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ORDER RESTRICTED STATISTICAL TESTS ON MULTINOMIAL AND POISSON PARAMETERS: THE STARSHAPED RESTRICTION[1]

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Likelihood ratio statistics for (i) testing the homogeneity of a collection of multinomial parameters against the alternative which accounts for the restriction that those parameters are starshaped (cf. Shaked, *Ann. Statist.*, 1979), and for (ii) testing the null hypothesis that this parameter vector is starshaped, are considered. For both tests the asymptotic distribution of the test statistic under the null hypothesis is a version of the Chi-bar-square distribution. Analogous tests on a collection of Poisson means are also found to have asymptotic Chi-bar-square distributions.

1. Introduction and summary. Shaked (1979) derived the maximum likelihood estimate of a vector of Poisson (normal) means subject to the restriction that this vector is "starshaped." A vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is said to be lower starshaped provided

$$\alpha_1 \geq \frac{\alpha_1 + \alpha_2}{2} \geq \cdots \geq \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_k}{k} \geq 0$$

with an analogous restriction defining an upper starshaped vector. Starshaped vectors arise naturally in reliability theory as well as in certain situations where finite populations are amalgamated. We refer the interested reader to Shaked (1979) and Dykstra and Robertson (1981) for further discussion on starshaped parameters.

In Section 2 we consider a multinomial sampling situation with probabilities $p_1, p_2, \cdots, p_k$. The maximum likelihood estimate of the vector $\mathbf{p} = (p_1, p_2, \cdots, p_k)$, subject to the restriction that it be lower starshaped, is derived. This derivation is quite direct and elegant in light of the complexity involved in finding the maximum likelihood estimate of $\mathbf{p}$ subject to other order restrictions (cf. Barlow, Bartholomew, Bremner and Brunk, 1972).

In addition, asymptotic distribution theory for the likelihood ratio test of the homogeneity of $p_1, p_2, \cdots, p_k$ against the alternative that $\mathbf{p}$ is starshaped and for testing that $\mathbf{p}$ is starshaped as a null hypothesis is also presented in Section 2. In both situations, the tail probabilities under the null hypothesis of this asymptotic distribution turn out to be of the form

$$\bar{\chi}^2_{k-1}(t) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell-1} (1/2)^{k-1} P(\chi^2_\ell \geq t),$$

where χ^2_ℓ denotes a standard Chi square random variable with ℓ degrees of freedom. A somewhat similar distribution is encountered in the problem of testing homogeneity when the alternative is restricted by $p_1 \geq p_2 \geq \cdots \geq p_k$ (cf. Chacko, 1966) and for testing $p_1 \geq p_2 \geq \cdots \geq p_k$ as a null hypothesis (cf. Robertson, 1978, and related results in Robertson and Wegman, 1978). Such weighted Chi square distributions are encountered in many order restricted inference problems (cf. Barlow et al., 1972). They were first encountered by Bartholomew (1959) and are usually called Chi-bar-square distributions.

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The lowered starshaped ordering, H_1, might be termed "decreasing on the average". This is somewhat similar to the restriction

$$H_2 : i^{-1} \sum_{j=1}^{i} \theta_j \geq (k-i)^{-1} \sum_{j=i+1}^{k} \theta_j, \quad i = 1, 2, \cdots, k-1.$$

An equivalent way of stating H_2 is $i^{-1} \sum_{j=1}^{i} \theta_j \geq k^{-1} \sum_{j=1}^{k} \theta_j$; $i = 1, 2, \cdots, k-1$. We note that the order restrictions specified in H_2 are less restrictive than those imposed by H_1 which in turn are less restrictive than $\theta_i \geq \theta_{i+1}$, $i = 1, \cdots, k-1$. In the multinomial setting, maximum likelihood estimates of p subject to H_2 and distribution theory for testing the homogeneity, H_0, of p vs. $H_2 - H_0$ and for testing H_2 as a null hypothesis can be found in Robertson and Wright (1982). Again, the asymptotic distribution is a Chi-bar-square.

In Section 3 we assume independent samples from each of k Poisson populations. The analysis in Section 2 together with the well known fact that the joint distribution of independent Poisson random variables, conditioned on the value of their sum, is multinomial, is used to derive maximum likelihood estimates under the starshaped restriction on the parameter values. Asymptotic distribution theory for likelihood ratio statistics used for testing homogeneity versus starshaped and for testing starshaped as a null hypothesis is also presented.

2. Multinomial problem. Suppose we have n independent trials of an experiment, the outcome of which must be one of k mutually exclusive events with corresponding probabilities, $p_1, p_2, \cdots, p_k$ $(\sum_{i=1}^{k} p_i = 1)$. We are concerned with two hypotheses, $H_0 : p_1 = p_2 = \cdots = p_k = 1/k$ and

$$H_1 : p_1 \geq \frac{p_1 + p_2}{2} \geq \cdots \geq \frac{p_1 + p_2 + \cdots + p_{k-1}}{k-1} \geq \frac{1}{k}. \tag{2.1}$$

It is convenient to define a one to one transformation of the parameter space by introducing new parameters $\theta_1, \theta_2, \cdots, \theta_{k-1}$ defined by

$$\theta_i = \left(\sum_{j=1}^{i} p_j\right) \Big/ \left(\sum_{j=1}^{i+1} p_j\right), \quad i = 1, 2, \cdots, k-1; \tag{2.2}$$

thus $p_1 = \prod_{j=1}^{k-1} \theta_j$, $p_i = (1 - \theta_{i-1}) \prod_{j=i}^{k-1} \theta_j$, $i = 2, 3, \cdots, k-1$ and $p_k = (1 - \theta_{k-1})$. In terms of the θ_i's, our hypotheses are $H_0 : \theta_i = i/(i+1)$, $i = 1, \cdots, k-1$ and $H_1 : \theta_i \geq i/(i+1)$, $i = 1, \cdots, k-1$. Consider first estimation of $\boldsymbol{\theta}$. The likelihood function can be written

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{k-1} \theta_i^{\,n \sum_{j=1}^{i} \hat{p}_j} (1 - \theta_i)^{n \hat{p}_{i+1}}, \quad 0 \leq \theta_i \leq 1, \tag{2.3}$$

where $\hat{p}_i$ is the relative frequency of the event having probability $p_i : i = 1, 2, \cdots, k$. It is easy to find the maximum of the function $\theta^a (1-\theta)^b$ subject to $\theta \geq c (0 \leq \theta \leq 1)$. This maximum is attained at $\bar{\theta} = \{a/(a+b)\} \vee c$, where $\vee$ denotes the larger of the two numbers. It follows that the maximum likelihood estimates which satisfy H_1 are given by

$$\bar{\theta}_i = \hat{\theta}_i \vee (i/i+1), \quad i = 1, 2, \cdots, k-1, \tag{2.4}$$

where $\hat{\theta}_i = (\sum_{j=1}^{i} \hat{p}_j)/(\sum_{j=1}^{i+1} \hat{p}_j)$. Evaluation of $\mathbf{p}$ at $\boldsymbol{\theta} = \bar{\boldsymbol{\theta}}$ gives the MLE of $\mathbf{p}$ under the restriction specified in (2.1).

Turning to the testing problem, we let Λ_{01} denote the likelihood ratio test statistic for testing H_0 against $H_1 - H_0$ and let $T_{01} = -2 \ln \Lambda_{01}$. Then

$$T_{01} = 2 \sum_{i=1}^{k-1} \{(n \sum_{j=1}^{i} \hat{p}_j)[\ln \bar{\theta}_i - \ln(i/i+1)] + n\hat{p}_{i+1}[\ln(1 - \bar{\theta}_i) - \ln(1/i+1)]\}. \tag{2.5}$$

Expanding $\ln \bar{\theta}_i$ and $\ln\{i/(i+1)\}$ about $\hat{\theta}_i$, and $\ln(1 - \bar{\theta}_i)$ and $\ln\{1/(i+1)\}$ about $1 - \hat{\theta}_i$ via Taylor's Theorem with a second degree remainder term, we obtain

$$\begin{aligned} T_{01} = 2 \sum_{i=1}^{k-1} \Bigg[& -\frac{n \sum_{j=1}^{i} \hat{p}_j}{2\alpha_i^2}(\bar{\theta}_i - \hat{\theta}_i)^2 + \frac{n \sum_{j=1}^{i} \hat{p}_j}{2\beta_i^2}\left(\hat{\theta}_i - \frac{i}{i+1}\right)^2 \\ & - \frac{n\hat{p}_{i+1}}{2\nu_i^2}(\hat{\theta}_i - \bar{\theta}_i)^2 + \frac{n\hat{p}_{i+1}}{2\gamma_i^2}\left(\hat{\theta}_i - \frac{i}{i+1}\right)^2 \Bigg], \end{aligned} \tag{2.6}$$

where α_i is between $\bar{\theta}_i$ and $\hat{\theta}_i$; β_i is between $\hat{\theta}_i$ and $i/(i+1)$; ν_i is between $(1-\bar{\theta}_i)$ and $(1-\hat{\theta}_i)$ and γ_i is between $(1-\hat{\theta}_i)$ and $1/(i+1)$. The law of large numbers implies that, under H_0, $\hat{\theta}_i$ converges to $i/i+1$.

To study the asymptotic power of the likelihood ratio test, we consider a sequence of alternatives $\mathbf{p}_n$ satisfying H_1 which converges to $\mathbf{p}$ where $p_i > 0$ for all i. We let $\hat{\mathbf{p}}_n$ denote a random vector corresponding to $\mathbf{p}_n$, i.e., $n\hat{\mathbf{p}}_n$ is multinomial $(n, \mathbf{p}_n)$; and $\boldsymbol{\theta}_n$, $\hat{\boldsymbol{\theta}}_n$ correspond to $\mathbf{p}_n$, $\hat{\mathbf{p}}_n$ via (2.2). Somewhat surprisingly, it can be shown, by conditioning on $\sum_{j=1}^{i+1} \hat{p}_{nj}$, that $E(\hat{\theta}_{n,i}) = \theta_{n,i}$.

A straightforward application of the CLT to $\sqrt{n}(\hat{\mathbf{p}}_n - \mathbf{p}_n)$ and then use of the delta method (Kepner, 1979) applied to $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) = \sqrt{n}\{g(\hat{\mathbf{p}}_n) - g(\mathbf{p}_n)\}$ shows that

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_n) \to \mathrm{MVN}(\mathbf{0}, \Psi), \tag{2.7}$$

where Ψ has elements

$$\Psi_{ij} = \begin{cases} \theta_i(1-\theta_i)/\sum_{j=1}^{i+1} p_j, & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore, fortunately, we have asymptotic independence among the $\sqrt{n}\hat{\theta}_{n,i}$'s.

If we recall that $\bar{\theta}_{n,i} = \hat{\theta}_{n,i} \vee 1/(i+1)$, we can express the likelihood ratio test statistic as

$$T_{01}^{(n)} = \textstyle\sum_{i=1}^{k-1} W_{n,i} = \sum_{i=1}^{k-1} (X_{n,i} + \delta_{n,i})^2 a_{n,i} - (X_{n,i} + \delta_{n,i})^2 b_{n,i} I_{[X_{n,i}+\delta_{n,i} \le 0]}, \tag{2.8}$$

where, with $c_i = (i+1)^3/ik$,

$$X_{n,i} = \sqrt{n}(\hat{\theta}_{n,i} - \theta_{n,i})c_i^{1/2}, \qquad \delta_{n,i} = \sqrt{n}\left(\theta_{n,i} - \frac{i}{i+1}\right)c_i^{1/2} \tag{2.9}$$

$$a_{n,i} = \left[\sum_{j=1}^{i} \frac{\hat{p}_{n,j}}{\beta_{n,i}^2} + \frac{\hat{p}_{n,i+1}}{\gamma_{n,i}^2}\right]c_i^{-1}, \quad \text{and} \quad b_{n,i} = \left[\sum_{j=1}^{i} \frac{\hat{p}_{n,j}}{\alpha_{n,i}^2} + \frac{\hat{p}_{n,i+1}}{\nu_{n,i}^2}\right]c_i^{-1}.$$

To obtain the limiting distribution of $T_{01}^{(n)}$, we shall apply (2.7) under the assumption $\delta_{n,i} \to \delta_i < \infty$, in which case $a_{n,i} \to_p 1$ and $b_{n,i} \to_p 1$ as $n \to \infty$.

In this situation, $X_{n,i} \to_{\mathscr{D}} Z_i$, where Z_i is a $N(0, 1)$ random variable. Using Theorems 4.9 and 5.1 of Billingsley (1968), we have

$$W_{n,i} \to_{\mathscr{D}} (Z_i + \delta_i)^2 - (Z_i + \delta_i)^2 I_{[Z_i+\delta_i \le 0]} = [(Z_i + \delta_i) \vee 0]^2.$$

In the event that $\delta_{n,i} \to \infty$, it can be shown that $a_{n,i}$ is bounded away from zero asymptotically while $X_{n,i}$ converges in distribution, so that $W_{n,i} \to_p \infty$. We have thus established the following theorem.

THEOREM 2.2. *If $\mathbf{p}_n$ satisfies H_1 and converges to $\mathbf{p}$ such that $p_i > 0$ for all i, and if $\delta_{n,i}$, as defined in* (2.9) *tends to δ_i (possibly ∞) for $i = 1, \cdots, k-1$, then $T_{01}^{(n)}$ is distributed asymptotically as*

$$U = \textstyle\sum_{i=1}^{k-1} [(Z_i + \delta_i) \vee 0]^2, \tag{2.10}$$

where $Z_1, \cdots, Z_{k-1}$ are independent $N(0, 1)$ random variables.

Of course the distribution of the random quantity in (2.10) is intractable, except under the null hypothesis $H_0(\delta_i = 0, i = 1, \cdots, k-1)$. To elaborate, suppose I is a subset of $\{1, 2, \cdots, k-1\}$ and let E_I be the event $E_I = [Z_i \ge 0, i \in I; Z_i < 0, i \notin I]$. Then, for any real number u,

$$\begin{aligned} P(U \ge u, E_I) &= P(\textstyle\sum_{i\in I} Z_i^2 \ge u, Z_i \ge 0, i \in I, Z_i < 0, i \notin I) \\ &= P(\textstyle\sum_{i\in I} Z_i^2 \ge u, Z_i \ge 0, i \in I) \cdot P(Z_i < 0, i \notin I) \end{aligned}$$

$$= P(\textstyle\sum_{i \in I} Z_i^2 \geq u \mid Z_i \geq 0, i \in I) \cdot (1/2)^{k-1}$$
$$= P(\chi_m^2 \geq u) \cdot (1/2)^{k-1},$$

where m is the number of elements in I. The last step follows from Lemma B on page 128 of Barlow, Bartholomew, Bremner and Brunk (1972). Partitioning the event $\{U \geq u\}$ by intersecting it with all such events, E_I, we obtain the following result.

THEOREM 2.3. *If H_0 is true then*

$$\lim_{n \to \infty} P(T_{01} \geq t) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \left(\frac{1}{2}\right)^{k-1} P(\chi_\ell^2 \geq t) = \bar{\chi}_{k-1}^2(t)$$

for all real $t(\chi_0^2 \equiv 0)$.

Critical values for this distribution for $k = 3, 4, \cdots, 15$ and $\alpha = .10, .05, .01$ are given in Table 1.

We note that the asymptotic distribution of $-2 \ln \Lambda^{(n)}$, where $\Lambda^{(n)}$ is the unrestricted likelihood ratio (or of the usual Pearson Chi-squared goodness of fit test) under the conditions of Theorem 2.2, is the same as $U' = \sum_{i=1}^{k-1} (Z_i + \delta_i)^2$. Clearly for the same size test, the critical point of U' must be substantially larger than the critical point of U. However, U and U' become equivalent as the $\delta_i \to \infty$, and hence the restricted test must eventually have greater power.

We now turn to the problem of testing H_1 as a null hypothesis when "not H_1" is the alternative. Since the unrestricted maximum likelihood estimate of θ_i is equal to $\hat{\theta}_i$, it follows directly, by writing the likelihood ratio in terms of $\hat{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\theta}}$ and expanding $\ln \bar{\theta}_i$ and $\ln(1 - \bar{\theta}_i)$ about $\hat{\theta}_i$ and $(1 - \hat{\theta}_i)$ respectively, that our test statistic can be written as

$$T_1 = -2 \ln \Lambda_1 = \sum_{i=1}^{k-1} \left[\frac{\sum_{j=1}^{i} \hat{p}_j}{\alpha_i^2} + \frac{\hat{p}_{i+1}}{\nu_i^2} \right] n(\hat{\theta}_i - \bar{\theta}_i)^2,$$

where α_i is between $\bar{\theta}$ and $\hat{\theta}_i$ (and thus converges a.s. to θ_i) and ν_i is between $1 - \bar{\theta}_i$ and $1 - \hat{\theta}_i$ (and thus converges a.s. to $1 - \theta_i$).

By employing arguments similar to those used in Theorem 2.2, we are led to the following Theorem.

TABLE 1

Critical values for $\bar{\chi}_{k-1}(t) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \left(\frac{1}{2}\right)^{k-1} P[\chi_\ell^2 \geq t]$

	α		
k	**.10**	**.05**	**.01**
3	2.95	4.23	7.28
4	4.01	5.44	8.77
5	4.95	6.50	10.02
6	5.84	7.48	11.18
7	6.67	8.41	12.26
8	7.48	9.29	13.31
9	8.26	10.15	14.29
10	9.02	10.99	15.29
11	9.76	11.79	16.21
12	10.49	12.59	17.12
13	11.22	13.38	18.01
14	11.93	14.15	18.91
15	12.63	14.91	19.78

THEOREM 2.4. *If* $\mathbf{p}_n$ *converges to* $\mathbf{p}$ *and if* $\delta_{n,i}$ *(as defined in* (2.9)*) converges to* δ_i *(*$\pm\infty$ *are possible values) for* $i = 1, \cdots, k-1$, *then* $T_1^{(n)}$ *is distributed asymptotically as*

$$V = \sum_{i=1}^{k-1} [(Z_i + \delta_i) \wedge 0]^2$$

where $Z_1, \cdots, Z_{k-1}$ *are independent* $N(0, 1)$ *random variables. Consequently, if we consider the asymptotic distribution of* $T_1^{(n)}$ *for* $\mathbf{p}_n \equiv \mathbf{p} \in H_1$ *then* $\lim_{n\to\infty} P(T_1^{(n)} \ge t) = \bar{\chi}_m^2(t)$, *where* m *is the number of subscripts* i *such that* $\theta_i = i/(i+1)$ *(under these circumstances* δ_i *is either zero or* $+\infty$*). Thus,*

$$\sup_{\mathbf{p}\in H_1} \lim_{n\to\infty} P_{\mathbf{p}}(T_1^{(n)} \ge t) = \lim_{n\to\infty} P_{H_0}(T_1^{(n)} \ge t) = \bar{\chi}_{k-1}^2(t),$$

where $P_{H_0}(T_1^{(n)} \ge t)$ *is the probability of the event* $\{T_1^{(n)} \ge t\}$ *computed under* H_0.

We note that Theorems 2.2 and 2.4 imply that the likelihood ratio tests considered here are consistent in the sense that for $\mathbf{p}$ lying in the region defined by the alternative hypothesis, the power function must converge to one.

3. Poisson problem. Suppose we have a random sample of size n from each of k Poisson populations having means $\lambda_1, \lambda_2, \cdots, \lambda_k$. Shaked (1979) found the maximum likelihood estimate of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ subject to the restriction H_1 requiring $\boldsymbol{\lambda}$ to be lower starshaped:

$$H_1 : \lambda_1 \ge \frac{\lambda_1 + \lambda_2}{2} \ge \cdots \ge \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{k} \ge 0. \tag{3.1}$$

This result can be found in a straightforward fashion using the results in Section 2 together with the fact that the conditional distribution of independent Poisson variables, given their sum, is multinomially distributed.

We first write the likelihood function in terms of the variables $\phi_1, \phi_2, \cdots, \phi_k$ defined by

$$\phi_i = \lambda_i / \textstyle\sum_{i=1}^k \lambda_i, \qquad i = 1, 2, \cdots, k-1 \quad \text{and} \quad \phi_k = \sum_{j=1}^k \lambda_i; \tag{3.2}$$

thus $\lambda_i = \phi_i \phi_k$, $i = 1, 2, \cdots, k-1$, and $\lambda_k = \phi_k - \sum_{i=1}^{k-1} \phi_i \phi_k$. H_1 is then equivalent to

$$H_1' : \phi_1 \ge \frac{\phi_1 + \phi_2}{2} \ge \cdots \ge (k-1)^{-1} \textstyle\sum_{i=1}^{k-1} \phi_i \ge k^{-1}, \tag{3.3}$$

and these restrictions do not involve ϕ_k. The likelihood function is proportional to

$$[(\textstyle\prod_{i=1}^{k-1} \phi_i^{n\bar{x}_i}) \cdot (1 - \sum_{i=1}^{k-1} \phi_i)^{n\bar{x}_k}] \cdot [e^{-n\phi_k} \cdot \phi_k^{n\sum_{i=1}^k \bar{x}_i}], \tag{3.4}$$

where $\bar{x}_i$ is the mean of the sample from the ith population. Because H_1 does not restrict ϕ_k, the two factors in brackets may be maximized independently. Using the results from Section 2, we obtain the restricted maximum likelihood estimates as follows:

$$\bar{\phi}_i = \hat{\theta}_i \vee \{i/(i+1)\}, \quad i = 1, 2, \cdots, k-1, \tag{3.5}$$

where $\hat{\phi}_i = \bar{x}_i / \sum_{j=1}^k \bar{x}_j$, and $\bar{\phi}_k = \sum_{j=1}^k \bar{x}_j = \hat{\phi}_k$. (Note that $\hat{\phi}_1, \hat{\phi}_2, \cdots, \hat{\phi}_k$ are the unrestricted maximum likelihood estimates of $\phi_1, \phi_2, \cdots, \phi_k$.) Using the invariance property of maximum likelihood estimation, we have the following theorem.

THEOREM 3.1. (Shaked, 1979). *The maximum likelihood estimates of* $\lambda_1, \lambda_2, \cdots, \lambda_k$ *subject to the restriction* H_1 *are given by*

$$\begin{aligned} \bar{\lambda}_i &= \left[\frac{\bar{x}_i}{\sum_{j=1}^k \bar{x}_j} \vee \frac{i}{i+1}\right] \cdot \textstyle\sum_{j=1}^k \bar{x}_j, \quad i = 1, 2, \cdots, k-1, \\ \bar{\lambda}_k &= (\textstyle\sum_{j=1}^k \bar{x}_j)\left[1 - \sum_{i=1}^{k-1} \left(\frac{\bar{x}_i}{\sum_{j=1}^k \bar{x}_j} \vee \frac{i}{i+1}\right)\right]. \end{aligned} \tag{3.6}$$

The likelihood ratio statistic for testing $H_0: \lambda_1 = \lambda_2 = \cdots = \lambda_k$ against the alternative $H_1 - H_0$ can be written as

$$\Lambda_{01} = \frac{(1/k)^{n\sum_{i=1}^{k}\bar{X}_i}}{\prod_{i=1}^{k-1} \bar{\phi}_i^{n\bar{X}_i}(1 - \sum_{i=1}^{k-1} \bar{\phi}_i)^{n\bar{X}_k}}. \tag{3.7}$$

If we let $S_{01} = -2 \ln \Lambda_{01}$ and let $Y = n \sum_{i=1}^{k} \bar{X}_i$, then the joint conditional distribution of $y\hat{\phi}_1, y\hat{\phi}_2, \cdots, y(1 - \sum_{j=1}^{k-1} \hat{\phi}_j)$ given $Y = y$ is multinomial with parameters y and $\phi_1, \phi_2, \cdots, 1 - \sum_{i=1}^{k-1} \phi_i$.

If we let $\boldsymbol{\lambda}_n$ (satisfying H_1) converge to $\boldsymbol{\lambda}$ ($\lambda_i > 0$ for all i) such that

$$\delta_{n,i} = \sqrt{n}\left(\sum_{j=1}^{i} \lambda_{n,j}/\sum_{j=1}^{i+1} \lambda_{n,j} - \frac{i}{i+1}\right)\left[\frac{(i+1)^3}{ik}\right]^{1/2} \to \delta_i \text{ (possibly } \infty) \tag{3.8}$$

and let $\bar{X}_{n,i}$ denote the corresponding independent sample means which occur in $S_{01}^{(n)} = -2 \ln \Lambda_{01}^{(n)}$, then using the Dominated Convergence Theorem we obtain

$$\begin{aligned} \lim_{n\to\infty} P(S_{01}^{(n)} \geq t) &= \lim_{n\to\infty} E\{P(S_{01}^{(n)} \geq t \mid Y_n)\} = E\{\lim_{n\to\infty} P(S_{01}^{(n)} \geq t \mid Y_n)\} \\ &= E\{P(U \geq t)\} = P(U \geq t), \end{aligned}$$

where U is distributed as in Theorem 2.2.

THEOREM 3.2. *Under the conditions in* (3.8), $S_{01}^{(n)}$ *is asymptotically distributed as* $T_{01}^{(n)}$ *in Theorem* 2.3. *In particular, if H_0 is true, then* $\lim_{n\to\infty} P(S_{01}^{(n)} \geq t) = \bar{\chi}_{k-1}^2(t)$.

In similar fashion, the problem of testing H_1 as a null hypothesis against "not H_1" as an alternative can be handled by conditioning on $Y = n \sum_{i=1}^{k} \bar{X}_{n,i}$ and using the multinomial results of Section 2.

In particular, if we write the likelihood ratio as

$$\Lambda_1 = \frac{(\prod_{i=1}^{k-1} \bar{\phi}_i^{n\bar{X}_i})(1 - \sum_{i=1}^{k-1} \bar{\phi}_i)^{n\bar{X}_k}}{(\prod_{i=1}^{k-1} \hat{\phi}_i^{n\bar{X}_i})(1 - \sum_{i=1}^{k-1} \hat{\phi}_i)^{n\bar{X}_k}} \tag{3.9}$$

and base our test upon $S_1 = -2 \ln \Lambda_1$, we obtain the following distributional result.

THEOREM 3.3. *Suppose $\boldsymbol{\lambda}_n$ is a vector of parameters converging to $\boldsymbol{\lambda}$ such that $\delta_{n,i}$ (defined in* (3.8)*) tends to δ_i (possibly $\pm\infty$). Then $S_1^{(n)}$ has the same asymptotic distribution as that given for $T_1^{(n)}$ in Theorem* 2.4. *Thus, the asymptotic distribution of $S_1^{(n)}$ for $\boldsymbol{\lambda}_n \equiv \boldsymbol{\lambda} \in H_1$ is given by $\lim_{n\to\infty} P_{\boldsymbol{\lambda}}(S_1^{(n)} \geq t) = \bar{\chi}_m^2(t)$ where m is the number of subscripts i such that $\sum_{j=1}^{i} \lambda_j/\sum_{j=1}^{i+1} \lambda_j = i/(i+1)$. It follows that*

$$\sup_{\boldsymbol{\lambda}\in H_1} \lim_{n\to\infty} P_{\boldsymbol{\lambda}}(S_1^{(n)} \geq t) = P_{H_0}(S_1 \geq t) = \bar{\chi}_{k-1}^2(t). \tag{3.10}$$

Note that Theorem 3.3 enables us to construct likelihood ratio tests of a particular size asymptotically when testing H_1 versus all other alternatives. Of course (3.8) and (3.10) assure us that our tests are asymptotically consistent in the sense that if $\boldsymbol{\lambda}$ is in the region of the alternate hypothesis, the probability of rejecting the null hypothesis converges to one as $n \to \infty$.

It should be noted that even though Shaked (1979) allows the more general starshaped ordering,

$$\lambda_1 \geq \sum_1^2 w_j\lambda_j/\sum_1^2 w_j \geq \sum_1^3 w_j\lambda_j/\sum_1^3 w_j \geq \cdots \geq \sum_1^k w_j\lambda_j/\sum_1^k w_j \geq 0,$$

his restriction that the sample size from the ith population be proportional to w_i effectively reduces the problem to the one considered earlier.

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