frac{1}{\varphi_n}(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_n) \tag{17'}$$

and then the LIL for $\Sigma_1^n X_t$ follows from the LIL for $\Sigma_1^n \varepsilon_t$ if

$$\varphi_n^{-1}\tilde{\varepsilon}_0 \to_{\text{a.s.}} 0 \quad \text{and} \quad \varphi_n^{-1}\tilde{\varepsilon}_n \to_{\text{a.s.}} 0.$$

These hold by the Borel–Cantelli lemma if

$$E|\tilde{\varepsilon}_n|^q < \infty \quad \text{for some } q > 2.$$

The latter condition holds under (16) and the strengthened moment condition $(\mathscr{A}_4)$ with $q < p$, leading to:

3.3 THEOREM (LIL). *Under* $(\mathscr{A}_4)$ *and* $(\mathscr{S}_1)$ *or* $(\mathscr{S}_2)$,

$$\limsup_{n\to\infty}, \liminf_{n\to\infty}\left[\frac{1}{\sigma_X \varphi_n}\left(\sum_1^n X_t\right)\right] = \pm 1 \quad a.s.,$$

where $\sigma_X^2 = \sigma_\varepsilon^2 C(1)^2$.

Lai and Wei [(1982), Theorem 3] obtained a comparable log log law for linear processes under (16) and $(\mathscr{A}_4)$ without the identical distribution assumption.

Continuing with quick results, we now use the BN decomposition to deliver a CLT and IP for partial sums of X_t. From (15),

$$\frac{1}{n^{1/2}}\sum_1^{[nr]} X_t = C(1)\frac{1}{n^{1/2}}\sum_1^{[nr]}\varepsilon_t + \frac{\tilde{\varepsilon}_0}{n^{1/2}} - \frac{\tilde{\varepsilon}_{[nr]}}{n^{1/2}}. \tag{18}$$

Under $(\mathscr{A}_2)$ we easily obtain a CLT and IP for the first term from Theorem 2.7. We need

$$\frac{1}{n}\sum_1^n \varepsilon_t^2 \to_p \sigma_\varepsilon^2$$

which holds by Theorem LLN; this ensures that (6) holds and (8) follows because

$$\frac{1}{n}\sum_1^n E\left[\varepsilon_i^2 1(\varepsilon_i^2 > n\delta)\right] = E\left[\varepsilon_0^2 1(\varepsilon_0^2 > n\delta)\right] \to 0 \quad \text{for any } \delta > 0$$

by dominated convergence. Thus,

$$\frac{1}{n^{1/2}}\sum_1^n \varepsilon_t \to_d N(0,\sigma_\varepsilon^2) \quad \text{and} \quad \frac{1}{n^{1/2}}\sum_1^{[nr]}\varepsilon_t \to_d \sigma_\varepsilon W(r)$$

by Theorems CLT and IP. To prove the CLT for X_t we see from (18) with $r = 1$, that it is sufficient to show that

$$n^{-1}\tilde{\varepsilon}_0^2 \to_p 0 \quad \text{and} \quad n^{-1}\tilde{\varepsilon}_n^2 \to_p 0.$$

These hold if $E(\tilde{\varepsilon}_n^2) = E(\tilde{\varepsilon}_0^2) < \infty$, which as before holds if $(\mathscr{S}_1)$ does.

For the IP for X_t we need [Billingsley (1968), Theorem 4.1]:

$$\sup_r\left|\frac{1}{n^{1/2}}\sum_1^{[nr]} X_t - C(1)\frac{1}{n^{1/2}}\sum_1^{[nr]}\varepsilon_t\right| \le \frac{\tilde{\varepsilon}_0}{n^{1/2}} + \sup_r\left|\frac{\tilde{\varepsilon}_{[nr]}}{n^{1/2}}\right| \to_p 0 \tag{19}$$

which holds if

$$\max_{1\le k\le n}\left(n^{-1}\tilde{\varepsilon}_k^2\right)\to_p 0.$$

This is equivalent to

$$J_n=\frac{1}{n}\sum_1^n\left[\tilde{\varepsilon}_k^2 1\left(\tilde{\varepsilon}_k^2>n\delta\right)\right]\to_p 0\quad\text{for any }\delta>0 \tag{20}$$

[cf. Hall and Heyde (1980), page 53, and (36) below]. But (20) holds because

$$E(J_n)=E\left[\tilde{\varepsilon}_0^2 1\left(\tilde{\varepsilon}_0^2>n\delta\right)\right]\to 0$$

by dominated convergence since $(\mathscr{S}_1)$ ensures that $E(\tilde{\varepsilon}_0^2)<\infty$. We therefore have the following:

3.4 THEOREM (CLT and IP for means). *Under* $(\mathscr{A}_2)$ *and* $(\mathscr{S}_1)$, (a) $n^{-1/2}\Sigma_1^n X_t\to_d N(0,\sigma_\varepsilon^2 C(1)^2)$, (b) $n^{-1/2}\Sigma_1^{[nr]}X_t\to_d\sigma_\varepsilon W(r)$.

3.5 REMARKS. (i) We have used $(\mathscr{S}_1)$ as the summability condition in Theorem 3.4 but it is clear from the proof that the results hold under (16), which is precisely the condition given for the use of the IP in [Hall and Heyde (1980), Theorem 5.5, pages 141 and 146] due to Heyde (1975). $(\mathscr{S}_1)$ may be preferable for applications because it is a little more concrete in terms of the coefficients of the process (13).

(ii) One advantage of Theorem 3.4(b) is that no proof of tightness for the partial sums of dependent sequences is required. Under (19) all we need to call upon is the IP for partial sums of i.i.d. sequences and here we can rely on existing tightness arguments with no difficulty [e.g., Billingsley (1968), pages 137–138].

Our next step is to use a second order BN decomposition to establish the limit theory for sample variances. We start by writing

$$X_t^2=\left(C(L)\varepsilon_t\right)^2=X_{at}+2X_{bt} \tag{21}$$

with

$$X_{at}=\sum_0^\infty c_j^2\varepsilon_{t-j}^2=f_0(L)\varepsilon_t^2,$$

$$X_{bt}=\sum_{r=1}^\infty\sum_{j=0}^\infty c_jc_{j+r}\varepsilon_{t-j}\varepsilon_{t-j-r}=\sum_1^\infty f_r(L)\varepsilon_t\varepsilon_{t-r},$$

$$f_j(L)=\sum_{k=0}^\infty c_kc_{k+j}L^k=\sum_{k=0}^\infty f_{jk}L^k. \tag{22}$$

Next, employ the BN decomposition to the lag polynomial $f_j(L)$ giving

$$f_j(L)=f_j(1)-(1-L)\tilde{f}_j(L) \tag{23}$$

with

$$\tilde{f}_j(L)=\sum_{k=0}^\infty\tilde{f}_{jk}L^k,\qquad\tilde{f}_{jk}=\sum_{s=k+1}^\infty f_{js}=\sum_{s=k+1}^\infty c_sc_{s+j}.$$

The validity of (23) follows from:

3.6 LEMMA.

(a) $$\sum_{k=0}^{\infty} \tilde{f}_{jk}^2 = \sum_{k=0}^{\infty} \left(\sum_{s=k+1}^{\infty} c_s c_{s+j} \right)^2 < \infty$$

and

(b) $$\sum_{j=0}^{\infty} \left(\sum_{s=0}^{\infty} c_s c_{s+j} \right)^2 < \infty$$

if

($\mathscr{S}_4$) $$\sum_1^{\infty} s^{1/2} c_s^2 < \infty.$$

We use the decomposition (22) on both components of (21), namely,

$$X_{at} = f_0(1)\varepsilon_t^2 - (1-L)\tilde{X}_{at}, \tag{24}$$

$$X_{bt} = \varepsilon_t \varepsilon_{t-1}^f - (1-L)\tilde{X}_{bt}, \tag{25}$$

where

$$\tilde{X}_{at} = \tilde{f}_0(L)\varepsilon_t^2; \qquad \tilde{f}_0(L) = \sum_0^{\infty} \tilde{f}_{0k} L^k, \qquad \tilde{f}_{0k} = \sum_{k+1}^{\infty} f_{0s} = \sum_{k+1}^{\infty} c_s^2,$$

$$\varepsilon_{t-1}^f = \sum_1^{\infty} f_j(1)\varepsilon_{t-j} = \sum_1^{\infty} \bar{\gamma}_j \varepsilon_{t-j}, \tag{26}$$

$$\tilde{X}_{bt} = \sum_1^{\infty} \tilde{f}_j(L)\varepsilon_t \varepsilon_{t-j}.$$

Observe that $\bar{\gamma}_j = f_j(1) = \Sigma_0^{\infty} c_s c_{s+j}$ and the autocovariance function of X_t is

$$\gamma_j = E(X_0 X_j) = \sigma_\varepsilon^2 \bar{\gamma}_j.$$

Finally, under ($\mathscr{S}_4$) by Lemma 3.6(b) we have:

$$\sigma_f^2 = E\left(\varepsilon_{t-1}^f\right)^2 = \sigma_\varepsilon^2 \sum_1^{\infty} \bar{\gamma}_j^2 < \infty. \tag{27}$$

As we did for the sample mean, the approach is now to develop a SLLN and a CLT for partial sums of X_t^2 by summing in (21), (24) and (25), using results for the innovations ε_t and disposing of the terms that involve $\tilde{X}_a$ and $\tilde{X}_b$. We obtain:

3.7 THEOREM (SLLN for variances). *Under ($\mathscr{A}_2$) and ($\mathscr{S}_5$),*

$$\frac{1}{n} \sum_1^n X_t^2 \to_{a.s.} \gamma_0 = E(X_0^2) = \sigma_\varepsilon^2 \sum_0^{\infty} c_s^2,$$

where

$$(\mathscr{S}_5) \qquad \sum_1^{\infty} s c_s^2 < \infty.$$

3.8 THEOREM (CLT and IP for variances). *Under* $(\mathscr{A}_3)$ *and* $(\mathscr{S}_5)$: (a) $n^{-1/2}\Sigma_1^n(X_t^2 - \gamma_0) \to_d N(0, v(0))$, *where* $v(0) = (2\sigma_\varepsilon^4 + \kappa_4) f_0(1)^2 + 4\sigma_\varepsilon^2\sigma_f^2 = \kappa_4\bar{\gamma}_0^2 + 2\sigma_\varepsilon^4\Sigma_{-\infty}^{\infty}\bar{\gamma}_j^2$. (b) $n^{-1/2}\Sigma_1^{[nr]}(X_t^2 - \gamma_0) \to_d v(0)^{1/2}W(r)$.

3.9 REMARKS. (i) Sample covariances may be treated in the same way as variances by using a second order BN decomposition. We write, treating c_j as zero for all $j < 0$,

$$\begin{aligned}
X_t X_{t+h} &= C(L)\varepsilon_t C(L)\varepsilon_{t+h} \\
&= \sum_0^{\infty} c_j c_{j+h}\varepsilon_{t-j}^2 + \sum_{j=0}^{\infty} \sum_{r=-h-j,\, \neq 0}^{\infty} c_j c_{h+j+r}\varepsilon_{t-j}\varepsilon_{t-j-r} \\
&= f_h(L)\varepsilon_t^2 + \sum_{r=-\infty,\, \neq 0}^{\infty} \sum_{j=-h-r}^{\infty} c_j c_{h+j+r}\varepsilon_{t-j}\varepsilon_{t-j-r} \\
&= f_h(L)\varepsilon_t^2 + \sum_{r=-\infty,\, \neq 0}^{\infty} f_{h+r}(L)\varepsilon_{t-r}\varepsilon_t \qquad (28)\\
&= f_h(L)\varepsilon_t^2 + \sum_{r=1}^{\infty} \left[f_{h+r}(L)\varepsilon_{t-r}\varepsilon_t + f_{h-r}(L)\varepsilon_{t+r}\varepsilon_t \right] \\
&= f_h(1)\varepsilon_t^2 + \sum_{r=1}^{\infty} \left[f_{h+r}(1)\varepsilon_{t-r}\varepsilon_t + f_{h-r}(1)\varepsilon_{t+r}\varepsilon_t \right] - (1-L)\tilde{f}_h(L)\varepsilon_t^2 \\
&\quad -(1-L)\sum_{r=1}^{\infty} \left[\tilde{f}_{h+r}(L)\varepsilon_{t-r}\varepsilon_t + \tilde{f}_{h-r}(L)\varepsilon_{t+r}\varepsilon_t \right].
\end{aligned}$$

Without detailing all the remainder algebra we now get, as in Theorem 3.7,

$$\frac{1}{n}\sum_1^n X_t X_{t+h} \to_{\text{a.s.}} f_h(1)\sigma_\varepsilon^2 = \gamma_h;$$

and, as in Theorem 3.8, we have

$$\begin{aligned}
\frac{1}{n^{1/2}}\sum_1^n (X_t X_{t+h} - \gamma_h) &\sim f_h(1)\left[\frac{1}{n^{1/2}}\sum_1^n (\varepsilon_t^2 - \sigma_\varepsilon^2)\right] \\
&\quad + \sum_{r=1}^{\infty} \left[f_{h+r}(1) + f_{h-r}(1) \right]\left[\frac{1}{n^{1/2}}\sum_1^n \varepsilon_t\varepsilon_{t-r}\right] \\
&\to_d N(0, v(h)),
\end{aligned}$$

with

$$v(h) = f_h(1)^2(2\sigma_\varepsilon^4 + \kappa_4) + \sum_{r=1}^{\infty} (f_{h+r}(1) + f_{h-r}(1))^2 \sigma_\varepsilon^4$$
$$= \kappa_4 \bar{\gamma}_h^2 + \sigma_\varepsilon^4 \sum_{r=-\infty}^{\infty} \left[\bar{\gamma}_{h+r}^2 + \bar{\gamma}_{h+r}\bar{\gamma}_{h-r}\right].$$

(ii) Results for sample correlations also follow easily. Set

$$r_h = \frac{\sum_1^n X_t X_{t+h}}{\sum_1^n X_t^2}, \qquad \rho_h = \frac{\gamma_h}{\gamma_0} = \frac{f_h(1)}{f_0(1)}.$$

Then, using (28), we obtain

$$(29)\qquad \begin{aligned} &n^{1/2}(r_h - \rho_h) \\ &\sim \left(\frac{1}{n}\sum_1^n X_t^2\right)^{-1} \left\{ \sum_{r=1}^{\infty} \left[f_{h+r}(1) + f_{h-r}(1) - \rho_h(f_r(1) + f_{-r}(1))\right] \times \frac{1}{n^{1/2}} \sum_1^n \varepsilon_t \varepsilon_{t-r} \right\} \\ &\to_d N(0, w(h)) \end{aligned}$$

with

$$w(h) = \sum_{r=1}^{\infty} (\rho_{h+r} + \rho_{h-r} - 2\rho_h \rho_r)^2.$$

The result for the limit distribution of serial correlations holds as in Theorem 3.8. But, in view of (29), we need only ($\mathscr{A}_2$) rather than ($\mathscr{A}_3$), thereby corresponding to the original result of Anderson and Walker (1964)—see Anderson [(1971), page 489] and Hall and Heyde (1980), page 188.

The results in this section are not, in general, the best possible. But the approach has the advantage that the results come very easily, it involves just algebraic calculation on top of i.i.d. limit theory, and the role of the summability conditions on the coefficients of the linear process is easily understood. For these reasons the approach seems to be quite useful for pedagogical purposes.

The price we pay for the convenience of the explicit use of the BN decomposition lies in the summability conditions that are employed in its justification. To obtain improved results we weaken these conditions and use the BN decomposition only indirectly, as we now demonstrate.

B. *BN in indirect mode and homogeneous innovations.* The idea behind the indirect approach is to use the BN decomposition to suggest an appropriate approximation and then to analyze the error of approximation rather than work directly with the remainder terms in the BN construction. Thus, in the case of the linear process (13), the BN decomposition gives $C(1)\varepsilon_t$ as an approximation to X_t. Instead of working with $\tilde{\varepsilon}_t$ as in the explicit construction (15), we now consider the remainder $X_t - C(1)\varepsilon_t$. For the CLT we have

$$\frac{w_n}{n^{1/2}} = \frac{1}{n^{1/2}} \sum_1^n [X_t - C(1)\varepsilon_t]$$

and the CLT follows if we can show $n^{-1}E(w_n^2) \to 0$. Similarly, the SLLN follows if we can show $n^{-1}w_n \to_{\text{a.s.}} 0$. This approach has been used before and is evident, for example, in Hannan [(1970), pages 246–248] and in Hall and Heyde [(1980), Theorem 5.3, pages 133–134] in the proof of time series CLT's. The following results make systematic use of the method.

3.10 THEOREM (SLLN). *Under* ($\mathscr{A}_2$) *and* ($\mathscr{S}_6$), $n^{-1}\Sigma_1^n X_t \to_{a.s.} 0$, *where*

$$(\mathscr{S}_6) \qquad \sum_1^\infty \ln s|c_s| < \infty.$$

3.11 THEOREM (CLT). *Under* ($\mathscr{A}_2$) *and* ($\mathscr{S}_7$), $n^{-1/2}\Sigma_1^n X_t \to_d N(0, \sigma_\varepsilon^2 C(1)^2)$, *where*

$$(\mathscr{S}_7) \qquad 0 < |C(1)| < \infty.$$

REMARKS. (i) A result that is very similar to Theorem 3.10 is obtained by using McLeish's (1975a) mixingale convergence theorem [Hall and Heyde (1980), pages 22 and 41]. Let $\mathscr{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \dots)$ be the natural filtration for (ε_t) and set

$$\psi_m = \| E(X_t|\mathscr{F}_{t-m})\|_2 = \left\{E\left[E(X_t|\mathscr{F}_{t-m})\right]^2\right\}^{1/2} = \left(\sum_m^\infty c_s^2\right)^{1/2} \sigma_\varepsilon.$$

Then, McLeish's SLLN [Corollary 1.9 and Example 1 of McLeish (1975a)] requires ψ_m to be of size $-1/2$, that is, $\psi_m = O(m^{-1/2}L(m)^{-1})$, where $L(m)$ is slowly varying at ∞ and satisfies the summability requirement $\Sigma_1^\infty m^{-1}L(m)^{-1} < \infty$. This leads to $\psi_m^2 = O(m^{-1}L(m)^{-2})$ and since $\Sigma_1^\infty m^{-1}L(m)^{-2} < \infty$, we deduce the implied summability condition

$$\sum_{m=1}^\infty \sum_{s=m}^\infty c_s^2 = \sum_0^\infty s c_s^2 < \infty.$$

This is our ($\mathscr{S}_5$) and is weaker than ($\mathscr{S}_6$), but only by a slowly varying factor. For example, $c_s = s^{-1}[\ln(1+s)]^{-1}$ satisfies ($\mathscr{S}_5$) but fails ($\mathscr{S}_6$). Thus McLeish's mixingale approach leads to a stronger result but involves more work and sophistication.

(ii) Theorem 3.11 offers a new proof of the minimal result given in Hall and Heyde [1980, Corollary 5.2, page 135]—see also Hannan [(1970), Theorem 11, page 221].

C. *BN in direct mode and heterogeneous innovations.* The explicit form of (15) makes it just as easy to work with heterogeneous as homogeneous innovations in the decomposition. Again the simplicity of the direct mode approach is that we can appeal immediately to established theory for i.n.i.d. and m.d.s. sequences and need only attend to the remainder terms to produce a rigorous theory for linear processes under assumptions like ($\mathscr{B}_1$) or ($\mathscr{B}_2$) on the innovations. The following are a selection of first and second moment results that are easy to obtain. The proofs are just like those for the homogeneous case.

3.13 THEOREM (SLLN). *Under* ($\mathscr{B}_2$) *and* ($\mathscr{S}_1$), $n^{-1}\Sigma_1^n X_t \to_{a.s.} 0$.

3.14 THEOREM (SLLN). *Under* ($\mathscr{B}_1$) *and* ($\mathscr{S}_3$), $n^{-1}\Sigma_1^n X_t \to_{a.s.} 0$.

3.15 THEOREM (CLT and IP). *Under* ($\mathscr{B}_2$) *and* ($\mathscr{S}_1$),

(a) $$\frac{1}{n^{1/2}}\sum_1^n X_t \to_d N\big(0, \sigma_\varepsilon^2 C(1)^2\big).$$

Under ($\mathscr{S}_3$) *and with* ($\mathscr{B}_2$) *strengthened so that* $E(Z^{2+\eta}) < \infty$ *for some* $\eta > 0$,

(b) $$\frac{1}{n^{1/2}}\sum_1^{[nr]} X_t \to_d \sigma_\varepsilon C(1)W(r).$$

3.16 THEOREM (SLLN for variances). *Under* ($\mathscr{S}_5$) *and with* ($\mathscr{B}_2$) *strengthened so that* $E(Z^4) < \infty$, *we have* $n^{-1}\Sigma_1^n X_t^2 \to_{a.s.} \gamma_0$.

3.17 REMARKS. (i) Theorem 3.13 is related to a result of Hannan and Heyde (1972) [see also Hall and Heyde (1980), page 184] who require only

($\mathscr{S}_8$) $$\sum_0^\infty |c_s| < \infty$$

in place of ($\mathscr{S}_1$) and only $E(Z^2)$ in ($\mathscr{B}_2$).

(ii) Theorem 3.14 gives us an extension of the Markov SLLN to linear processes. The theorem continues to hold, by virtue of theorem LLN, if we replace ($\mathscr{B}_1$) with:

($\mathscr{B}_1'$) (ε_t) is an independent sequence, is s.u.i. with dominating r.v. Z and $E|Z| < \infty$.

Thus, all that is needed to extend theorem LLN from independent sequences to linear processes is ($\mathscr{S}_3$). This result and Theorem 3.14 would appear to be new.

(iii) Theorem 3.16 is also related to Hannan and Heyde (1972) [see also Hall and Heyde (1980), page 184]. Again, they require only $E(Z^2) < \infty$ in our ($\mathscr{B}_2$) and only ($\mathscr{S}_8$) in place of our ($\mathscr{S}_1$); but they show convergence in probability not a.s. convergence in this case.

(iv) Hannan and Heyde (1972) also extend the Anderson and Walker (1964) limit theory for autocorrelations to the heterogeneous case. Their Theorem 2 may be obtained using our approach.

D. *BN in direct mode and stable limit laws.* Stable limit laws for the linear process X_t given by (13) can be deduced in much the same way as the classical SLLN and CLT asymptotics. We rely again on the BN decomposition that leads to (15). First observe that, if ε_t satisfies condition ($\mathscr{C}_1$), $X_t = \sum_0^\infty c_j \varepsilon_{t-j}$ is convergent a.s. provided

$$(\mathscr{S}_9) \qquad \sum_0^\infty |c_j|^p < \infty$$

for $0 < p < \alpha$ and $p \leq 1$ [e.g., Brockwell and Davies (1987), page 480]. Similarly, $\tilde{\varepsilon}_t = \sum_0^\infty \tilde{c}_j \varepsilon_{t-j}$ in (15) is convergent a.s. provided $\sum_0^\infty |\tilde{c}_j|^p < \infty$ and this holds by the BN lemma if

$$(\mathscr{S}_{10}) \qquad \sum_1^\infty j|c_j|^p < \infty \quad \text{for } 0 < p < \alpha \text{ and } p \leq 1.$$

With the validity of (15), in hand, it is a simple matter to deduce asymptotics for standardized sums and cross products of X_t. We give the following two useful results.

3.18 THEOREM (Stable limit for means). *Under* ($\mathscr{C}_1$) *and* ($\mathscr{S}_{10}$),

$$\frac{1}{a_n} \sum_1^n X_t \to_d C(1) U_\alpha(1).$$

3.19 THEOREM (Stable limit for covariances). *Under* ($\mathscr{C}_1$) *and* ($\mathscr{S}_{10}$),

$$\text{(a)} \qquad \frac{1}{a_n^2} \sum_1^n \left[X_t^2, X_t X_{t+1}, \ldots, X_t X_{t+h} \right] \to_d \left[f_0(1), f_1(1), \ldots, f_h(1) \right] \int_0^1 (dU_\alpha)^2,$$

$$\text{(b)} \qquad r_h \to_p \rho_h = f_h(1)/f_0(1).$$

3.20 REMARKS. (i) Theorem 3.18 gives a result that seems first to have been established by Davis and Resnick [(1985a), Theorem 4.1, page 189]. Their proof uses truncation arguments and point process theory and is more involved than ours; but they need only ($\mathscr{S}_9$) in place of our ($\mathscr{S}_{10}$). As in Section 3(a), the explicit construction (15) leads to a substantial simplification but is achieved at the cost of somewhat stronger conditions on the coefficients of $C(L)$.

(ii) Theorem 3.19 also gives results that appear in Davis and Resnick [(1985a), Theorem 4.2, page 192]. Again, they require ($\mathscr{S}_9$) rather than ($\mathscr{S}_{10}$) and they obtain the asymptotic distribution of r_h, so our results are therefore mainly of pedagogical interest.

(iii) Interestingly, Theorem 3.18 does not extend directly to a functional version, as it does in the case of finite variance innovations (cf. Theorem 3.3). This has been discovered by Avram and Taqqu (1986, 1989). In the present context, we can explain the failure in terms of the BN decomposition. We have, as before,

$$\frac{1}{a_n}\sum_1^{[nr]} X_t = C(1)\frac{1}{a_n}\sum_1^{[nr]}\varepsilon_t + \frac{1}{a_n}\left(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_{[nr]}\right)$$

and, according to (11), $a_n^{-1}\Sigma_1^{[nr]} \to_d U_\alpha(r)$ in the space $D[0,1]$ with the Skorohod topology. However, the remainder term does not vanish in probability in general. For instance, the distance between $a_n^{-1}\tilde{\varepsilon}_{[nr]}$ and the zero function in the Skorohod J_1 metric is simply

$$\sup_r \left|a_n^{-1}\tilde{\varepsilon}_{[nr]}\right| = \max_{0\le k\le n} a_n^{-1}|\tilde{\varepsilon}_k|. \tag{30}$$

But, under $(\mathscr{S}_9)$, $\tilde{\varepsilon}_k \in \mathscr{D}(\alpha)$ and thus, when $\alpha < 2$, (30) does not converge in probability to zero [Breiman (1965), Theorem 2, page 323]. So the functional law $a_n^{-1}\Sigma_1^{[nr]}X_t \to_d C(1)U_\alpha(r)$ does not apply in $D[0,1]$ endowed with the usual Skorohod topology, even though all the finite-dimensional distributions converge.

4. Supplementary remarks. (i) The BN Lemma 2.1 continues to hold for matrix polynomials using conventional matrix norms in the summability conditions. Thus, the decomposition (15) also applies to vector linear processes. The limit theory of Sections 3A–C can then be extended to the multivariate case.

(ii) The decomposition (15) is important in the vector case to the theory of cointegration—see Engle and Granger (1987). Suppose $(1-L)Y_t = X_t = C(L)\varepsilon_t = \Sigma_{j=0}^{\infty}C_jL^j\varepsilon_t$, where ε_t is an m.d.s. and $\Sigma_0^\infty j^2\|C_j\|^2 < \infty$ with $\|C_j\| = [\mathrm{tr}(C_j'C_j)]^{1/2}$. Then Y_t is stationary and if $C(1) \neq 0$, at least some components of Y_t are integrated processes. The components of Y_t are cointegrated if $C(1)$ is a singular matrix. Using (15) we have

$$(1-L)Y_t = C(1)\varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t$$

and by summation and with $Y_0 = 0$ we get

$$Y_t = C(1)S_t + \tilde{\varepsilon}_0 - \tilde{\varepsilon}_t, \qquad S_t = \sum_1^t \varepsilon_s.$$

Thus, if α is a vector for which $\alpha'C(1) = 0$, then α' annihilates the integrated element $C(1)S_t$ of Y_t (i.e., the martingale approximation to Y_t) and we have

$$\alpha'Y_t = \alpha'\left(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t\right) \quad \text{a.s.}, \tag{31}$$

which is a stationary time series under the stated summability condition. In econometric models, the equation (31) is interpreted as describing stationary deviations about a long-run equilibrium relation $\alpha'Y_t = 0$. Phillips (1991), Johansen (1988) and Phillips and Loretan (1991) provide further discussion and develop optimal inference procedures.

(iii) The BN lemma also has a version that is suitable for frequency domain applications. In effect, we can expand the polynomial $C(L)$ in a Taylor series about an arbitrary point on the unit circle, say $e^{i\lambda}$ rather than unity, giving

$$C(L) = C(e^{i\lambda}) - (1 - e^{-i\lambda}L)\tilde{C}_\lambda(L) \tag{32}$$

with

$$\tilde{C}_\lambda(L) = \sum_0^\infty \tilde{c}_{s\lambda} L^s, \qquad \tilde{c}_{s\lambda} = e^{-i\lambda s} \sum_{s+1}^\infty c_k e^{i\lambda k}.$$

This leads to the following decomposition for the discrete Fourier transform (d.f.t.), $(2\pi n)^{-1/2}\sum_1^n X_t e^{it\lambda}$, of X_t in terms of the d.f.t. of ε_t and a residual:

$$w_X(\lambda) = C(e^{i\lambda})w_\varepsilon(\lambda) + (2\pi n)^{-1/2}\left[\tilde{X}_{0\lambda} - e^{i\lambda n}\tilde{X}_{n\lambda}\right]$$

with

$$\tilde{X}_{t\lambda} = \tilde{C}_\lambda(L)\varepsilon_t = \sum_0^\infty \tilde{c}_{s\lambda}\varepsilon_{t-s}.$$

This decomposition leads in much the same way as Theorem 3.4 to a CLT for $w_X(\lambda)$ and joint CLT's for the d.f.t's at many frequencies. In the above form, the decomposition (32) is used by Hannan and Deistler [(1988), page 156] and is apparently due to Bouaziz. It is justified, as in Lemma BN, by conditions such as $(\mathscr{S}_1)$. This frequency domain BN approach may also be applied in what we have termed the indirect mode in Section 3B. As such, the idea appears in Hannan [(1970), Theorem 1, page 248].

5. Proofs. We start with some useful bounds provided by the following lemma.

5.1 LEMMA.

(a) $$\sum_{t+1}^\infty u^{-1-b} \le b^{-1}t^{-b}, \qquad b > 0.$$

(b) $$\sum_1^t u^{c-1} \le c^{-1}t^c, \qquad 0 < c < 1.$$

(c) $$\sum_1^t u^{-1} \le 1 + \ln t.$$

PROOF. If $0 < u \le s$, then $s^{-1} \le u^{-1}$ and

$$s^{-1} \le \int_{s-1}^s u^{-1}\,du = \ln s - \ln(s-1)$$

so that $\sum_{s=2}^t s^{-1} \le \ln t$ and (c) follows. Results (b) and (a) follow by similar arguments. □

5.2 PROOF OF LEMMA 2.1. The case $p = 1$ is obvious so we take $p > 1$. Then for a suitably chosen constant a and by Hölder's inequality we have

$$\begin{aligned}
\sum_1^\infty \left|\sum_{j+1}^\infty c_k\right|^p &\le \sum_1^\infty \left(\sum_{j+1}^\infty k^a |c_k| k^{-a}\right)^p \\
&\le \sum_1^\infty \left(\sum_{j+1}^\infty k^{ap}|c_k|^p\right)\left(\sum_{j+1}^\infty k^{-aq}\right)^{p/q} \qquad (1/p + 1/q = 1) \\
&\le \frac{1}{aq-1}\sum_1^\infty \left(\sum_{j+1}^\infty k^{ap}|c_k|^p\right)(j^{1-aq})^{p/q} \\
&\qquad [\text{using 5.1 (a) with } 1/q < a < 1] \\
&= \frac{1}{aq-1}\sum_1^\infty k^{ap}|c_k|^p\left(\sum_1^{k-1} j^{p/q-ap}\right) \\
&\le \frac{1}{(aq-1)(1+p/q-ap)}\sum_1^\infty k^{ap}|c_k|^p k^{1+p/q-ap} \\
&\qquad [\text{using 5.1 (b) with } a < 1/p + 1/q] \\
&\le \frac{1}{(aq-1)(1+p/q-ap)}\sum_1^\infty k^p |c_k|^p
\end{aligned}$$

for $1/q < a < 1/p + 1/q = 1$. To prove (3) we note that for $p < 1$,

$$\sum_0^\infty |\tilde{c}_j|^p = \sum_0^\infty \left|\sum_{j+1}^\infty c_k\right|^p \le \sum_0^\infty \sum_{j+1}^\infty |c_k|^p = \sum_1^\infty k|c_k|^p.$$

Finally,

$$|C(1)| \le |c_0| + \left|\sum_1^\infty \frac{c_j j}{j}\right| \le |c_0| + \left(\sum_1^\infty (|c_j| j)^p\right)^{1/p}\left(\sum_1^\infty \frac{1}{j^q}\right)^{1/q} < \infty. \qquad \square$$

5.3 PROOF OF THEOREM 2.6. This follows from McLeish's (1974) theorem quoted in Hall and Heyde [(1980), page 58]. That theorem also requires that $E(\max_{1\le i\le n} Z_{ni}^2)$ is bounded uniformly in n. But in our context this can be dispensed with since it is bounded by $E(\sum_1^n Z_{ni}^2) = 1$. $\square$

5.4 PROOF OF THEOREM 2.7. This follows from Brown (1971) [see also Hall and Heyde (1980), page 99]. To show that (7) implies (8) we note that

$0 \le \Sigma_1^n Z_{ni}^2 1(|Z_{ni}| > \varepsilon) \le \Sigma_1^n Z_{ni}^2$. If each of the following are true:

$$\sum_1^n Z_{ni}^2 1(|Z_{ni}| > \varepsilon) \to_p 0 \quad \text{for any } \varepsilon > 0, \tag{33}$$

$$\sum_1^n Z_{ni}^2 \to_p 1, \tag{34}$$

$$E\left(\sum_1^n Z_{ni}^2\right) \to 1, \tag{35}$$

then, by a version of the dominated convergence theorem [cf. Hall and Heyde (1980), page 281], (8) immediately follows. But (34) is simply (6), while (35) is trivially true. Finally (33) is equivalent to (8) since [cf. Hall and Heyde (1980), page 53]

$$P\left(\max_{1 \le i \le n} |Z_{ni}| > \varepsilon\right) = P\left(\sum_1^n Z_{ni}^2 1(|Z_{ni}| > \varepsilon) > \varepsilon^2\right). \tag{36}$$

To show (8) implies (7) we have, in view of (36), only to show (33). But, of course, (8) implies (33). Thus, when (6) holds, conditions (7) and (8) are equivalent. □

5.5 PROOF OF THEOREM 3.2. From (16),

$$\frac{1}{n}\sum_1^n X_t = C(1)\frac{1}{n}\sum_1^n \varepsilon_t + \frac{1}{n}\tilde{\varepsilon}_0 - \frac{1}{n}\tilde{\varepsilon}_n. \tag{37}$$

By Theorem LLN the first term $\to_{\text{a.s.}} 0$. Because $|\tilde{\varepsilon}_0| < \infty$ a.s., the second term $\to_{\text{a.s.}} 0$ also. The third term of (37) is

$$\frac{1}{n}\tilde{\varepsilon}_n = \frac{1}{n}\sum_0^\infty \tilde{c}_j \varepsilon_{n-j} = \frac{1}{n}\sum_{-\infty}^n \tilde{c}_{n-s}\varepsilon_s = \frac{1}{n}\sum_0^n \tilde{c}_{n-s}\varepsilon_s + \frac{1}{n}\sum_1^\infty \tilde{c}_{n+t}\varepsilon_{-t}.$$

Introduce $\bar{c}_t = \Sigma_{t+1}^\infty |c_s|$ and note that

$$\sum_1^\infty |\tilde{c}_t| \le \sum_1^\infty \bar{c}_t = \sum_1^\infty \sum_{t+1}^\infty |c_s| \le \sum_1^\infty s|c_s| < \infty$$

under $(\mathscr{S}_3)$. Then

$$\begin{aligned} \frac{1}{n}|\tilde{\varepsilon}_n| &\le \frac{1}{n}\sum_0^n |\tilde{c}_t||\varepsilon_{n-t}| + \frac{1}{n}\sum_1^\infty \bar{c}_{n+t}|\varepsilon_{-t}| \\ &\le \frac{1}{n}\left(\max_{0 \le t \le n} |\varepsilon_t|\right)\left(\sum_0^\infty |\tilde{c}_t|\right) + \frac{1}{n}\sum_1^\infty \bar{c}_t|\varepsilon_{-t}|. \end{aligned} \tag{38}$$

Now $\max_{0 \le t \le n}(n^{-1}|\varepsilon_t|) \to_{\text{a.s.}} 0$ if $n^{-1}|\varepsilon_n| \to_{\text{a.s.}} 0$, which holds if $n^{-1}\varepsilon_n \to_{\text{a.s.}} 0$, which holds since

$$\frac{1}{n}\varepsilon_n = \frac{1}{n}\sum_1^n \varepsilon_j - \frac{1-n^{-1}}{n-1}\sum_1^{n-1}\varepsilon_j \to_{\text{a.s.}} 0 - 0 = 0.$$

Next $\Sigma_1^\infty \bar{c}_t|\varepsilon_t| < \infty$ a.s. since its expectation is finite. Thus, both terms of (38) $\to_{\text{a.s.}}$ 0 and the result follows from (37). □

5.6 PROOF OF THEOREM 3.3. By stationarity we have

$$E|\tilde{\varepsilon}_n|^q = E|\tilde{\varepsilon}_0|^q = E\left|\sum_0^\infty \tilde{c}_j\varepsilon_{-j}\right|^q.$$

Let $u_N = \Sigma_0^N \tilde{c}_j\varepsilon_{-j}$ and require $2 < q \le p$. By Burkholder's inequality [Hall and Heyde (1980), page 23], there is a constant c_q for which

$$E|u_N|^q \le c_q E\left(\sum_0^N \tilde{c}_j^2\varepsilon_{-j}^2\right)^{q/2}.$$

Applying Minkowski's inequality to the right side we find

$$E|u_N|^q \le c_q\left(\sum_0^N \tilde{c}_j^2\right)^{q/2} E|\varepsilon_0|^q \le c_q\left(\sum_0^\infty \tilde{c}_j^2\right)^{q/2} E|\varepsilon_0|^q = d_q, \quad \text{say}.$$

Now choose q such that $2 < q < p$ and we have

$$E\left(|u_N|^q\right)^{p/q} = E|u_N|^p \le d_p.$$

Since $p/q > 1$ it follows that $\{|u_N|^q \colon N = 1, 2, \dots\}$ is uniformly integrable. But $u_N \to_p \tilde{\varepsilon}_0$ so that

$$E|u_N|^q \to E|\tilde{\varepsilon}_0|^q \le d_p < \infty.$$

We deduce that $\varphi_n^{-1}\tilde{\varepsilon}_0, \varphi_n^{-1}\tilde{\varepsilon}_n \to_{\text{a.s.}} 0$ and then by (17)′,

$$\limsup_{n\to\infty}\left[\frac{1}{\sigma_X\varphi_n}\left(\sum_1^n X_t\right)\right] = \limsup_{n\to\infty}\left[\frac{1}{\sigma_X\varphi_n}C(1)\left(\sum_1^n \varepsilon_t\right)\right].$$

By the Hartman–Wintner LIL for iid r.v.'s [Hall and Heyde (1980), page 116] we have

$$\limsup_{n\to\infty}, \qquad \liminf_{n\to\infty}\frac{1}{\sigma_\varepsilon\varphi_n}\left(\sum_1^n \varepsilon_t\right) = \pm 1 \quad \text{a.s.}$$

Noting that $\sigma_X^{-1}C(1)\sigma_\varepsilon = \text{sgn}\{C(1)\}$ the LIL for $\Sigma_1^n X_t$ follows directly. □

5.7 PROOF OF LEMMA 3.6.

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\sum_{s=k+1}^{\infty} c_s c_{s+j}\right)^2 &= \sum_{0}^{\infty}\left(\sum_{k+1}^{\infty} s^{1/4} c_s c_{s+j} s^{-1/4}\right)^2\\ &\le \sum_{0}^{\infty}\left(\sum_{k+1}^{\infty} s^{1/2} c_s^2\right)\left(\sum_{k+1}^{\infty} c_{s+j}^2 s^{-1/2}\right)\\ &\le \left(\sum_{1}^{\infty} s^{1/2} c_s^2\right)\left(\sum_{k=0}^{\infty}\sum_{s=k+1}^{\infty} s^{-1/2} c_{s+j}^2\right)\\ &\le \left(\sum_{1}^{\infty} s^{1/2} c_s^2\right)\left(\sum_{s=1}^{\infty} s^{-1/2} c_{s+j}^2 \sum_{k=0}^{s-1} 1\right)\\ &\le \left(\sum_{1}^{\infty} s^{1/2} c_s^2\right)\left(\sum_{1}^{\infty} (s+j)^{1/2} c_{s+j}^2\right) \le \left(\sum_{1}^{\infty} s^{1/2} c_s^2\right)^2 < \infty.\end{aligned}$$

The proof of (b) follows in the same way. □

5.8 PROOF OF THEOREM 3.7. From (21) we have to show

$$\frac{1}{n}\sum_{1}^{n} X_{at} \to_{\text{a.s.}} \gamma_0, \tag{39}$$

$$\frac{1}{n}\sum_{1}^{n} X_{bt} \to_{\text{a.s.}} 0. \tag{40}$$

We shall prove (40) first. From (25) this follows if

$$\frac{1}{n}\tilde{X}_{bn} \to_{\text{a.s.}} 0 \tag{41}$$

and

$$\frac{1}{n}\sum_{1}^{n} \varepsilon_t \varepsilon_{t-1}^f \to_{\text{a.s.}} 0. \tag{42}$$

By Kronecker's lemma, (42) holds if $T_n = \sum_1^n t^{-1}\varepsilon_t\varepsilon_{t-1}^f$ converges a.s. But T_n is a martingale so by the L_2 martingale convergence theorem [e.g., Hall and Heyde (1980), page 18], T_n converges a.s. if $\sup_n E(T_n^2) < \infty$, which holds because $\sup_n E(T_n^2) \le \sum_1^\infty(\sigma_\varepsilon^2\sigma_f^2 t^{-2}) < \infty$. Next (41) holds if $\sum_1^\infty E(\tilde{X}_{bn}^2)n^{-2} < \infty$, which holds if $E(\tilde{X}_{bn}^2) < \infty$, which holds if $(\mathscr{S}_5)$ holds, as shown in Lemma 5.9 below.

To prove (39) we note from (24) that this holds if the following are true:

$$\frac{1}{n}\tilde{X}_{an} \to_{\text{a.s.}} 0, \tag{43}$$

$$\frac{1}{n}\sum_{1}^{n} \varepsilon_t^2 \to_{\text{a.s.}} \sigma_\varepsilon^2. \tag{44}$$

But (44) follows from Theorem LLN under $(\mathscr{A}_2)$ so we need only to prove (43). As in (38) above we have

$$\frac{1}{n}\tilde{X}_{an} = \frac{1}{n}\sum_0^\infty \tilde{f}_{0k}\varepsilon_{n-k}^2 \le \frac{1}{n}\Big(\max_{0\le t\le n}\varepsilon_t^2\Big)\Big(\sum_0^\infty \tilde{f}_{0k}\Big) + \frac{1}{n}\sum_1^\infty \tilde{f}_{1k}\varepsilon_{-k}^2. \tag{45}$$

Now $n^{-1}\max_{0\le t\le n}\varepsilon_t^2 \to_{\text{a.s.}} 0$ if $n^{-1}\varepsilon_n^2 \to_{\text{a.s.}} 0$ which holds because

$$\frac{1}{n}\varepsilon_n^2 = \frac{1}{n}\sum_1^n \varepsilon_j^2 - \frac{1-n^{-1}}{n-1}\sum_1^{n-1}\varepsilon_j^2 \to_{\text{a.s.}} 0.$$

Also $\sum_0^\infty \tilde{f}_{0k} = \sum_0^\infty \sum_{k+1}^\infty c_s^2 \le \sum_0^\infty s c_s^2 < \infty$ so that the first term of (45) converges a.s. to zero. Moreover, this ensures that $E(\sum_1^\infty \tilde{f}_{0k}\varepsilon_{-k}^2) < \infty$ so that $\sum_1^\infty \tilde{f}_{0k}\varepsilon_{-k}^2 < \infty$ a.s. and the second term of (45) converges a.s. to zero. □

5.9 LEMMA. *Under* $(\mathscr{A}_3)$ *and* $(\mathscr{S}_5)$, $E(\tilde{X}_{bn}^2) < \infty$.

PROOF.

$$\begin{aligned} E(\tilde{X}_{bn}^2) &= \sum_{j,j'=1}^\infty \sum_{k,k'=0}^\infty \tilde{f}_{jk}\tilde{f}_{j'k'}E(\varepsilon_{n-k}\varepsilon_{n-j-k}\varepsilon_{n-k'}\varepsilon_{n-j'-k'}) \\ &= \sum_{j,j'}\sum_{k,k'} \tilde{f}_{jk}\tilde{f}_{j'k'}\big(\delta_{jj'}\delta_{kk'}\sigma_\varepsilon^4 + \delta_{k,k'+j'}\delta_{k',k+j}\sigma_\varepsilon^4\big) \\ &= \sum_{j=1}^\infty\sum_{k=0}^\infty \tilde{f}_{jk}^2\sigma_\varepsilon^4 + \sum_{k=0}^\infty\sum_{k'=0}^\infty \tilde{f}_{k'-k,k}\tilde{f}_{k-k',k'}\sigma_\varepsilon^4 - \sum_{k=0}^\infty \tilde{f}_{0k}^2\sigma_\varepsilon^4. \end{aligned} \tag{46}$$

But the first term of (46) is bounded since

$$\begin{aligned} \sum_{j=1}^\infty\sum_{k=0}^\infty \tilde{f}_{jk}^2 &= \sum_{j=1}^\infty\sum_{k=0}^\infty\Big(\sum_{s=k+1}^\infty c_s c_{s+j}\Big)^2 \\ &\le \sum_{j=1}^\infty\sum_{k=0}^\infty\Big(\sum_{s=k+1}^\infty c_s^2\Big)\Big(\sum_{s=k+1}^\infty c_{s+j}^2\Big) \\ &\le \Big(\sum_{k=0}^\infty\sum_{s=k+1}^\infty c_s^2\Big)\Big(\sum_{j=1}^\infty\sum_{s=1}^\infty c_{s+j}^2\Big) \\ &= \Big(\sum_{s=1}^\infty c_s^2\sum_{k=1}^{s-1}1\Big)^2 = \Big(\sum_1^\infty s c_s^2\Big)^2 < \infty. \end{aligned} \tag{47}$$

Next, using the Cauchy inequality for double sums, we get for the second term

$$\begin{aligned} \Big(\sum_{k,k'}\tilde{f}_{k'-k,k}\tilde{f}_{k-k',k'}\Big)^2 &\le \Big(\sum_{k,k'}\tilde{f}_{k'-k,k}^2\Big)^2 \\ &= \Big(\sum_{k'=0}^\infty\sum_{k=0}^{k'}\tilde{f}_{k'-k,k}^2\Big)^2 \le \Big(\sum_{j=1}^\infty\sum_{k=0}^\infty \tilde{f}_{jk}^2 + \tilde{f}_{00}^2\Big)^2 < \infty, \end{aligned}$$

since (47) is finite. Finally

$$\sum_{k=0}^{\infty} \tilde{f}_{0k}^2 = \sum_{k=0}^{\infty}\left(\sum_{s=k+1}^{\infty} c_s^2\right)^2 \leq \left(\sum_1^{\infty} c_s^2\right)\left(\sum_{k=0}^{\infty}\sum_{s=k+1}^{\infty} c_s^2\right)$$

$$\leq \left(\sum_1^{\infty} c_s^2\right)\left(\sum_{s=1}^{\infty} c_s^2 \sum_{k=0}^{s-1} 1\right) = \left(\sum_1^{\infty} c_s^2\right)\left(\sum_1^{\infty} s c_s^2\right) < \infty.$$

Thus all terms of (46) are finite and the required result follows. □

5.10 Proof of Theorem 3.8. From (21), (24) and (25) we have the decomposition

$$(48)\qquad \begin{aligned}\frac{1}{n^{1/2}}\sum_1^n (X_t^2 - \gamma_0) &= \frac{1}{n^{1/2}}\sum_1^n \left[\bar{\gamma}_0(\varepsilon_t^2 - \sigma_\varepsilon^2) + 2\varepsilon_t\varepsilon_{t-1}^f\right]\\ &\quad + \frac{1}{n^{1/2}}\left(\tilde{X}_{a0} - \tilde{X}_{an}\right) + \frac{2}{n^{1/2}}\left(\tilde{X}_{b0} - \tilde{X}_{bn}\right).\end{aligned}$$

From Lemma 5.9, $E(\tilde{X}_{bn}^2) < \infty$ and the final term of (48) converges in probability to zero. Further

$$E\left(\tilde{X}_{a0}\right), \qquad E\left(\tilde{X}_{an}\right) = \sigma_\varepsilon^2 \sum_0^{\infty} \tilde{f}_{0k} = \sigma_\varepsilon^2 \sum_{k=0}^{\infty}\sum_{k+1}^{\infty} c_s^2 = \sigma_\varepsilon^2 \sum_{s=1}^{\infty} s c_s^2 < \infty$$

so that the second term on the right side of (48) also converges in probability to zero. It remains to show that the first term of (48) converges weakly to $N(0,v(0))$. We apply theorem CLT and need only verify

$$(49)\qquad \frac{1}{n}\sum_1^n Z_t^2 \to_p v(0),$$

$$(50)\qquad \frac{1}{n}\sum_1^n E\left[Z_t^2 1(Z_t^2 > \varepsilon n)\right] \to 0$$

with

$$Z_t = \bar{\gamma}_0\left(\varepsilon_t^2 - \sigma_\varepsilon^2\right) + 2\varepsilon_t\varepsilon_{t-1}^f.$$

Now (50) is just $E[Z_0^2 1(Z_0^2 > \varepsilon n)] \to 0$, which follows by dominated convergence since $E(Z_0^2) = v(0) < \infty$, by ($\mathscr{A}_3$). Next we apply theorem LLN to (49) giving

$$\frac{1}{n}\sum_1^n \left[Z_t^2 - E\left(Z_t^2 | \mathscr{F}_{t-1}\right)\right] \to_p 0.$$

So we need only show that

$$\frac{1}{n}\sum_1^n E\left(Z_t^2 | \mathscr{F}_{t-1}\right) \to_p v(0)$$

which follows directly if

$$\frac{1}{n}\sum_{1}^{n}\left(\varepsilon_{t-1}^{f}\right)^{2} \to_{p} \sigma_{f}^{2}. \tag{51}$$

This holds by the pointwise ergodic theorem since $\sigma_f^2 < \infty$ but we can also establish it by appealing to our own SLLN, Theorem 3.7—see Lemma 5.11 below.

To prove the IP [part (b)] we again employ the decomposition

$$\frac{1}{n^{1/2}}\sum_{1}^{[nr]}\left(X_t^2 - \gamma_0\right)$$
$$= \frac{1}{n^{1/2}}\sum_{1}^{[nr]} Z_t + \frac{1}{n^{1/2}}\left(\tilde{X}_{a0} - \tilde{X}_{a[nr]}\right) + \frac{2}{n^{1/2}}\left(\tilde{X}_{b0} - \tilde{X}_{b[nr]}\right).$$

Since (50) and (51) hold, we have $n^{-1/2}\sum_1^{[nr]}Z_t \to_d v(0)^{1/2}W(r)$ directly from Theorem IP. The IP for $X_t^2 - \gamma_0$ then follows provided

$$\max_{1\le k\le n}\left(n^{-1}\tilde{X}_{ak}^2\right) \to_p 0, \qquad \max_{1\le k\le n}\left(n^{-1}\tilde{X}_{bk}^2\right) \to_p 0.$$

But these are equivalent [cf. (20)] to

$$J_{an} = \frac{1}{n}\sum_{1}^{n}\left[\tilde{X}_{ak}^2 1\left(\tilde{X}_{ak}^2 > n\delta\right)\right] \to_p 0,$$

$$J_{bn} = \frac{1}{n}\sum_{1}^{n}\left[\tilde{X}_{bk}^2 1\left(\tilde{X}_{bk}^2 > n\delta\right)\right] \to_p 0$$

which in turn hold because

$$E(J_{an}) = E\left[\tilde{X}_{a0}^2 1\left(\tilde{X}_{a0}^2 > n\delta\right)\right] \to 0; \qquad E(J_{bn}) = E\left[\tilde{X}_{b0}^2 1\left(\tilde{X}_{b0}^2 > n\delta\right)\right] \to 0$$

for any $\delta > 0$ by dominated convergence. □

5.11 LEMMA. *Under* $(\mathscr{A}_2)$ *and* $(\mathscr{S}_5)$, $n^{-1}\sum_1^n(\varepsilon_{t-1}^f)^2 \to_{a.s.} \sigma_f^2$.

PROOF. We appeal to Theorem 3.7. Because $\varepsilon_{t-1}^f = \sum_1^\infty \bar{\gamma}_s \varepsilon_{t-s}$, we require $\sum_1^\infty s\bar{\gamma}_s^2 < \infty$ for this theorem to be used. Observe that

$$\sum_{1}^{\infty} s\bar{\gamma}_s^2 = \sum_{s=1}^{\infty} s\left(\sum_{t=1}^{\infty} c_t c_{t+s}\right)^2$$
$$< 2\sum_{s=1}^{\infty} s\left(\sum_{t=1}^{s} c_t c_{t+1}\right)^2 + 2\sum_{s=1}^{\infty} s\left(\sum_{t=s+1}^{\infty} c_t c_{t+s}\right)^2.$$

The second term in the above expression is dominated by twice times

$$\begin{aligned}\sum_{s=1}^{\infty} s \sum_{t=s+1}^{\infty} c_t^2 t \sum_{t=s+1}^{\infty} c_{t+s}^2 t^{-1} &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{\infty} \frac{1}{t} \sum_{s=1}^{t-1} sc_{s+t}^2 \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} c_{s+t}^2 \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{\infty} \sum_{r=t+1}^{\infty} c_r^2 \\ &= \left(\sum_1^{\infty} tc_t^2\right)^2 < \infty \quad \text{under } (\mathscr{S}_5).\end{aligned}$$

For the first term, consider

$$\begin{aligned}R_N &= \sum_{s=1}^{N} s\left(\sum_{t=1}^{s} c_t c_{t+s}\right)^2 \leq \sum_{s=1}^{N} s\left(\sum_{t=1}^{s} t^2 c_t^2\right)\left(\sum_{t=1}^{s} c_{t+s}^2 t^{-2}\right) \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{s=1}^{N} s^2 \sum_{t=1}^{s} c_{t+s}^2 t^{-2} = \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{N} t^{-2} \sum_{s=t}^{N} s^2 c_{t+s}^2 \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{N} t^{-1} \sum_{r=2t}^{N+t} r^2 c_r^2 \leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{N} t^{-2} \sum_{r=t}^{2N} r^2 c_r^2 \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{t=1}^{2N} t^{-2} \sum_{r=t}^{2N} r^2 c_r^2 = \left(\sum_1^{\infty} tc_t^2\right) \sum_{r=1}^{2N} r^2 c_r^2 \sum_{t=1}^{r} t^{-2} \\ &\leq \left(\sum_1^{\infty} tc_t^2\right) \sum_{r=1}^{2N} rc_r^2 \\ &\leq \left(\sum_1^{\infty} tc_t^2\right)^2.\end{aligned}$$

Since R_N is nondecreasing, $R_N \to R_\infty < \infty$. Thus, the first term is also finite if $(\mathscr{S}_5)$ holds, thereby proving the lemma. □

5.12 PROOF OF THEOREM 3.10. Set $n^{-1}w_n = n^{-1}\Sigma_1^n[X_t - C(1)\varepsilon_t]$. We need to show

$$\sum_1^{\infty} \frac{1}{n^2} E(w_n^2) < \infty \tag{52}$$

so that $n^{-1}w_n \to_{\text{a.s.}} 0$ and thus $n^{-1}\Sigma_1^n X_t \to_{\text{a.s.}} 0$ since $n^{-1}\Sigma_1^n \varepsilon_t \to_{\text{a.s.}} 0$ under $(\mathscr{A}_2)$. Now

$$\begin{aligned}E\left(\sum_1^n X_t \sum_1^n \varepsilon_s\right) &= \sum_1^n E\left(X_t \sum_1^t \varepsilon_s\right) = \sum_1^n \sum_1^t c_{t-s}\sigma_\varepsilon^2 \\ &= \sum_1^n \left(\sum_0^{t-1} c_j\right)\sigma_\varepsilon^2 = \sigma_\varepsilon^2 \sum_1^n \left[C(1) - \tilde{c}_{t-1}\right]\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{n^2}E\left(w_n^2\right) &= E(\tilde{X}^2) + \frac{1}{n}C(1)^2\sigma_\varepsilon^2 - \frac{2}{n^2}C(1)\sigma_\varepsilon^2\sum_1^n\left[C(1) - \tilde{c}_{t-1}\right] \\ &= \left[E(\tilde{X}^2) - \frac{1}{n}\sigma_\varepsilon^2 C(1)^2\right] + 2\sigma_\varepsilon^2 C(1)\left[\frac{1}{n^2}\sum_0^{n-1}\tilde{c}_t\right] \qquad (53) \\ &= a_n + 2\sigma_\varepsilon^2 C(1) b_n, \quad \text{say.}\end{aligned}$$

Next

$$\begin{aligned}\sum_1^\infty |b_n| &\le \sum_1^\infty \frac{1}{n^2}\sum_0^{n-1}|\tilde{c}_t| \le \sum_0^\infty |\tilde{c}_t|\sum_{t+1}^\infty \frac{1}{n^2} \\ &\le \sum_1^\infty |\tilde{c}_t|\sum_{t+1}^\infty \frac{1}{n(n-1)} + |\tilde{c}_0|\sum_1^\infty \frac{1}{n^2} \\ &\le \sum_1^\infty \frac{|\tilde{c}_t|}{t} + |\tilde{c}_0|\sum_1^\infty \frac{1}{n^2} \\ &\le \sum_1^\infty \frac{1}{t}\sum_{t+1}^\infty |c_s| + |\tilde{c}_0|\sum_1^\infty \frac{1}{n^2} \\ &= \sum_1^\infty |c_s|\sum_1^{s-1}\frac{1}{t} + |\tilde{c}_0|\sum_1^\infty \frac{1}{n^2} \\ &\le \sum_1^\infty |c_s|(1 + \ln s) + \left(\sum_1^\infty |c_s|\right)\left(\sum_1^\infty \frac{1}{n^2}\right)\end{aligned}$$

by Lemma 5.1(c). Further

$$\begin{aligned}a_n &= \frac{1}{n}\left(\gamma_0 + 2\sum_1^{n-1}\gamma_r\right) - \frac{2}{n^2}\sum_1^{n-1} r\gamma_r - \frac{1}{n}\left(\gamma_0 + 2\sum_1^\infty \gamma_r\right) \qquad (54) \\ &= -\frac{2}{n}\sum_n^\infty \gamma_r - \frac{2}{n^2}\sum_1^{n-1} r\gamma_r\end{aligned}$$

and, then, by use of Lemma 5.1 again we deduce that

$$\begin{aligned}\sum_1^\infty |a_n| &\le 2\sum_1^\infty \frac{1}{n}\sum_n^\infty |\gamma_r| + 2\sum_1^\infty \frac{1}{n^2}\sum_1^{n-1} r|\gamma_r| \\ &\le 2\sum_1^\infty |\gamma_r|(\ln r + 1) + 2\sum_1^\infty r|\gamma_r|\sum_{n=r+1}^\infty \frac{1}{n^2} \\ &\le 2\sum_1^\infty |\gamma_r|(\ln r + 1) + 2\sum_1^\infty |\gamma_r|.\end{aligned}$$

But

$$\begin{aligned}\frac{1}{\sigma_\varepsilon^2}\sum_1^\infty \ln r|\gamma_r| &\le \sum_{r=1}^\infty\sum_{s=0}^\infty \ln r|c_s||c_{s+r}| \\ &\le \sum_{s=0}^\infty |c_s| \sum_{r=1}^\infty \ln(r+s)|c_{s+r}| \\ &= \sum_{s=0}^\infty |c_s| \sum_{p=s+1}^\infty \ln p|c_p| \\ &\le \sum_{p=1}^\infty \ln p|c_p| \sum_{s=0}^{p-1}|c_s| \\ &\le \sum_{p=1}^\infty \ln p|c_p| \sum_{s=0}^\infty |c_s|(\ln s+1) < \infty\end{aligned}$$

under $(\mathscr{S}_6)$. Thus $\Sigma_1^\infty|a_n| < \infty$, $\Sigma_1^\infty|b_n| < \infty$ and (52) follows. □

5.13 PROOF OF THEOREM 3.11. From (21) and (54) we have

$$\frac{1}{n}E(w_n^2) = -2\sum_n^\infty \gamma_r - \frac{2}{n}\sum_1^{n-1} r\gamma_r + \frac{2\sigma_\varepsilon^2 C(1)}{n}\sum_0^{n-1}\tilde{c}_t.$$

But $\Sigma_1^n\gamma_r$ converges to $\Sigma_1^\infty\gamma_r$, which is finite under $(\mathscr{S}_7)$ since $\gamma_0 + 2\Sigma_1^\infty\gamma_r = C(1)^2\sigma_\varepsilon^2$. Thus, $\Sigma_n^\infty\gamma_r \to 0$ as $n\to\infty$. Further, by Kronecker's lemma, $n^{-1}\Sigma_1^n r\gamma_r \to 0$. Finally, if $\tilde{c}_t \to 0$ as $t\to\infty$, then, by the Toeplitz lemma, $n^{-1}\Sigma_1^n\tilde{c}_t \to 0$. However, $\tilde{c}_t = \Sigma_{t+1}^\infty c_s \to 0$ as $t\to\infty$ because $\Sigma_0^\infty c_s = C(1)$ is convergent under $(\mathscr{S}_7)$. Hence $n^{-1/2}w_n \to_p 0$ and $n^{-1/2}\Sigma_1^n X_t = C(1)n^{-1/2}\Sigma_1^n\varepsilon_t + o_p(1) \to_d N(0, \sigma_\varepsilon^2 C(1)^2)$ as required. □

5.14 PROOF OF THEOREM 3.13. This follows the proof of Theorem 3.1. By Remark 2.9(ii), $n^{-1}\Sigma_1^n\varepsilon_t \to_{\text{a.s.}} 0$ under $(\mathscr{B}_2)$. The result then follows if $E(\tilde{\varepsilon}_n^2)$ is bounded above uniformly in n, which it is in view of $(\mathscr{S}_1)$ and $(\mathscr{B}_2)$. □

5.15 PROOF OF THEOREM 3.14. This follows the proof of Theorem 3.2. We just note that Theorem LLN ensures that $n^{-1}\Sigma_1^n\varepsilon_t \to_{\text{a.s.}} 0$ while $(\mathscr{B}_1)$ ensures that $E|\varepsilon_t| \le E|Z| < \infty$ so that $\Sigma_1^\infty\tilde{c}_t|\varepsilon_t| < \infty$ a.s. □

5.16 PROOF OF THEOREM 3.15. The proof is similar to that of Theorem 3.4. We work from equation (18). The CLT for $n^{-1/2}\Sigma_1^n\varepsilon_t$ follows from theorem CLT if

$$\frac{1}{n}\sum_1^n \varepsilon_t^2 \to_p \sigma_\varepsilon^2 \tag{55}$$

and

$$\frac{1}{n}\sum_{1}^{n} E\left[\varepsilon_t^2 1\left(\varepsilon_t^2 > n\varepsilon\right)\right] \to 0. \tag{56}$$

Now (55) follows from ($\mathscr{B}_2$), as indicated in Remark 2.9(ii). Next, from Billingsley [(1968), page 223] we have

$$\begin{aligned} E\left[\varepsilon_t^2 1\left(\varepsilon_t^2 > n\varepsilon\right)\right] &= \varepsilon n P\left(\varepsilon_t^2 > n\varepsilon\right) + \int_{n\varepsilon}^{\infty} P\left(\varepsilon_t^2 > s\right) ds \\ &\le c\left[\varepsilon n P(Z^2 > n\varepsilon) + \int_{n\varepsilon}^{\infty} P(Z^2 > s)\, ds\right] \\ &= cE\left[Z^2 1(Z^2 > n\varepsilon)\right]. \end{aligned}$$

Thus, dominated convergence and $E(Z^2) < \infty$ yield (56). This establishes that $n^{-1/2}\sum_1^{[nr]}\varepsilon_t \to_d \sigma_\varepsilon W(r)$. Theorem 3.15(a) now holds because $n^{-1/2}\tilde{\varepsilon}_0$, $n^{-1/2}\tilde{\varepsilon}_n \to_p 0$ since

$$E\left(\frac{\tilde{\varepsilon}_n^2}{n}\right) = \frac{1}{n}\sum_{0}^{\infty} \tilde{c}_j^2 E\left(\varepsilon_{n-j}^2\right) \le \frac{1}{n}\sum_{0}^{\infty} \tilde{c}_j^2 E(Z^2) \to 0,$$

under ($\mathscr{S}_1$) and ($\mathscr{B}_2$).

To prove the IP [part (b)] we need to verify (19) or equivalently (20), that is,

$$\frac{1}{n}\sum_{1}^{n} \left[\tilde{\varepsilon}_k^2 1\left(\tilde{\varepsilon}_k^2 > n\delta\right)\right] \to_p 0 \quad \text{for any } \delta > 0.$$

This holds if

$$\frac{1}{n}\sum_{1}^{n} E\left[\tilde{\varepsilon}_k^2 1\left(\tilde{\varepsilon}_k^2 > n\delta\right)\right] \to 0$$

which holds if the variables $\tilde{\varepsilon}_k^2$ are uniformly integrable, which holds by Minkowski's inequality applied to $E|\tilde{\varepsilon}_k|^{2+\eta}$ and ($\mathscr{S}_3$) when $\sup_t E|\varepsilon_t|^{2+\eta} < \infty$ for some $\eta > 0$. Thus, ($\mathscr{S}_3$) and the strengthened version of ($\mathscr{B}_2$) suffice to establish the IP. □

5.17 Proof of Theorem 3.16. This follows the proof of Theorem 3.7. We need to verify (41)–(44). As noted in Remark 2.9(ii), ($\mathscr{B}_2$) gives (44). For (43) the same proof as in 5.7 applies and we just need

$$E\left(\sum_{1}^{\infty} \tilde{f}_{0k}\varepsilon_{-k}^2\right) < \infty.$$

But $\sum_1^\infty \tilde{f}_{0k} < \infty$ under ($\mathscr{S}_5$), and by ($\mathscr{B}_2$), we have

$$E\left(\varepsilon_{-k}^2\right) = \int_0^\infty P\left(\varepsilon_{-k}^2 > s\right) ds \le c\int_0^\infty P(Z^2 > s)\, ds = cE(Z^2) < \infty.$$

For (41) the same proof as in 5.7 again applies. All we need is the result of Lemma 5.8 and this holds under ($\mathscr{S}_5$) and the strengthened version of ($\mathscr{B}_2$), since $\sup_t E(\varepsilon_t^4) \le E(Z^4) < \infty$. Finally, we consider (42). As in Section 5.7, $T_n = \sum_1^n t^{-1}\varepsilon_t \varepsilon_{t-1}^f$ is still a martingale and we find

$$E(T_n^2) = \sum_1^\infty \frac{1}{t^2} E\left[\varepsilon_t^2 (\varepsilon_{t-1}^f)^2\right] = \sum_1^\infty \frac{1}{t^2} \sum_1^\infty \bar{\gamma}_j^2 E\left(\varepsilon_t^2 \varepsilon_{t-j}^2\right)$$
$$\le \sup_t E(\varepsilon_t^4) \sum_1^\infty \frac{1}{t^2} \sum_1^\infty \bar{\gamma}_j^2 < \infty. \qquad \square$$

5.18 PROOF OF THEOREM 3.18. From (15),

$$\frac{1}{a_n}\sum_1^n X_t = C(1)\frac{1}{a_n}\sum_1^n \varepsilon_t + \frac{1}{a_n}(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_n)$$

and $a_n^{-1}\sum_1^n \varepsilon_t \to_d U_\alpha(1)$ under $(\mathscr{C}_1)$ by (10) so the result follows provided

$$a_n^{-1}(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_n) \to_p 0.$$

But under $(\mathscr{C}_1)$ and $(\mathscr{S}_9)$, $\tilde{\varepsilon}_n$ is strictly stationary and in $\mathscr{D}(\alpha)$ [e.g., Brockwell and Davis (1987), page 481]. Thus for any $\delta > 0$ we have

$$P(a_n^{-1}|\tilde{\varepsilon}_n| > \delta) = P(|\tilde{\varepsilon}_n| > a_n\delta) = O(n^{-1})$$

and $a_n^{-1}|\tilde{\varepsilon}_n| \to_p 0$ as required. $\square$

5.19 LEMMA. (a) *Under* $(\mathscr{S}_9)$, $\sum_{j=0}^\infty |\tilde{f}_{kj}|^{p/2} < \infty$ *and* $\sum_{r=1}^\infty |\sum_{k=0}^\infty \tilde{f}_{h+r,k}|^p < \infty$. (b) *Under* $(\mathscr{S}_{10})$, $\sum_{r=-\infty,\neq 0}^\infty |f_{h+r}(1)|^p < \infty$.

PROOF.

$$\sum_{j=0}^\infty |\tilde{f}_{kj}|^{p/2} = \sum_{j=0}^\infty \left|\sum_{s=j+1}^\infty c_s c_{s+k}\right|^{p/2} \le \sum_{j=0}^\infty \sum_{s=j+1}^\infty |c_s|^{p/2}|c_{s+k}|^{p/2}$$
$$= \sum_{s=1}^\infty |c_s|^{p/2}|c_{s+k}|^{p/2}\sum_{j=0}^{s-1} 1 = \sum_{s=1}^\infty s|c_s|^{p/2}|c_{s+k}|^{p/2}$$
$$= \sum_{s=1}^\infty s^{1/2}|c_s|^{p/2}s^{1/2}|c_{s+k}|^{p/2}$$
$$< \left(\sum_{s=1}^\infty s|c_s|^p\right)^{1/2}\left(\sum_{s=1}^\infty (s+k)|c_{s+k}|^p\right)^{1/2}$$
$$< \sum_{s=1}^\infty s|c_s|^p < \infty$$

under $(\mathscr{S}_9)$, giving (a). Use $\sum_r'$ to signify the summation $\sum_{r=-\infty, r\neq 0}^\infty$ and we have

$$\sum_r{}' |f_{h+r}(1)|^p = \sum_r{}' \left|\sum_{s=0}^\infty c_s c_{s+h+r}\right|^p \le \sum_r{}' \sum_0^\infty |c_s|^p |c_{s+h+r}|^p$$
$$= \sum_{s=0}^\infty |c_s|^p \sum_r{}' |c_{s+h+r}|^p < \left(\sum_0^\infty |c_s|^p\right)^2 < \infty$$

under $(\mathscr{S}_{10})$. $\square$

5.20 PROOF OF THEOREM 3.19. We use (28) and write

$$X_t X_{t+h} = f_h(1)\varepsilon_t^2 + \varepsilon_{t+h}\varepsilon_t^h - (1-L)\tilde{f}_h(L)\varepsilon_t^2 \tag{57}$$
$$-(1-L)\sum_{r=1}^{\infty}\left[\tilde{f}_{h+r}(L)\varepsilon_{t+h-r}\varepsilon_{t+h} + \tilde{f}_{h-r}(L)\varepsilon_{t+h+r}\varepsilon_{t+h}\right],$$

where $\varepsilon_t^h = \Sigma_{r=1}^{\infty}[f_{h+r}(1)\varepsilon_{t+h-r} + f_{h-r}(1)\varepsilon_{t+h+r}]$. In view of Lemma 5.19(b), $\varepsilon_t^h \in \mathscr{D}(\alpha)$ and, because ε_t is independent of ε_t^h, we have $\varepsilon_{t+h}\varepsilon_t^h \in \mathscr{D}(\alpha)$. Thus,

$$\frac{1}{a_n^2}\sum_1^n \varepsilon_{t+h}\varepsilon_t^h \to_p 0. \tag{58}$$

Next set $\tilde{\varepsilon}_n^2 = \tilde{f}_n(L)\varepsilon_n^2 = \Sigma_{j=0}^{\infty}\tilde{f}_{hj}\varepsilon_{n-j}^2$. By Lemma 5.19(a) the series for $\tilde{\varepsilon}_n^2$ converges a.s. and $\tilde{\varepsilon}_n^2 \in \mathscr{D}(\alpha/2)$. Hence for any $\delta > 0$, $P(a_n^{-2}\tilde{\varepsilon}_n^2 > \delta) = O(n^{-1})$ and so

$$\frac{\tilde{\varepsilon}_n^2}{a_n^2} \to_p 0. \tag{59}$$

Next note that under $(\mathscr{S}_9)$,

$$\sum_{r=1}^{\infty}\sum_{k=0}^{\infty}|\tilde{f}_{h+r,k}|^p \le \sum_{r=1}^{\infty}\sum_{k=0}^{\infty}\sum_{s=k+1}^{\infty}|c_s c_{s+h+r}|^p$$
$$= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} s|c_s|^p|c_{s+h+r}|^p < \left(\sum_{s=1}^{\infty} s|c_s|^p\right)\left(\sum_{r=0}^{\infty}|c_s|^p\right) < \infty,$$

so that $\Sigma_{r=1}^{\infty}\Sigma_{k=0}^{\infty}\tilde{f}_{h+r,k}\varepsilon_{t+h-r-k}\varepsilon_{t+h-k}$ converges a.s. and is in $\mathscr{D}(\alpha)$. Thus

$$\frac{1}{a_n^2}\sum_{r=1}^{\infty}\tilde{f}_{h+r}(L)\varepsilon_{n+h-r}\varepsilon_{n+h} \to_p 0 \tag{60}$$

and similarly

$$\frac{1}{a_n^2}\sum_{r=1}^{\infty}\tilde{f}_{h-r}\varepsilon_{n+h+r}\varepsilon_{n+h} \to_p 0. \tag{61}$$

Summing (57), scaling by a_n^{-2} and using (58)–(61) we deduce that

$$\frac{1}{a_n^2}\sum_1^n X_t X_{t+h} = \frac{f_h(1)}{a_n^2}\sum_1^n \varepsilon_t^2 + o_p(1) \to_d f_h(1)\int_0^1 (dU_\alpha)^2 \tag{62}$$

by (12). The joint convergence of $a_n^{-2}[\Sigma_1^n X_t^2, \dots, \Sigma_1^n X_t X_{t+h}]$ follows directly. Part (b) follows since

$$r_h = \frac{\Sigma_1^n X_t X_{t+h}}{\Sigma_1^n X_t^2} = \frac{f_h(1)}{f_h(0)} + o_p(1). \qquad \square$$

Acknowledgments. Our thanks go to an Associate Editor, two referees and Peter M. Robinson for helpful comments, to the NSF for research support and to Glena Ames for keyboarding the manuscript.

REFERENCES

ANDERSON, T. W. (1971). *The Statistical Analysis of Time Series*. Wiley, New York.

ANDERSON, T. W. and WALKER, A. M. (1964). On the asymptotic distribution of the autocorrelations of a sample from a linear stochastic process. *Ann. Math. Statist.* **35** 1296–1303.

AVRAM, F. and TAQQU, M. S. (1986). Weak convergence of moving averages with infinite variance. In *Dependence in Probability and Statistics: A Survey of Recent Results* (E. Eberlain and M. S. Taqqu, eds.) 399–415. Birkäuser, Boston.

AVRAM, F. and TAQQU, M. S. (1989). Weak convergence of sums of moving averages in the α-stable domain of attraction. Unpublished manuscript.

BEVERIDGE, S. and NELSON, C. R. (1981). A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the 'business cycle.' *Journal of Monetary Economics* **7** 151–174.

BEWLEY, R. A. (1979). The direct estimation of the equilibrium response in a linear dynamic model. *Econom. Lett.* **3** 357–361.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

BREIMAN, L. (1965). On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.* **10** 323–331.

BROCKWELL, P. J. and DAVIS, R. A. (1987). *Time Series: Theory and Methods*. Springer, New York.

BROWN, B. M. (1971). Martingale central limit theorems. *Ann. Math. Statist.* **42** 59–66.

DAVIS, R. A. and RESNICK, S. I. (1985a). Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann. Probab.* **13** 179–195.

DAVIS, R. A. and RESNICK, S. I. (1985b). More limit theory for sample correlation functions of moving averages. *Stochastic Process. Appl.* **20** 257–279.

DAVIS, R. A. and RESNICK, S. I. (1986). Limit theory for the sample covariance and correlation functions of moving averages. *Ann. Statist.* **14** 533–558.

ENGLE, R. F. and GRANGER, C. W. J. (1987). Cointegration and error correction: Representation, estimating and testing. *Econometrica* **55** 251–276.

FULLER, W. A. (1976). *Introduction to Statistical Time Series*. Wiley, New York.

GALLANT, A. R. (1987). *Nonlinear Statistical Models*. Wiley, New York.

GORDIN, M. I. (1969). The central limit theorem for stationary processes. *Soviet Math. Dokl.* **10** 1174–1176.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Applications*. Academic, New York.

HANNAN, E. J. (1970). *Multiple Time Series*. Wiley, New York.

HANNAN, E. J. and DEISTLER, M. (1988). *The Statistical Theory of Linear Systems*. Wiley, New York.

HANNAN, E. J. and HEYDE, C. C. (1972). On limit theorems for quadratic functions of discrete time series. *Ann. Math. Statist.* **43** 2058–2066.

HEYDE, C. C. (1975). On the central limit theorem and iterated logarithm law for stationary processes. *Bull. Austral. Math. Soc.* **12** 1–18.

IBRAGIMOV, I. A. and LINNIK, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhof, Groningen.

JOHANSEN, S. (1988). Statistical analysis of cointegration vectors. *J. Econom. Dynamics Control* **12** 231–254.

LAI, T. L. and WEI, C. Z. (1982). A law of the iterated logarithm for double arrays of independent random variables with applications to regression and time series models. *Ann. Probab.* **10** 320–335.

MCLEISH, D. L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probab.* **2** 620–628.

MCLEISH, D. L. (1975a). A maximal inequality and dependent strong laws. *Ann. Probab.* **3** 829–839.

MCLEISH, D. L. (1975b). Invariance principles for dependent variables. *Z. Wahrsch. Verw. Gebiete* **32** 165–178.

McLeish, D. L. (1977). On the invariance principle for nonstationary mixingales. *Ann. Probab.* **5** 616–621.

Phillips, P. C. B. (1991). Optimal inference in cointegrated systems. *Econometrica* **59** 283–306.

Phillips, P. C. B. and Loretan, M. (1991). Estimating long-run economic equilibrium. *Rev. Econom. Stud.* **58** 407–436.

Resnick, S. I. (1986). Point processes regular variation and weak convergence. *Adv. in Appl. Probab.* **18** 66–138.

Solo, V. (1982, 1986). Topics in advanced time series. *Lectures in Probability and Statistics. Lecture Notes in Math.* **1215**. Springer, New York.

Solo, V. (1989). A simple derivation of the Granger representation theorem. Unpublished manuscript.

Department of Economics
Yale University
P. O. Box 2125, Yale Station
New Haven, Connecticut 06520-2125

Department of Statistics
Macquarie University
Sydney NSW 2109
Australia