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DISTRIBUTION FUNCTIONS OF MEANS OF A DIRICHLET PROCESS[1]

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Let χ be a random probability measure chosen by a Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter α and such that $\int x\chi(dx)$ turns out to be a (finite) random variable. The main concern of this paper is the statement of a suitable expression for the distribution function of that random variable. Such an expression is deduced through an extension of a procedure based on the use of generalized Stieltjes transforms, originally proposed by the present authors in 1978.

0. Introduction. The present paper deals with the probability distribution function $\mathscr{M}$ of $Y := \int x\,d\chi$, χ being a random probability measure chosen by a Dirichlet process with parameter α, on the σ-field of Borel subsets of $\mathbb{R}$, $\mathscr{B}$. Section 1 includes a brief note about these concepts. The Dirichlet process was introduced and studied by Ferguson (1973) in view of its applications to Bayesian nonparametric statistics. In that framework, the assessment of $\mathscr{M}$ represents a very useful tool in order to produce any Bayesian inference concerning the mean of a statistical population. Apropos of this use of $\mathscr{M}$, we recall that the posterior distribution of χ, given the sample $X_1, \dots, X_n$ [cf. Definition 2 in Ferguson (1973)], is also a Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter $\alpha + \sum_1^n \delta_{X_i}$, where δ_x denotes the measure giving mass 1 to the point x [Ferguson (1973), Theorem 1]. Therefore, the expression of $\mathscr{M}$ can be employed for both prior and posterior Bayesian analysis.

The present authors (1978, 1979a, b) provided the distribution function of Y, under the following rather restrictive hypotheses.

H1. The support of α, $S(\alpha)$, is included in $[0, \infty)$.

H2. α is absolutely continuous with respect to Lebesgue measure λ on $(\mathbb{R}, \mathscr{B})$.

H3. $\int |x|\,d\alpha < \infty$.

Taking that result as a starting point, and under the same hypotheses, Cifarelli and Merlini (1979) determined the probability distribution function of $\{\chi((t, \infty))\}^{-1}\int_{(t,\infty)} x\chi(dx)$, $t > 0$. More recently, Hannum, Hollander and

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Langberg (1981), in all probability being unacquainted with our previous works, introduced a family of random variables $\{T^x;\ x \in \mathbb{R}\}$ in such a way that, for each $x \in \mathbb{R}$, $\mathscr{M}(x)$ turns out to coincide with the probability of the event $\{T^x \leq 0\}$. Moreover, for every parameter α satisfying H3, the abovementioned authors have written the characteristic function ϕ_{T^x} of T^x,

$$\phi_{T^x}(t) = \exp\left\{ -\int_{-\infty}^{\infty} \ln[1 - it\{\xi - x\}]\,\alpha(d\xi)\right\}.$$

Hence, one could try to deduce the probability distribution function in question by means of Zolotarev's inversion formula

$$\mathscr{M}(x) = \tfrac{1}{2} - \pi^{-1} \lim_{c \to \infty} \int_0^c [\mathrm{Im}(\phi_{T^x}(t))/t]\, dt;$$

see Zolotarev (1957). In any case, those authors do not provide any explicit expression for $\mathscr{M}$. Recently, Tamura (1988) used Zolotarev's formula in order to invert ϕ_{T^x} numerically. Consequently, our main concern is the statement of that expression by fitting the procedure conceived in our previous papers to the case of an arbitrary parameter α. It is worth recalling that such a procedure is based on a generalized Stieltjes transform. Unfortunately, our ignorance of general, though classical, inversion formulas had prevented us from providing any satisfactory result whenever α did not satisfy H1, H2 and H3. Recently learning of some of these formulas has made it possible to develop our previous research, and the resulting conclusions seem to represent a useful complement of the most recent general results relative to the topic at issue.

The organization of the present paper is as follows. Section 1 includes a few basic definitions and elementary results concerning Dirichlet processes and establishes notation for the subsequent sections. Section 2 introduces a random functional which turns out to be intimately connected to Y. A useful recurrence relation for the moments of the new functional is deduced and, based on that, a generalized Stieltjes transform of the probability distribution function of the same functional is determined. Section 3 provides explicit expressions for $\mathscr{M}$, by means of inversion formulas developed by Sumner (1949) and Hirschman and Widder (1950, 1955). Finally, Section 4 includes a few applications of the main result of the paper. Because of space limitations, some technical details are omitted; in any case, they are extensively explained in Cifarelli and Regazzini (1988).

1. Preliminaries. Let $Z_1, \dots, Z_k$ be independent random variables (rv's), where Z_j has a gamma probability distribution function (pdf) with scale parameter 1 and shape parameter $\alpha_j \geq 0$, and with the proviso that, when $\alpha_j = 0$, then the pdf of Z_j is degenerate at zero. If $\alpha_j > 0$ for some $j \in \{1, \dots, k\}$, then the pdf of the random vector $(Y_1, \dots, Y_{k-1})$, where $Y_j = Z_j / \sum_1^k Z_i$ for $j = 1, \dots, k-1$ and $k \geq 2$, is said to be a *Dirichlet pdf with parameter* $(\alpha_1, \dots, \alpha_k)$; the value of such a distribution at $(x_1, \dots, x_{k-1})$ will be denoted by $\mathscr{D}(x_1, \dots, x_{k-1}; \alpha_1, \dots, \alpha_k)$, for each $(x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1}$. Let us now consider a finite, positive measure α on $(\mathbb{R}, \mathscr{B})$ and, to each finite partition $\{B_1, \dots, B_k\}$ of $\mathbb{R}$ in $\mathscr{B}$, let us associate a Dirichlet pdf with parameter $(\alpha(B_1),\dots,\alpha(B_k))$. Given any finite class $\{E_1, E_2,\dots, E_n\}$ of Borel subsets of $\mathbb{R}$ and the family $\{B_1,\dots, B_k\}$ of its constituents, let us then write the pdf, calculated at $(y_1,\dots, y_n) \in \mathbb{R}^n$,

$$\int_{\Xi(\mathbf{y})} d\mathscr{D}(x_1,\dots,x_{k-1};\ \alpha(B_1),\dots,\alpha(B_k)), \tag{1.1}$$

where $\Xi(\mathbf{y}) := \{(x_1,\cdots,x_{k-1}) \in \mathbb{R}_+^{k-1}\colon\ \Sigma_1^{k-1}x_i \le 1, \Sigma_{i\in C(j)}x_i \le y_j$ for $j = 1,\cdots,n$, and $x_k = 1 - \Sigma_1^{k-1}x_i\}$ and $C(j)$ designates the set of i's for which $B_i \subset E_j$. Ferguson (1973) showed that (1.1) determines a consistent family of pdf's and that, by virtue of Kolmogorov's extension theorem, there exists a unique probability measure $\mathscr{P}$ on the σ-algebra of cylinders, $\sigma([0,1]^{\mathscr{B}})$, with n-dimensional Borel bases, such that the pdf of the coordinate rv's $(\chi(E_1),\dots,\chi(E_n))$ is given by (1.1) for all n and $E_1,\dots, E_n$. Under these conditions, the class of rv's $\{\chi(E);\ E \in \mathscr{B}\}$ is said to be a *Dirichlet process on* $(\mathbb{R}, \mathscr{B})$ *with parameter* α. Denoting $\chi((-\infty, x])$ and $\alpha((-\infty, x])$ by $P(x)$ and $A(x)$, respectively, it is easy to show that for any $n \ge 2$ and $-\infty < t_1 < t_2 < \cdots < t_n < \infty$, the family of rv's $\{P(t);\ t = t_1,\dots, t_n\}$ is a Markov process and that, consequently:

If χ *is a Dirichlet process on* $(\mathbb{R}, \mathscr{B})$, *then*

$\{P(t);\ t \in \mathbb{R}\}$ *is a Markov process;*

$\{P(t);\ t < \tau\}$ *and* $\{P(t);\ t \ge \tau\}$ *are conditionally independent given* $P(\tau)$.

Doksum [(1974), Proposition 3.1] showed that there exists a separable version P such that $\mathscr{P}(P\colon P$ is a probability distribution function$) = 1$. From now on, we will confine ourselves to considering parameters α such that:

CONDITION $(*)$. $\mathscr{P}(P\colon P$ is a pdf and $\int|x|\,dP < \infty) = 1$ is satisfied. By virtue of this hypothesis, we can suppose that each of the P's we will take into consideration is a pdf with finite expectation.

According to our procedure, the starting point for the determination of $\mathscr{M}$ is the assessment of the pdf of the random functional

$$U(\tau, T) := \int_\tau^T \{1 - P(x)\}\,dx \qquad -\infty < \tau < T < \infty,$$

which is linked to Y by the relation

$$\lim_{\substack{\tau\to-\infty\\ T\to+\infty}} \{\tau + U(\tau, T)\} = Y \quad \text{a.s.-}\mathscr{P}. \tag{1.2}$$

After denoting expectations assessed according to $\mathscr{P}$ by $\mathscr{E}$, we will designate the conditional moment of order n of $U(\tau, T)$, given $P(\tau)$, by

$$\mu_n(\tau, P(\tau); T) := \mathscr{E}\big(U(\tau, T)^n | P(\tau)\big), \qquad n = 0,1,\dots .$$

Finally, the pdf of $U(\tau, T)|P(\tau)$ [resp. $U(\tau, T)$] will be denoted by $M_T(\tau, P(\tau);\ \cdot\ )$ [resp. $M_T(\tau;\ \cdot\ )$].

2. The generalized Stieltjes transform of order $\alpha(\mathbb{R})$ of $M_T(\tau;\cdot)$. This section is, in practice, a more general and precise draft of the arguments developed in Section 2 of Cifarelli and Regazzini (1979b).

The well-known relation

$$\left\{\int_\tau^T [1-P(x)]\,dx\right\}^n = n!\int_\tau^T d\tau_1 \int_{\tau_1}^T d\tau_2 \cdots \int_{\tau_{n-1}}^T \prod_1^n [1-P(\tau_j)]\,d\tau_n$$

and Markov property of P yield, a.s.-$\mathscr{P}$,

(2.1)
$$\mu_n(\tau, P(\tau); T) = n\int_\tau^T d\tau_1 \int_I (1-x)\mu_{n-1}(\tau_1, x; T)\mathscr{P}(\tau, P(\tau); \tau_1, dx),$$

where $I = [P(\tau), 1]$, and $\mathscr{P}(\tau, P(\tau); \tau_1, \cdot\,)$ denotes the conditional distribution of $P(\tau_1)$ given $P(\tau)$ with $\tau < \tau_1$. Setting $\tau^* = \inf\{x\colon x \ge \tau,\ x \in S(\alpha)\}$, if $\tau^* \ge T$ one obtains

$$\mu_n(\tau, P(\tau), T) = \{1-P(\tau)\}^n (T-\tau)^n \quad \text{a.s.-}\mathscr{P}; \tag{2.2}$$

in fact, from Proposition 1 in Ferguson (1973), $\alpha(A) = 0 \Rightarrow \mathscr{P}(\chi(A) = 0) = 1$, $\alpha(A) > 0 \Rightarrow \mathscr{P}(\chi(A) > 0) = 1$. On the other hand, if $\tau^* \in [\tau, T)$, it is worth noticing the following relations, which hold a.s.-$\mathscr{P}$:

$$\mathscr{P}(\tau, P(\tau); \tau_1, \{P(\tau)\}) = 1 \quad \text{if } \tau_1 \in (\tau, \tau^*);$$

$$\mathscr{P}(\tau, P(\tau); \tau_1, dx) = \frac{\{x-P(\tau)\}^{A(\tau_1)-A(\tau)-1}(1-x)^{\alpha^*-A(\tau_1)-1}}{B(A(\tau_1)-A(\tau), \alpha^*-A(\tau_1))} \times \{1-P(\tau)\}^{A(\tau)+1-\alpha^*} I_{(P(\tau),1)}(x)\,dx \quad \text{if } \tau_1 > \tau^*;$$

from now on, α^* will designate $\alpha(\mathbb{R})$. These expressions together with (2.1) yield

$$\mu_n(\tau, P(\tau); T) = \{1-P(\tau)\}^n \mu_n^*(\tau, T) \quad \text{a.s.-}\mathscr{P}, \tag{2.3}$$

where $\mu_0^*(\tau, T) = 1$ and

$$\mu_n^*(\tau, T) = n\int_\tau^T \frac{B(\alpha^*-A(\tau), n)}{B(\alpha^*-A(t), n)}\mu_{n-1}^*(t, T)\,dt, \qquad n \ge 1. \tag{2.4}$$

It is easy to show that (2.3)–(2.4) yield (2.2) whenever $\tau^* \ge T$. We are now able to prove

LEMMA 1. *For each $s > 0$ and $x \in [0,1]$, $T \ge \tau$,*

$$\int_{[0,\infty)} (s+y)^{A(\tau)-\alpha^*} d_y M_T(\tau, x; y) = s^{A(\tau)-\alpha^*} \exp\left\{-(1-x)\int_\tau^T \frac{\alpha^*-A(v)}{s+(1-x)(v-\tau)}\,dv\right\}.$$

Moreover, for $s > 0$ and $T \geq \tau$,

$$\int_{[0,\infty)} (s+y)^{-\alpha^*} d_y M_T(\tau; y) = s^{-\alpha^*} \exp\left\{ -\int_\tau^T \frac{\alpha^* - A(v)}{s+v-\tau} dv \right\}$$

and the left-hand side of this equality represents the generalized Stieltjes transform, of order α^*, of $M_T(\tau; \cdot)$.

PROOF. In order to prove the first part, it is sufficient to consider the case when $\alpha^* - A(\tau) > 0$ and $x \in [0,1)$. For each $z \geq 0$, define

$$\begin{aligned} G_T(\tau; z) &:= \int_{[0,\infty)} \left\{1 + \frac{yz}{1-x}\right\}^{A(\tau)-\alpha^*} d_y M_T(\tau, x; y) \\ &= \frac{1}{\Gamma(\alpha^* - A(\tau))} \int_0^\infty e^{-u} u^{\alpha^* - A(\tau) - 1} g_T(\tau; uz)\, du, \end{aligned}$$

where

$$\begin{aligned} g_T(\tau; z) &= \sum_0^\infty \frac{(-1)^n z^n \mu_n^*(\tau, T)}{n!} \\ &= \int_{[0,\infty)} \left\{ \exp\left[-\frac{zy}{1-P(\tau)} \right] \right\} d_y M_T(\tau, P(\tau); y) \quad \text{a.s.-}\mathscr{P}. \end{aligned}$$

In view of standard arguments explained in Cifarelli and Regazzini (1988), jointly with (2.4),

$$\begin{aligned} G_T(\tau; z) = 1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!} z^n n \int_\tau^T \{\alpha^* - A(t) + n - 1\} \\ \times \mu_{n-1}^*(t, T) \left\{ \int_0^\infty \frac{e^{-u} u^{\alpha^* - A(t) + n - 2}}{\Gamma(\alpha^* - A(t))} du \right\} dt \end{aligned}$$

whenever $0 \leq z < 1/(T - \tau)$ and $\tau < T$.

Term by term differentiation, with respect to $\tau > a > -\infty$, is valid at each continuity point of A; hence the relation

$$D_\tau G_T = z\{\alpha^* - A(\tau)\} G_T + z^2 D_z G_T$$

holds for all pairs (τ, z) such that τ is a continuity point of A and $a < \tau < T$, $0 < z < 1/(T - a)$. The general solution of this equation is

$$G_T = C \exp\left\{ -\int_\tau^{T'} \frac{z[\alpha^* - A(v)]}{[1 + z(v - \tau)]} dv \right\}. \tag{2.5}$$

We can use the definition of G_T together with (2.5) and write

$$\begin{aligned} &\int_{[0,\infty)} (s+y)^{A(\tau)-\alpha^*} d_y M_T(\tau, x; y) \\ &\quad = C s^{A(\tau)-\alpha^*} \exp\left\{ -(1-x) \int_\tau^T \frac{\alpha^* - A(v)}{s + (1-x)(v-\tau)} dv \right\}, \end{aligned} \tag{2.6}$$

for $\tau < T$ and $s > (1 - x)(T - a)$. The identity theorem for analytic functions can be used to deduce that (2.6) holds for every $s > 0$, with $C = 1$. This proves the first part of the lemma. As far as the second part is concerned, define

$$\mu_n(\tau, T) = \mathscr{E}\big(U(\tau, T)^n\big), \qquad n \geq 1;$$

then

$$\begin{aligned}\mu_n(\tau, T) &= \int_{[0,1]} \mu_n(\tau, x; T)\, d\mathscr{D}(x; A(\tau), \alpha^* - A(\tau))\\ &= \int_{[0,1]} (1 - x)^n \mu_n^*(\tau, T)\, d\mathscr{D}(x; A(\tau), \alpha^* - A(\tau))\\ &= \frac{\Gamma(\alpha^*)\Gamma(\alpha^* - A(\tau) + n)}{\Gamma(\alpha^* + n)\Gamma(\alpha^* - A(\tau))} \mu_n^*(\tau, T)\end{aligned}$$

and, for $z \in (0, 1/(T - \tau))$ and $T > \tau$,

$$\exp\left\{ - \int_\tau^T \frac{z\{\alpha^* - A(v)\}}{1 + z(v - \tau)}\, dv \right\} = \int_{[0,\infty)} (1 + zy)^{-\alpha^*} d_y M_T(\tau; y).$$

Hence, for $s := 1/z$ and $s > T - a > 0$,

$$s^{-\alpha^*} \exp\left\{ - \int_\tau^T \frac{\alpha^* - A(v)}{s + v - \tau}\, dv \right\} = \int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M_T(\tau; y),$$

and the thesis can be obtained by arguments similar to those developed to prove the first part. □

A few remarks, designated by $R_1, \dots, R_6$, will conclude the present section.

R_1. Denoting the pdf of $\int_\tau^T P(x)\, dx$ by $M_\tau^*(T; \cdot)$, one obtains for $s > 0$,

$$\int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M_\tau^*(T; y) = s^{-\alpha^*} \exp\left\{ - \int_\tau^T \frac{A(v)}{s + T - v}\, dv \right\}.$$

R_2. Suppose that Condition (*) holds and designate the pdf's of $\int_{-\infty}^T P\, dx$ and $\int_\tau^\infty \{1 - P\}\, dx$ by $M^*(T; \cdot)$ and $M(\tau; \cdot)$, respectively. Then

$$\begin{aligned} M_T(\tau; \cdot) &\to_w M(\tau; \cdot) && \text{as } T \to \infty\\ M_\tau^*(T; \cdot) &\to_w M^*(T; \cdot) && \text{as } \tau \to -\infty.\end{aligned}$$

R_3. If Condition (*) holds, then from R_2 and well-known characterization of weak convergence of probability measures, for $s > 0$ we have

$$\int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M_T(\tau; y) \to \int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M(\tau; y) \qquad \text{as } T \to +\infty,$$

$$\int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M_\tau^*(T; y) \to \int_{[0,\infty)} (s + y)^{-\alpha^*} d_y M^*(T; y) \quad \text{as } \tau \to -\infty.$$

R_4. Under Condition $(*)$ and for $s > 0$,

$$\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\, dv < \infty, \qquad \int_{-\infty}^T \frac{A(v)}{s+T-v}\, dv < \infty,$$

$$\int_{[0,\infty)} (s+y)^{-\alpha^*}\, d_y M(\tau; y) = s^{-\alpha^*} \exp\left\{ -\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\, dv \right\},$$

$$\int_{[0,\infty)} (s+y)^{-\alpha^*}\, d_y M^*(T; y) = s^{-\alpha^*} \exp\left\{ -\int_{-\infty}^T \frac{A(v)}{s+T-v}\, dv \right\}.$$

R_5. Under Condition $(*)$ and for $s > 0$,

$$s^{-\alpha^*} \exp\left\{ -\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\, dv \right\} = s^{-A(\tau)} \exp\left\{ -\int_{(\tau,\infty)} \ln(s+v-\tau)\, dA(v) \right\},$$

$$s^{-\alpha^*} \exp\left\{ -\int_{-\infty}^T \frac{A(v)}{s+T-v}\, dv \right\} = s^{A(T)-\alpha^*} \exp\left\{ -\int_{(-\infty,T)} \ln(s+T-v)\, dA(v) \right\}.$$

R_6. Under Condition $(*)$ there exists a Borel set Q such that $\lambda(Q) = 0$ and

$$\int_{\mathbb{R}} |\ln|x-s||\, dA(x) < \infty \quad \text{for each } s \in Q^c \cap [0,\infty).$$

3. The pdf of random means of a Dirichlet process. This section deals with the inversion of generalized Stieltjes transform of order α^* of $M_T(\tau; \cdot)$ [see Lemma 1]. In order to avail ourselves of a few well-known convenient inversion formulas, we preliminarily state that $M(\tau; \cdot)$ and $M^*(T; \cdot)$ are absolutely continuous pdf's.

LEMMA 2. *$M_T(\tau; \cdot)$ and $M_\tau^*(T; \cdot)$ are absolutely continuous pdf's whenever $-\infty < \tau < T < \infty$, A is not degenerate and $\alpha([\tau, T]) > 0$. Moreover, if Condition $(*)$ holds, A is not degenerate and $\alpha([\tau,\infty)) > 0$ [resp. $\alpha((-\infty, T]) > 0$], then $M(\tau; \cdot)$ [resp. $M^*(T; \cdot)$] is an absolutely continuous pdf.*

PROOF. See Cifarelli and Regazzini (1988). □

From now on, the probability density function of $M(\tau; \cdot)$ will be designated by $f(\cdot; \tau)$. Observe that, by virtue of the previous lemma, remark R_4 of Section 2 can be restated:

If Condition $(*)$ holds, A is not degenerate and $\alpha([\tau,\infty)) > 0$, then

$$\int_{[0,\infty)} (s+y)^{-\alpha^*} f(y; \tau)\, dy = s^{-\alpha^*} \exp\left\{ -\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\, dv \right\} \tag{3.1}$$

for each $s > 0$.

In other words, the generalized Stieltjes transform, of order α^*, of f is given by the right-hand side of (3.1). We will act according to the following procedure: firstly, we determine f through inversion of the above-stated transform; then, we evaluate $\mathscr{M}$ based on f, R_2 and (1.2).

The problem of inverting a generalized Stieltjes transform has been solved by Widder (1938), Sumner (1949) and Hirschman and Widder (1950, 1955). To invert the right-hand side of (3.1) we will avail ourselves of both Hirschman–Widder and Sumner according to the value of $A(\tau)$, where $\tau := \inf S(\alpha)$. Then, the main result of the present paper can be condensed into the following:

THEOREM 1. *Let χ be a random probability measure chosen by a Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter α, and satisfying Condition $(*)$. Write $\mathscr{M}$ for the pdf of $Y = \int_{\mathbb{R}} x\chi(dx)$, $S(\alpha)$ for the support of α and $A(\cdot)$ for the corresponding distribution function (df). Then, if α is degenerate at ξ, $\mathscr{M}$ is also degenerate at the same point. On the other hand, if α is not degenerate we obtain*

(i) *for* $\inf S(\alpha) = \tau > -\infty$ *and* $A(\tau) \geq 1$,

$$\mathscr{M}(x) = \begin{cases} 0 & \text{if } x < \tau, \\ \displaystyle\int_\tau^x \frac{2^{\alpha^*-3}(\alpha^*-1)}{\pi(u-\tau)}\,du & \\ \displaystyle\times \int_{-\pi}^{\pi} \left\{\cos\left(\frac{y}{2}\right)\right\}^{\alpha^*-2} \cos\left\{\int_\tau^\infty q(v;u,y)(u-\tau)\sin y\,dv - \frac{\alpha^* y}{2}\right\} & \\ \displaystyle\quad \times \exp\left\{-\int_\tau^\infty q(v;u,y)[(u-\tau)\cos y + v - \tau]\,dv\right\} dy & \text{if } x \geq \tau, \end{cases}$$

where

$$q(v;u,y) = \frac{\alpha^* - A(v)}{(u-\tau)^2 + (v-\tau)^2 + 2(v-\tau)(u-\tau)\cos y};$$

(ii) *for* $\inf S(\alpha) = \tau \geq -\infty$ *and* $A(\tau) \in [0,1)$,

$$\mathscr{M}(x) = \begin{cases} 0 & \text{if } x \leq \tau, \\ \displaystyle\frac{1}{\pi}\int_\tau^x \frac{(x-u)^{\alpha^*-1}}{(u-\tau)^{A(\tau)}} \sin\{\pi A(u)\} h(u; -\infty)\,du & \text{if } x > \tau, \end{cases}$$

where $(u-\tau)^{A(\tau)} = 1$ if $A(\tau) = 0$ and h is any function from $\mathbb{R}$ to $[0, \infty]$ such that

$$h(y; -\infty) = \exp\left\{-\int_{(\tau,\infty)} \ln|v-y|\,dA(v)\right\} \quad a.e.\text{-}\lambda.$$

PROOF. From (3.1) and R_5 with $s = e^z$, one obtains

$$\exp\left\{-zA(\tau) - \int_{(\tau,\infty)} \ln(e^z + v - \tau)\, dA(v)\right\}$$
$$= \int_0^{\infty} (y + e^z)^{-\alpha^*} f(y; \tau)\, dy$$
$$= \int_{-\infty}^{\infty} \exp\left\{-\alpha^*\left(\frac{t+z}{2}\right)\right\} f(e^t; \tau) e^t \left[\operatorname{sech}\left(\frac{z-t}{2}\right)\right]^{\alpha^*} \frac{dt}{2^{\alpha^*}},$$

where the latter equality follows from the change $y = e^t$; hence,

$$\int_{-\infty}^{\infty} \exp\left\{t\left(1 - \frac{\alpha^*}{2}\right)\right\} f(e^t; \tau) \left[\operatorname{sech}\left(\frac{z-t}{2}\right)\right]^{\alpha^*} dt$$
$$= 2^{\alpha^*} \exp\left\{z\left(\frac{\alpha^*}{2} - A(\tau)\right) - \int_{(\tau,\infty)} \ln(e^z + v - \tau)\, dA(v)\right\}.$$

At this point one can determine f from the final part of Section 7 of Chapter 9 in Hirschman and Widder [(1955), page 235], and obtain, for $\alpha^* > 1$,

$$f(u; \tau) = \frac{\alpha^* - 1}{\pi} 2^{\alpha^* - 3} u^{\alpha^* - A(\tau) - 1}$$
$$\times \lim_{\rho \to 1^-} \int_{-\pi}^{\pi} \left\{\cos\left(\frac{y}{2}\right)\right\}^{\alpha^* - 2} \exp\left\{\frac{i\rho y[\alpha^* - 2A(\tau)]}{2} - \int_{(\tau,\infty)} \ln(u e^{i\rho y} + v - \tau)\, dA(v)\right\} dy, \tag{3.2}$$

which holds for almost all $u > 0$ in view of condition 5 of Theorem 7.1b in Hirschman and Widder (1955). Expression (3.2) can be simplified by evaluating the limit involved as $\rho \to 1^-$; in fact, repeated application of Lebesgue's dominated convergence theorem yields

$$f(u; \tau) = \frac{\alpha^* - 1}{\pi} 2^{\alpha^* - 3} u^{\alpha^* - A(\tau) - 1}$$
$$\times \int_{-\pi}^{\pi} \left\{\cos\left(\frac{y}{2}\right)\right\}^{\alpha^* - 2} \exp\left\{\frac{iy[\alpha^* - 2A(\tau)]}{2} - \int_{(\tau,\infty)} \ln(u e^{iy} + v - \tau)\, dA(v)\right\} dy, \tag{3.3}$$

which holds for almost all $u > 0$. Therefore, if $\inf S(\alpha) > -\infty$,

$$\mathscr{M}(x) = \mathscr{P}\left(\int_{\tau}^{\infty} \{1 - P(u)\}\, du + \tau \le x\right) = M(\tau; x - \tau)$$

represents the pdf of Y. Hence, after elementary manipulations one deduces Theorem 1(i). When $\alpha^* > 1$, $\tau > -\infty$ and $A(\tau) \in [0, 1)$, one can obtain a more elegant expression for f, by virtue of Theorem 4a in Sumner (1949). In fact, (3.3) states that f is continuous for almost all $u > 0$ and, therefore, Sumner's inversion formula yields

$$f(u;\tau) = \lim_{\eta\to 0^+} \frac{-1}{2i\pi}\int_{C_{\eta u}} (z+u)^{\alpha^*-1} g'(z)\,dz \quad \text{a.e.-}\lambda,$$

where $g(z)$ is the right-hand side of (3.1), with z in place of s, and $C_{\eta u}$ is the contour which starts at the point $-u - i\eta$, proceeds along the straight line $\mathrm{Im}(z) = -\eta$ to the point $-i\eta$, then along the semicircle $|z| = \eta$, $\mathrm{Re}(z) \ge 0$, to the point $i\eta$, and, finally, along the line $\mathrm{Im}(z) = \eta$ to the point $-u + i\eta$. Hence,

$$\begin{aligned}\int_{C_{\eta u}} &(z+u)^{\alpha^*-1} g'(z)\,dz \\ &= (\alpha^*-1)\int_0^u \Big\{(u-\xi+i\eta)^{\alpha^*-2} g(-\xi+i\eta) \\ &\qquad\qquad -(u-\xi-i\eta)^{\alpha^*-2} g(-\xi-i\eta)\Big\}\,d\xi \\ &\quad -(\alpha^*-1)i\eta\int_{-\pi/2}^{\pi/2} g(\eta\, e^{i\sigma})(\eta\, e^{i\sigma}+u)^{\alpha^*-2} e^{i\sigma}\,d\sigma \\ &\quad + g(-u+i\eta)(i\eta)^{\alpha^*-1} - g(-u-i\eta)(-i\eta)^{\alpha^*-1},\end{aligned}$$

where

$$\begin{aligned} g(s) &= s^{-\alpha^*}\exp\left\{-\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\,dv\right\} \\ &= s^{-A(\tau)}\exp\left\{-\int_{(\tau,\infty)} \ln(s+v-\tau)\,dA(v)\right\}\end{aligned}$$

for $s \in D$, D being the complex plane cut from the origin along the negative real axis. In view of R_5 and R_6, one can determine Q, with $\lambda(Q) = 0$, such that the last three addenda converge to 0 as $\eta \to 0^+$ on $[0,\infty)\setminus Q$. As far as the first addendum is concerned, by virtue of R_6 and Lebesgue's dominated convergence theorem, one can show that it converges to

$$\begin{aligned}2i(\alpha^*-1)\int_0^u \frac{(u-\xi)^{\alpha^*-2}}{\xi^{A(\tau)}}&\left\{\exp\left(-\int_{(\tau,\infty)} \ln|v-\tau-\xi|\,dA(v)\right)\right\} \\ &\times\left[-\sin(\pi A(\tau) + \pi\{A(\tau+\xi) - A(\tau)\}\right]\,d\xi\end{aligned}$$

for every $u \in (0,\infty)\setminus Q$. At this point it is immediate to deduce

$$\begin{aligned}\frac{d\mathscr{M}(x)}{dx} = \frac{\alpha^*-1}{\pi}\int_0^{x-\tau} &\frac{(x-\tau-\xi)^{\alpha^*-2}}{\xi^{A(\tau)}} \\ &\times\left[\exp\left\{-\int_{(\tau,\infty)} \ln|v-\tau-\xi|\,dA(v)\right\}\right]\sin\{\pi A(\tau+\xi)\}\,d\xi,\end{aligned}$$

and, consequently, one can determine $\mathscr{M}$ according to expression (ii) of Theorem 1. When $\tau = -\infty$ and $\alpha^* > 1$, there is $\tau^* > -\infty$ such that $A(\tau^*) < 1$. Consequently, one can determine $\mathscr{M}$ arguing as in the previous case with τ^* in place of τ and then passing to the limit as $\tau^* \to -\infty$. In order to deal with the case when $\alpha^* \in (0, 1)$, one may start by stating the equality

$$(3.4)\quad \int_0^\infty (s+y)^{-\alpha^*-1} f^*(y;\tau)\,dy = \frac{1}{\alpha^*} s^{-\alpha^*} \exp\left\{-\int_\tau^\infty \frac{\alpha^* - A(v)}{s+v-\tau}\,dv\right\},$$

where

$$f^*(y;\tau) = \int_0^y f(x;\tau)\,dx \quad \text{for } y \geq 0.$$

Consequently, f^* may be obtained from the inversion of the right-hand side of (3.4), by arguments similar to those expounded in connection with the previous case when $A(\tau) \in [0, 1)$.

Finally, when $\alpha^* = 1$, one can use the classical inversion formula of Stieltjes [cf. Widder (1946), Theorem 7b(3), page 340], which yields a result agreeing with Theorem 1(ii). □

The previous theorem can be applied in order to determine the pdf of

$$Y_\psi = \int_{\mathbb{R}} \psi(x)\chi(dx),$$

where ψ is a measurable function, χ is a random probability measure chosen by the Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter α and satisfying

CONDITION $(*)_\psi$. $\mathscr{P}(\{\chi: \int |\psi|\,d\chi < \infty\}) = 1.$

Indeed, under these conditions, $\chi\psi^{-1}(\cdot) := \chi(\psi^{-1}(\cdot))$ turns out to be a random probability measure from the Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter $\alpha_\psi := \alpha(\psi^{-1}(\cdot))$. Hence, the previous theorem yields

COROLLARY 1. *Let χ be a random probability measure chosen by a Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter α; let ψ: $\mathbb{R} \to \mathbb{R}$ be a measurable function satisfying Condition $(*)_\psi$ and write $\mathscr{M}_\psi$ for the pdf of Y_ψ. Under these conditions, $\mathscr{M}_\psi$ coincides with $\mathscr{M}$ in Theorem 1 upon replacement of A by the df corresponding to α_ψ.*

4. Applications of the previous result. This section is devoted to a few applications of Theorem 1.

The first application relates to the case when the parameter α is proportional to a Cauchy pdf. Apropos of this kind of parameter, it is well known that:
If $A(x) = \alpha^* \int_{-\infty}^x \lambda\{\lambda^2 + (t-\mu)^2\}^{-1} dt/\pi$ for all x in $\mathbb{R}$, then

$$\mathscr{M}(x) = A(x)/\alpha^* \quad \textit{for all } x \in \mathbb{R},$$

whatever $\alpha^* > 0$ may be.

This statement, which could be deduced from our Theorem 1(ii), is due to Yamato (1984). Here the inverse statement is proved, that is:
Let Q be an assigned nondegenerate pdf on $\mathbb{R}$ and let $\mathscr{M}$ be the pdf of the mean of a Dirichlet process with parameter $\alpha(\cdot) = \alpha^* Q(\cdot)$. Then

$$\mathscr{M}(x) = Q(x) \tag{4.1}$$

holds for all x in $\mathbb{R}$ and $\alpha^* > 0$, if and only if Q is Cauchy.

In fact, in view of Lemma 2 of Section 3, if (4.1) holds, then Q is absolutely continuous and $A(\tau) = 0$. Hence, from our Theorem 1(ii),

$$\begin{aligned} Q'(x) &= \frac{1}{\pi}\int_\tau^x \exp\left\{-2\int_{\mathbb{R}}[\log|v-u|]Q'(v)\,dv\right\}\sin\{2\pi Q(u)\}\,du \\ &\qquad\qquad\qquad\qquad \text{by taking } \alpha^* = 2 \\ &= \frac{1}{\pi}\exp\left\{-\int_{\mathbb{R}}[\log|v-x|]Q'(v)\,dv\right\}\sin\{\pi Q(x)\} \\ &\qquad\qquad\qquad\qquad \text{by taking } \alpha^* = 1, \end{aligned} \tag{4.2}$$

which holds a.e.-λ. Moreover, (4.2) entails

$$Q'(x) = \frac{1}{\pi}\int_\tau^x \frac{\pi^2}{\sin^2\{\pi Q(u)\}} Q'(u)^2 \sin\{2\pi Q(u)\}\,du;$$

hence,

$$Q''(x) = 2\pi(Q'(x))^2 \cot\{\pi Q(x)\} \quad \text{a.e.-}\lambda,$$

whose solution is given by $Q(x) = \frac{1}{2} + \{\arctan(ax+b)\}/\pi$, with $a > 0$.

We now mention an application to the limit distribution of $\{\sum_{k=1}^n X_k/n\}_{n=1}^\infty$, as $n \to \infty$, when $\{X_n\}_{n=1}^\infty$ is a sequence of exchangeable random variables which, conditionally on χ, are iid, χ being a Dirichlet process on $(\mathbb{R}, \mathscr{B})$ with parameter α. In fact, under Condition $(*)$, it is easy to show that

$$\mathscr{P}\left(\sum_{k=1}^n \frac{X_k}{n} \le x\right) \to_{\mathrm{w}} \mathscr{M}(x), \qquad n \to \infty,$$

$\mathscr{M}$ being the pdf determined in Section 3. Compare in connection with this, Klass and Teicher (1987).

The last application we give deals with the posterior distribution of χ, given a random sample [see Ferguson (1973), page 216] of size $n > 1$ with realizations $\xi_1 \le \cdots \le \xi_n$. It is well known that the influence of the mean $A(\cdot)/\alpha^*$ of the prior distribution of P, on any inference based on posterior Dirichlet process, "vanishes" as $\alpha^* \to 0$. On the other hand, the influence of the empirical distribution $\sum_{k=1}^n \delta_{\xi_k}/n$ reaches its "upper bound" as $\alpha^* \to 0$. We are analyzing the behavior of the pdf of $Y|\xi_1, \dots, \xi_n$ as $\alpha^* \to 0$. It will be shown that a limit pdf exists. Hence such a limit, in view of the previous remarks, can be considered to be a posterior pdf of the random mean Y when the prior knowledge is vague. If

$\mathcal{M}'_{\xi_1,\dots,\xi_n}$ denotes the density of $Y|\xi_1,\dots,\xi_n$, then (3.3) yields

$$\mathcal{M}'_{\xi_1,\dots,\xi_n}(x) = \frac{\alpha^* + n - 1}{\pi} 2^{\alpha^*+n-3}(x-\tau)^{\alpha^*+n-A_n(\tau)-1} \int_{-\pi}^{\pi} \left\{\cos\frac{y}{2}\right\}^{\alpha^*+n-2} \times \exp\left\{ \frac{iy[\alpha^* + n - 2A_n(\tau)]}{2} - \int_{(\tau,\infty)} \log\{(x-\tau)e^{iy} + v - \tau\}\, dA_n(v)\right\} dy \quad \text{a.e.-}\lambda,$$

where

$$A_n(v) := A(v) + n(v), \qquad n(v) := \#\{k: \xi_k \le v\},$$

which, as $\alpha^* \to 0$, converges to

$$\begin{aligned} I(x) &= \frac{n-1}{\pi} 2^{n-3}(x-\tau)^{n-n(\tau)-1} \\ &\quad \times \int_{-\pi}^{\pi} \left(\cos\frac{y}{2}\right)^{n-2} \exp\left\{ iy\left[\frac{n}{2} - n(\tau)\right] - \sum_{\xi_i > \tau} \log[(x-\tau)e^{iy} + \xi_i - \tau]\right\} dy \\ &= \frac{n-1}{2\pi i} \int_{|z|=x-\tau} \frac{(z+x-\tau)^{n-2}}{(z+\xi_1-\tau)\cdots(z+\xi_n-\tau)}\, dz \quad \text{for } x \in (\xi_1, \xi_n). \end{aligned}$$

The latter integral can be evaluated by means of Cauchy's residue theorem. For the sake of simplicity, we will consider the case in which $\tau < \xi_1 < \xi_2 < \cdots < \xi_n$. Then

$$I(x) = (n-1) \sum_{p=1}^{k} \frac{(x-\xi_p)^{n-2}}{\prod_{i\neq p}|\xi_i - \xi_p|}(-1)^{p-1}$$

for $x \in [\xi_k, \xi_{k+1})$, $k = 1,\dots, n-1$ and

$$\lim_{\alpha^*\to 0} \mathcal{P}(Y \le x|\xi_1,\dots,\xi_n) = \begin{cases} 0 & \text{if } x < \xi_1, \\ \displaystyle\sum_{p=1}^{k} \frac{(x-\xi_p)^{n-1}}{\prod_{i\neq p}|\xi_i-\xi_p|}(-1)^{p-1} & \text{if } x \in [\xi_k, \xi_{k+1}) \\ & \quad k = 1,\dots, n-1 \\ 1 & \text{if } x \ge \xi_n. \end{cases}$$

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