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EDGEWORTH EXPANSIONS FOR LINEAR COMBINATIONS OF ORDER STATISTICS WITH SMOOTH WEIGHT FUNCTIONS[1]

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Edgeworth expansions for linear combinations of order statistics are established. The theorems require smooth weight functions and the underlying distribution function must possess a finite fourth moment. In addition a local smoothness condition is imposed on the underlying distribution function.

1. Introduction. Statistics of the form $T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{in}$, $n \geq 1$, where X_{in} $i = 1, 2, \cdots, n$ denotes the ith order statistic of a random sample $X_1, \cdots, X_n$ of size n from a distribution with distribution function (df)F and the c_{in}, $i = 1, 2, \cdots, n$ are known real numbers (weights), are said to be linear combinations of order statistics. In the last decade there has been considerable interest in these statistics with regard to the problem of their asymptotic normality, which has been investigated under different sets of conditions by many authors in this area. We refer to the important papers of Shorack (1972) and Stigler (1974) and the references given in these papers. More recently attention has been paid to the rate of convergence problem. Berry-Esseen type bounds for linear combinations of order statistics were established by Bjerve (1977) and by the author (1977), (1980).

The purpose of this paper is to establish Edgeworth expansions for linear combinations of order statistics with remainder $o(n^{-1})$ for the case of smooth weights. Our method of proof was outlined by van Zwet (1977). In his paper he obtained a bound on the characteristic function of a linear combination of order statistics which solves a crucial part of our problem. A drawback of the approach followed in the present paper is that our results do not include trimmed means. However Bjerve (1974) has shown that trimmed means admit asymptotic expansions. In Helmers (1979) Edgeworth expansions for trimmed linear combinations of order statistics are established. The results of this paper as well as related ones are summarized in Helmers (1980).

The paper is organized as follows: in Section 2 we state our results in the form of two theorems. Section 3 contains a few preliminaries. Theorem 2.1 is proved in Section 4 and Theorem 2.2 in Section 5.

2. The Results. Let J be a bounded function on (0, 1), which is three times differentiable with first, second and third derivative J', J'' and J''' on (0, 1). Let J''' be bounded on (0, 1) and let F be a df with finite fourth moment. The inverse of a df will always be the left-continuous one. χ_E denotes the indicator of a set E. Let $\| h \| = \sup_{0<s<1} | h(s) |$ for any function h on (0, 1). Introduce functions h_1, h_2 and h_3 by

$$h_1(u) = -\int_0^1 J(s)(\chi_{(0,s]}(u) - s)\, dF^{-1}(s) \tag{2.1}$$

$$h_2(u, v) = -\int_0^1 J'(s)(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)\, dF^{-1}(s) \tag{2.2}$$

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$$h_3(u, v, w) = -\int_0^1 J''(s)(\chi_{(0,s]}(u) - s)(\chi_{(0,s]}(v) - s)(\chi_{(0,s]}(w) - s)\, dF^{-1}(s) \tag{2.3}$$

for $0 < u, v, w < 1$. Furthermore define, for each $n \geq 1$ and real x, the function K_n by

$$K_n(x) = \Phi(x) - \varphi(x)\left[\frac{\kappa_3}{6n^{1/2}}(x^2 - 1) + \frac{\kappa_4}{24n}(x^3 - 3x) + \frac{\kappa_3^2}{72n}(x^5 - 10x^3 + 15x)\right] \tag{2.4}$$

where Φ and φ denote the df and the density of the standard normal distribution. The quantities $\kappa_3 = \kappa_3(J, F)$ and $\kappa_4 = \kappa_4(J, F)$ are given by

$$\kappa_3 = \kappa_3(J, F) = \frac{1}{\sigma^3(J, F)}\left[\int_0^1 h_1^3(u)\, du + 3\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u, v)\, du\, dv\right] \tag{2.5}$$

and

$$\begin{aligned} \kappa_4 &= \kappa_4(J, F) \\ &= \frac{1}{\sigma^4(J, F)}\left[\int_0^1 h_1^4(u)\, du - 3\sigma^4(J, F)\right. \\ &\quad + 12\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u, v)\, du\, dv \\ &\quad \left. + \int_0^1\int_0^1\int_0^1 (4h_1(u)h_1(v)h_1(w)h_3(u, v, w) + 12h_1(u)h_1(v)h_2(u, w)h_2(v, w))\, du\, dv\, dw\right] \end{aligned} \tag{2.6}$$

where

$$\sigma^2 = \sigma^2(J, F) = \int_0^1 h_1^2(u)\, du. \tag{2.7}$$

In our first theorem we shall establish an asymptotic expansion with remainder $o(n^{-1})$ for the df $F_n^*(x) = P(T_n^* \leq x)$ for $-\infty < x < \infty$ where

$$T_n^* = (T_n - E(T_n))/\sigma(T_n) \tag{2.8}$$

for the case of smooth weights.

THEOREM 2.1. *Suppose that*

$$c_{in} = J\left(\frac{i}{n+1}\right) \quad \textit{for} \quad i = 1, 2, \cdots, n, \quad n = 1, 2, \cdots; \tag{2.9}$$

J is three times differentiable on $(0, 1)$ with first, second and third bounded derivative J', J'' and J''' on $(0, 1)$. Suppose further that there exists an open neighbourhood in $[0, 1]$ in which the density $f(F^{-1}(t))$ and the second derivative $f'(F^{-1}(t))$ exist and are bounded and in which $J(t)$ and $f(F^{-1}(t))$ are bounded away from zero. Suppose also that F possesses a finite fourth moment $\beta_4 = EX_1^4$. Then we have that

$$\lim_{n\to\infty} n \sup_x | F_n^*(x) - K_n(x)| = 0.$$

Our second theorem is a modification of Theorem 2.1 which lends itself better to applications. We shall establish an asymptotic expansion with remainder $o(n^{-1})$ for the df $G_n(x) = P(n^{1/2}(T_n - \mu)/\sigma \leq x)$ for $-\infty < x < \infty$ where

$$\mu = \mu(J, F) = \int_0^1 F^{-1}(s)J(s)\, ds \tag{2.10}$$

and $\sigma^2 = \sigma^2(J, F)$ as in (2.7). Introduce a function h_4 by

$$(2.11)\qquad h_4(u) = -\int_0^1 (\tfrac{1}{2} - s)J'(s)(\chi_{(0,s]}(u) - s)\, dF^{-1}(s),$$

for $0 < u < 1$. Furthermore, quantities $a = a(J, F)$ and $b = b(J, F)$ are given by

$$(2.12)\qquad a = a(J, F) = \frac{1}{\sigma(J, F)}\left[2^{-1}\int_0^1 s(1-s)J'(s)\, dF^{-1}(s) - \int_0^1 F^{-1}(s)(\tfrac{1}{2} - s)J'(s)\, ds\right]$$

and

$$(2.13)\qquad \begin{aligned} b = b(J, F) = \frac{1}{2\sigma^2(J, F)}\Bigg[&\int_0^1 (h_1(u)h_2(u, u) + 2h_1(u)h_4(u))\, du \\ &+ \int_0^1\int_0^1 (2^{-1}h_2^2(u, v) + h_1(u)h_3(u, v, v))\, du\, dv\Bigg]. \end{aligned}$$

Finally define, for each $n \geq 1$ and real x, the function L_n by

$$(2.14)\qquad L_n(x) = K_n(x) - \varphi(x)\left[-\frac{a}{n^{1/2}} + \frac{(a\kappa_3 + a^2 + 2b)}{2n}x - \frac{a\kappa_3}{6n}x^3\right].$$

THEOREM 2.2. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. Then we have that*

$$\lim_{n\to\infty} n \sup_x |G_n(x) - L_n(x)| = 0.$$

It may be useful to comment briefly on these results. In the first place we remark that it is not difficult to check from our proofs that assumption (2.9) (see Chernoff, et al., (1967) and Stigler (1974) where the same type of weights are considered) can be replaced by the weaker condition that

$$\max_{1\leq i\leq n} \left| c_{in} - n\int_{(i-1)/n}^{i/n} J(s)\, ds - \int_{(i-1)/n}^{i/n} M(s)\, ds \right| = O(n^{-\gamma}), \qquad \text{as} \quad n \to \infty,$$

for some $\gamma > \frac{3}{2}$ and smooth functions J and M ($J(M)$ must have bounded third (first) derivative on (0, 1)), provided we replace the factor $(\frac{1}{2} - s)J'(s)$ appearing in the integrands of (2.11) and (2.12) by $M(s)$. In particular this condition is satisfied in either one of the following cases: $c_{in} = J(i/n)$ (see Moore (1968)) or

$$c_{in} = n\int_{(i-1)/n}^{i/n} J(s)\, ds$$

(see Bickel (1967)) with $M(s) = \frac{1}{2}J'(s)$ and $M(s) = 0$ respectively.

Secondly we note that the assumptions required for J and F to hold on an open neighbourhood in [0, 1] are needed to ensure sufficient smoothness of F_n^* and G_n, which is what Cramér's condition (C) (see Cramér (1962)) does in the classical case of sums of independent and identically distributed random variables (cf. the proof of relation (4.2); see also van Zwet (1977)).

Next we give a numerical example which indicates that the expansions given in this paper perform well as approximations of the finite sample exact df's. It also shows that they can be much better than the usual normal approximation.

We consider the asymptotically first order efficient estimator, based on linear combinations of order statistics with weights $c_{in} = J(i/(n+1))$, of the centre θ of the logistic distribution

$$F(x) = [1 + e^{-(x-\theta)}]^{-1} \qquad \text{for} \quad -\infty < x < \infty$$

which is (see, e.g., David (1970), page 224) given by the weight function

$$J(s) = 6s(1-s), \qquad 0 < s < 1.$$

As F is symmetric about its expectation and J is symmetric about ½ we can check that in such a case $\kappa_3(J, F) = a(J, F) = 0$ (cf. (2.5) and (2.12)); i.e., there is no term of order $n^{-1/2}$ in the expansion. After long but straightforward computations we find that

$$L_n(x) = \Phi(x) - \varphi(x)\left[\frac{1}{20n}(x^3 - 3x) + \frac{(11 - \pi^2)}{n}x\right].$$

In the following table we give the numerical results. The exact df $G_n(x)$ is computed by numerical integration of the multiple integrals involved in this computation for $n = 3$ and $n = 4$ and by Monte-Carlo simulation based on 25,000 samples for $n = 10$ and $n = 25$.

The agreement between G_n and L_n is very reasonable. Already for sample size $n = 3$ the expansion performs much better than the normal approximation as approximations of the finite sample exact df's. It seems useful to add a comment on how the related Cornish-Fisher expansion improves accuracy as regards the percentiles of the exact df: for the sample size $n = 3$ the normal 95% percentile 1.6449 underestimates the actual percentile 2.1609 (obtained by a simulation based on 25,000 samples) by 24 percent. Applying the Cornish-Fisher expansion for the percentage points we get 2.2566 as our estimate for the actual percentage.

To conclude this section it may be mentioned that an important application of the asymptotic expansions established in this paper lies in the computation of asymptotic deficiencies in the sense of Hodges and Lehmann (1970) for estimators and tests based on linear combinations of order statistics. These computations are given in Helmers (1980). Here we note only (we omit the details) that in the asymmetric case the phenomenon, first noted by Pfanzagl (1979), that "first order efficiency implies second order efficiency" (see also Bickel and van Zwet (1978), p 940 and 988) also holds true for linear combinations of order statistics.

3. Preliminaries. In this section we present a few preliminary results which will be needed in our proofs.

Let, for each $n \geq 1$, $U_1, \cdots, U_n$ be independent uniform (0, 1) rv's and let $U_{in}(1 \leq i \leq n)$ denote the ith order statistic of $U_1, \cdots, U_n$. It is well known that the joint distribution of $X_1, \cdots, X_n$ is the same as that of $(F^{-1}(U_1), \cdots, F^{-1}(U_n))$ for any df F. Therefore we shall identify X_i with $F^{-1}(U_i)$ and also X_{in} with $F^{-1}(U_{in})$. The empirical df based on $U_1, \cdots, U_n$ will be denoted by Γ_n. Throughout this paper we shall assume that all rv's are defined on the same probability space (Ω, A, P). For any rv X with $0 < \sigma(X) < \infty$ we write $\hat{X} = X - \mathscr{E}(X)$ and $X^* = \hat{X}/\sigma(X)$. For any positive number l the lth absolute moment of F will be denoted by β_l. We start by stating an obvious result concerning the finiteness of certain integrals.

TABLE 1

x	G_3	L_3	G_4	L_4	G_{10}	L_{10}	G_{25}	L_{25}	Φ
0.0	.5000	.5000	.5000	.5000	.5000	.5000	.4991	.5000	.5000
0.2	.5640	.5536	.5663	.5601	.5734	.5716	.5758	.5762	.5793
0.4	.6262	.6069	.6307	.6190	.6445	.6409	.6492	.6495	.6554
0.6	.6850	.6592	.6919	.6759	.7089	.7058	.7152	.7177	.7257
0.8	.7391	.7100	.7469	.7318	.7680	.7647	.7728	.7788	.7881
1.0	.7875	.7583	.7963	.7790	.8196	.8164	.8295	.8314	.8413
1.2	.8248	.8032	.8391	.8236	.8629	.8604	.8756	.8752	.8849
1.4	.8658	.8439	.8752	.8627	.8985	.8966	.9100	.9102	.9192
1.6	.8958	.8797	.9049	.8960	.9275	.9256	.9376	.9374	.9452
1.8	.9202	.9100	.9287	.9234	.9486	.9478	.9580	.9576	.9641
2.0	.9397	.9347	.9474	.9454	.9646	.9645	.9732	.9711	.9772
2.2	.9550	.9543	.9618	.9622	.9764	.9766	.9830	.9824	.9861
2.4	.9669	.9691	.9726	.9748	.9845	.9850	.9895	.9890	.9918
2.6	.9758	.9798	.9807	.9837	.9905	.9907	.9942	.9934	.9953
2.8	.9825	.9873	.9865	.9899	.9937	.9945	.9963	.9963	.9974
3.0	.9875	.9863	.9907	.9939	.9959	.9968	.9982	.9879	.9987

LEMMA 3.1. (a) *Let l be a number > 1 and let, for some $\delta > 0$, $\beta_{l+\delta} < \infty$. Then there exists $A > 0$, depending on l and δ, such that*

$$\int_0^1 (s(1-s))^{1/l}\, dF^{-1}(s) \le A\beta_{l+\delta}^{1/(l+\delta)} < \infty; \tag{3.1}$$

(b) *If $l = 1$ and $\delta = 0$ then* (3.1) *holds with $A = 1$.*

PROOF. Applying integration by parts we obtain

$$\int_0^1 (s(1-s)^{1/l}\, dF^{-1}(s) \tag{3.2}$$
$$= (s(1-s))^{1/l}F^{-1}(s)\Big|_0^1 - l^{-1}\int_0^1 F^{-1}(s)(s(1-s))^{(1/l)-1}(1-2s)\, ds.$$

Note that under the assumptions (a) and (b) the first term on the right of (3.2) is easily seen to be zero. To conclude the proof of part (a) we apply Hölder's inequality to the second term on the right of (3.2):

$$\left| l^{-1}\int_0^1 F^{-1}(s)(s(1-s))^{(1/l)-1}\, ds \right| \le \int_0^1 |F^{-1}(s)|(s(1-s))^{(1/l)-1}\, ds \tag{3.3}$$
$$\le \beta_{l+\delta}^{1/(l+\delta)}\left(\int_0^1 (s(1-s))^{-1+\delta/l(l+\delta-1)}\, ds\right)^{(l+\delta-1)/(l+\delta)} < \infty.$$

The proof of part (b) is immediate from (3.2) and the remark made after it. This completes the proof of the lemma. □

The second lemma of this section will enable us to estimate certain moments.

LEMMA 3.2. *Let l be a positive integer and let, for some $\delta > 0$, $\beta_{l+\delta} < \infty$. Then for any number p, for which $pl \ge 2$, there exists $A > 0$ depending only on p, l and δ, such that*

$$\mathscr{E}\left(\int_0^1 |\Gamma_n(s) - s|^p\, dF^{-1}(s)\right)^l \le A\beta_{l+\delta}^{l/(l+\delta)} n^{-pl/2}. \tag{3.4}$$

PROOF. By Fubini's theorem we have

$$\mathscr{E}\left(\int_0^1 |\Gamma_n(s) - s|^p\, dF^{-1}(s)\right)^l = \int_0^1 \cdots \int_0^1 \mathscr{E}\prod_{i=1}^l |\Gamma_n(s_i) - s_i|^p\, dF^{-1}(s_1)\cdots dF^{-1}(s_l).$$

An application of Hölder's inequality shows that

$$\mathscr{E}\prod_{i=1}^l |\Gamma_n(s_i) - s_i|^p \le \prod_{i=1}^l (\mathscr{E}|\Gamma_n(s_i) - s_i|^{pl})^{1/l}$$

for all $0 < s_1, \cdots, s_l < 1$. Hence we know that

$$\mathscr{E}\left(\int_0^1 |\Gamma_n(s) - s|^p\, dF^{-1}(s)\right)^l \le \left(\int_0^1 (\mathscr{E}|\Gamma_n(s) - s|^{pl})^{1/l}\, dF^{-1}(s)\right)^l.$$

Since $\Gamma_n(s) = n^{-1}\sum_{i=1}^n \chi_{(0,s]}(U_i)$ for all $0 < s < 1$ and $n \ge 1$ the Marcinkievitz, Zygmund, Chung inequality (see Chung (1951)) yields for $pl \ge 2$, $n \ge 1$ and $0 < s < 1$

$$\mathscr{E}|\Gamma_n(s) - s|^{pl} \le Bn^{-pl/2}s(1-s)$$

where $B > 0$ depends only on p and l. It follows that

$$\mathscr{E}\left(\int_0^1 |\Gamma_n(s) - s|^p\, dF^{-1}(s)\right)^l \le Bn^{-pl/2}\left(\int_0^1 (s(1-s))^{1/l}\, dF^{-1}(s)\right)^l.$$

An application of Lemma 3.1 completes our proof. □

To formulate the third and final lemma of this section we introduce a function H by

$$H(u) = \int_0^1 |\chi_{(0,s]}(u) - s|\, dF^{-1}(s) \tag{3.5}$$

for $0 < u < 1$.

LEMMA 3.3. (a) *Let $l \geq 1$ and suppose that $\beta_l < \infty$. Then $\mathscr{E} H^l(U_1) < \infty$.*

(b). *Let $\beta_1 < \infty$ and suppose that J'' is bounded on $(0, 1)$. Then $\mathscr{E} h_1(U_i) = 0$ for any i and with probability one $\mathscr{E}(h_2(U_i, U_j) | U_j) = 0$ for $i \neq j$ and $\mathscr{E}(h_3(U_i, U_j, U_k) | U_j, U_k) = 0$ if $i \neq j$ and $i \neq k$.*

PROOF. (a) It is immediate from (3.5) and the c_r – inequality that

$$\mathscr{E} H^l(U_1) \leq 2^{l-1}\left[\mathscr{E}\left(\int_{(0,U_1)} s\, dF^{-1}(s)\right)^l + \mathscr{E}\left(\int_{(U_1,1)} (1-s)\, dF^{-1}(s)\right)^l\right]. \tag{3.6}$$

Using integration by parts, the finiteness of β_l and applying the c_r – inequality once more we easily check that the moments appearing on the right-hand side of (3.6) are both finite. This proves part (a) of the lemma. To see that part (b) of the lemma is also true we shall only prove the last statement of part (b). The other two statements are easier and can be proved similarly. Using Fubini's theorem and applying part (a) of the lemma we see that with probability one

$$\mathscr{E}\left(\int_0^1 |J''(s)|\, |\chi_{(0,s]}(U_i) - s|\, |\chi_{(0,s]}(U_j) - s|\, |\chi_{(0,s]}(U_k) - s|\, dF^{-1}(s) |\, U_j, U_k\right)$$
$$\leq \| J'' \| \cdot \mathscr{E} H(U_1) < \infty.$$

Therefore the conditional expectation $\mathscr{E}(h_3(U_i, U_j, U_k) | U_j, U_k)$ is well-defined and Fubini's theorem can be applied to see that $\mathscr{E}(h_3(U_i, U_j, U_k) | U_j, U_k) = 0$ with probability one. □

4. Proof of Theorem 2.1. Since our proofs will depend on characteristic function (c.f.) arguments let us denote by $\rho_n^*(t)$ the c.f. of T_n^* and by $\tilde{\rho}_n(t)$ the Fourier-Stieltjes transform $\tilde{\rho}_n(t) = \int_{-\infty}^{\infty} \exp(itx)\, dK_n(x)$ of K_n (see (2.4)).

We shall show that for some sufficiently small $\epsilon > 0$

$$\int_{|t| \leq n^\epsilon} |\rho_n^*(t) - \tilde{\rho}_n(t)|\, |t|^{-1}\, dt = o(n^{-1}) \tag{4.1}$$

and that

$$\int_{n^\epsilon \leq |t| \leq n^{3/2}} |\rho_n^*(t)|\, |t|^{-1}\, dt = o(n^{-1}) \tag{4.2}$$

and

$$\int_{|t| > \log(n+1)} |\tilde{\rho}_n(t)|\, |t|^{-1}\, dt = o(n^{-1}) \tag{4.3}$$

hold as $n \to \infty$. An application of Esseen's smoothness lemma (Esseen (1945)) will then complete our proof.

We first prove (4.1). We shall essentially have to expand $\rho_n^*(t)$ for these "small" values of $|t|$. To start with we define for $0 < u < 1$

$$\psi_1(u) = \int_u^1 J(s)\, ds - (1-u)\bar{J}_1 \tag{4.4}$$

and

$$\psi_2(u) = \int_u^1 (\tfrac{1}{2} - s)J'(s)\,ds - (1-u)\bar{J}_2$$

where $\bar{J}_1 = \int_0^1 J(s)ds$ and $\bar{J}_2 = \int_0^1 (\tfrac{1}{2} - s)J'(s)\,ds$. Then it is easy to check (see Shorack (1972) for a similar approach) that with probability one

$$\begin{aligned} T_n = &\int_0^1 (\psi_1(\Gamma_n(s)) + n^{-1}\psi_2(\Gamma_n(s)))\,dF^{-1}(s) + (\bar{J}_1 + n^{-1}\bar{J}_2)n^{-1}\textstyle\sum_{i=1}^n F^{-1}(U_i) \\ &+ n^{-1}\textstyle\sum_{i=1}^n \left(c_{in} - n\int_{(i-1)/n}^{i/n} J(s)\,ds - \int_{(i-1)/n}^{i/n} (\tfrac{1}{2}-s)J'(s)\,ds\right)F^{-1}(U_{in}). \end{aligned} \tag{4.5}$$

Introduce, for each $n \geq 1$, the rv S_n by (a prime denoting differentiation),

$$\begin{aligned} S_n = &\int_0^1 \Big\{ \psi_1(s) + n^{-1}\psi_2(s) + (\Gamma_n(s) - s)(\psi_1'(s) + n^{-1}\psi_2'(s)) \\ &+ \frac{(\Gamma_n(s)-s)^2}{2}\psi_1''(s) + \frac{(\Gamma_n(s)-s)^3}{6}\psi_1'''(s)\Big\}\,dF^{-1}(s) \\ &+ (\bar{J}_1 + n^{-1}\bar{J}_2)n^{-1}\textstyle\sum_{i=1}^n F^{-1}(U_i). \end{aligned} \tag{4.6}$$

Note that $|\psi_1(u)| \leq 4\|J\|u(1-u)$ and $|\psi_2(u)| \leq 4\|J'\|u(1-u)$ for $0 < u < 1$, and that $\psi_1'(s) = -J(s) + \bar{J}_1$, $\psi_2'(s) = (s - \tfrac{1}{2})J'(s) + \bar{J}_2$, $\psi_1''(s) = -J'(s)$ and $\psi_1'''(s) = -J''(s)$ on $(0, 1)$ so that it is easily verified that S_n is a well-defined rv.

We introduce some more notation. Define rv's I_{mn} for $m = 1, 2, 3, 4$ and $n \geq 1$ by

$$I_{1n} = -\int_0^1 J(s)(\Gamma_n(s) - s)\,dF^{-1}(s) = n^{-1}\textstyle\sum_{i=1}^n h_1(U_i) \tag{4.7}$$

$$I_{2n} = -\int_0^1 J'(s)\frac{(\Gamma_n(s)-s)^2}{2}\,dF^{-1}(s) = 2^{-1}n^{-2}\textstyle\sum_{i=1}^n\sum_{j=1}^n h_2(U_i, U_j) \tag{4.8}$$

$$I_{3n} = -\int_0^1 J''(s)\frac{(\Gamma_n(s)-s)^3}{6}\,dF^{-1}(s) = 6^{-1}n^{-3}\textstyle\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n h_3(U_i, U_j, U_k) \tag{4.9}$$

$$I_{4n} = -n^{-1}\int_0^1 (\tfrac{1}{2} - s)J'(s)(\Gamma_n(s) - s)\,dF^{-1}(s) = n^{-2}\textstyle\sum_{i=1}^n h_4(U_i) \tag{4.10}$$

where the functions h_1, h_2, h_3 and h_4 are given by (2.1), (2.2), (2.3) and (2.11). It is easily checked that

$$\hat{S}_n = S_n - \mathscr{E}S_n = \textstyle\sum_{m=1}^4 \hat{I}_{mn} = \sum_{m=1}^4 (I_{mn} - \mathscr{E}I_{mn}). \tag{4.11}$$

Furthermore define rv's J_{mn} for $m = 1, 2, 3, 4$ and $n \geq 1$ by

$$J_{mn} = \hat{I}_{mn}/\sigma(S_n) = (I_{mn} - \mathscr{E}I_{mn})/\sigma(S_n), \tag{4.12}$$

so that

$$S_n^* = \textstyle\sum_{m=1}^4 J_{mn}. \tag{4.13}$$

The proof of (4.1) will now be split up in a number of steps. We begin by deriving an asymptotic expansion for the variance of S_n; i.e., we shall first prove

$$|\sigma^2(S_n) - n^{-1}\sigma^2 - 2n^{-2}\sigma^2 b| = O(n^{-5/2}) \quad \text{as} \quad n \to \infty \tag{4.14}$$

with $\sigma^2 = \sigma^2(J, F)$ and $b = b(J, F)$ as in (2.7) and (2.13). To see this we first note that (cf. (4.11)) $\sigma^2(S_n) = \sigma^2(\sum_{m=1}^4 I_{mn})$. It follows directly from (4.7) and (2.7) that $\sigma^2(I_{1n}) = n^{-1}\sigma^2$.

Also note that it is immediate from (4.7), (4.8) and an application of Lemma 3.3 that

$$2\,\mathrm{Cov}(I_{1n}, I_{2n}) = 2\mathscr{E} I_{1n} I_{2n} = n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathscr{E} h_1(U_i) h_2(U_j, U_k)$$
$$= n^{-2} \int_0^1 h_1(u) h_2(u, u)\, du.$$

Next we consider $\sigma^2(I_{2n})$. Using Lemma 3.1 and Lemma 3.3 once more we directly find that

$$\mathscr{E} I_{2n}^2 = 4^{-1} n^{-2} (\mathscr{E} h_2(U_1, U_1))^2 + 2^{-1} n^{-1} \mathscr{E} h_2^2(U_1, U_2) + O(n^{-3}) \quad \text{as} \quad n \to \infty.$$

Because we also know that $(\mathscr{E} I_{2n})^2 = 4^{-1} n^{-2} (\mathscr{E} h_2(U_1, U_1))^2$ we have shown that

$$\sigma^2(I_{2n}) = 2^{-1} n^{-2} \int_0^1 \int_0^1 h_2^2(u, v)\, du\, dv + O(n^{-3}) \quad \text{as} \quad n \to \infty.$$

Similarly we can prove that

$$2\,\mathrm{Cov}(I_{1n}, I_{3n}) = n^{-2} \int_0^1 \int_0^1 h_1(u) h_3(u, v, v)\, du\, dv + O(n^{-3}) \quad \text{as} \quad n \to \infty$$

and also that

$$2\,\mathrm{Cov}(I_{1n}, I_{4n}) = 2n^{-2} \int_0^1 h_1(u) h_4(u)\, du.$$

Finally we remark that we can prove using similar arguments as above that

$$\sigma^2(I_{3n}) + \sigma^2(I_{4n}) = O(n^{-3}) \quad \text{as} \quad n \to \infty$$

and also that

$$|\mathrm{Cov}(I_{2n}, I_{3n}) + \mathrm{Cov}(I_{2n}, I_{4n}) + \mathrm{Cov}(I_{3n}, I_{4n})| = O(n^{-5/2}) \quad \text{as} \quad n \to \infty.$$

Combining all these results we have proved (4.14). It follows directly from (4.14) that

$$(4.15) \qquad |\sigma^{-m}(S_n) - n^{m/2} \sigma^{-m}| = O(n^{m/2-1}) \quad \text{as} \quad n \to \infty$$

with $\sigma^2 = \sigma^2(J, F)$ as in (2.7).

To proceed we shall prove that $T_n^* - S_n^*$ is of negligible order for our purposes. Let τ_n^* denote the c.f. of S_n^*. We shall show that for every $\varepsilon > 0$

$$(4.16) \qquad \int_{|t| \le n^{\varepsilon}} |\rho_n^*(t) - \tau_n^*(t)|\, |t|^{-1}\, dt = O(n^{-3/2+\varepsilon}) \quad \text{as} \quad n \to \infty.$$

To prove this we first use Lemma X.V.4.1 of Feller (1966) to see that

$$(4.17) \qquad |\rho_n^*(t) - \tau_n^*(t)| \le |t|\, \mathscr{E} |T_n^* - S_n^*|$$

for all t and $n \ge 1$. Using (4.5), (4.6), the boundedness of J''' on (0, 1) and applying Taylor's theorem we see directly that

$$(4.18) \qquad \begin{aligned} \sigma^2(T_n - S_n) \le\ & 3 \| J''' \|^2 \mathscr{E} \left(\int_0^1 (\Gamma_n(s) - s)^4\, dF^{-1}(s) \right)^2 \\ & + 3 \| J'' \|^2 n^{-2} \mathscr{E} \left(\int_0^1 (\Gamma_n(s) - s)^2\, dF^{-1}(s) \right)^2 \\ & + 3\sigma^2 \Bigg(n^{-1} \sum_{i=1}^{n} \Bigg(J\left(\frac{i}{n+1}\right) - n \int_{(i-1)/n}^{i/n} J(s)\, ds \\ & - \int_{(i-1)/n}^{i/n} \left(\frac{1}{2} - s\right) J'(s)\, ds \Bigg) F^{-1}(U_{in}) \Bigg). \end{aligned}$$

Applications of Lemma 3.2 with $l = 2$ and $p = 2$ and $p = 4$ respectively implies that the sum of the first two terms on the right of (4.18) is $O(n^{-4})$, as $n \to \infty$. To treat the third term on the right of (4.18) we need the following simple inequality: $\sigma^2 (\sum_{i=1}^n a_i X_{in}) \le \sigma^2 (\sum_{i=1}^n b_i X_{in})$, provided $a_i a_j \le b_i b_j$ for all $1 \le i, j \le n$. (cf. Helmers (1977), page 943). Using this and the fact that it is easily verified that

$$(4.19)\quad \max_{1\le i\le n} \left| J\left(\frac{i}{n+1}\right) - n\int_{(i-1)/n}^{i/n} J(s)\,ds - \int_{(i-1)/n}^{i/n} \left(\frac{1}{2} - s\right) J'(s)\,ds \right| = O(n^{-2}) \quad \text{as} \quad n \to \infty$$

we find that the third term on the right of (4.18) is $O(n^{-5})$ as $n \to \infty$. Combining these results it is easy to conclude that

$$(4.20)\qquad \sigma^2(T_n - S_n) = O(n^{-4}) \qquad \text{as} \quad n \to \infty.$$

To complete our proof of (4.16) we remark that it follows now from (4.20), (4.15) (with $m = -2$) and the fact that the conditions of Theorem 2.1 can be shown to imply the positivity of $\sigma^2(J, F)$ that $\sigma^2(T_n^* - S_n^*) = O(n^{-3})$ as $n \to \infty$. This combined with (4.17) proves (4.16).

Next we define for real t and $n \ge 1$

$$(4.21)\qquad \tau_{1n}(t) = \mathscr{E} e^{itJ_{1n}}(1 + it(J_{2n} + J_{3n} + J_{4n}) + \frac{(it)^2}{2} J_{2n}^2).$$

We shall show that τ_n^* can be approximated by τ_{1n} for all $| t \le n^\varepsilon$: i.e., for every $\varepsilon > 0$

$$(4.22)\qquad \int_{|t|\le n^\varepsilon} | \tau_n^*(t) - \tau_{1n}(t) | \, |t|^{-1}\,dt = O(n^{-3/2+3\varepsilon}) \qquad \text{as} \quad n \to \infty.$$

To prove this we first use Lemma X.V.4.1 of Feller (1966) once more to find that

$$| \tau_n^*(t) - \tau_{1n}(t)| = \left| \mathscr{E} e^{itJ_{1n}}\left(e^{it(J_{2n}+J_{3n}+J_{4n})} - 1 - it(J_{2n} + J_{3n} + J_{4n}) - \frac{(it)^2}{2} J_{2n}^2\right)\right|$$

$$\le t^2(\mathscr{E}\,| J_{2n} J_{3n} | + \mathscr{E}\,| J_{2n} J_{4n} | + \mathscr{E}\,| J_{3n} J_{4n} | + \mathscr{E} J_{3n}^2 + \mathscr{E} J_{4n}^2) + |t|^3 \mathscr{E}\,| J_{2n} + J_{3n} + J_{4n} |^3,$$

for all t and $n \ge 1$. It is not difficult to verify from the proof of (4.14) and from (4.15) that the coefficient of t^2 in the above inequality is $O(n^{-3/2})$ as $n \to \infty$. An application of the c_r – inequality Lemma 3.2 with $l = 3$ and $p = 2$, 3 and 4 respectively and of (4.15) shows that also $\mathscr{E}\,| J_{2n} + J_{3n} + J_{4n} |^3 = O(n^{-3/2})$ as $n \to \infty$. Combining these results we can check that (4.22) is proved.

We continue with the analysis of $\tau_{1n}(t)$. For convenience we write σ_n^2 to indicate $n\sigma^2(S_n)$ and we denote the c.f. of $h_1(U_1)$ by ρ. To start with we remark that it follows from (4.21) that

$$\tau_{1n}(t) = \rho^n\left(\frac{t}{n^{1/2}\sigma_n}\right)$$
$$+ \frac{it}{2n^{3/2}\sigma_n} \rho^{n-2}\left(\frac{t}{n^{1/2}\sigma_n}\right) n(n-1)\,\mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right)$$
$$\cdot h_2(U_1, U_2)$$
$$+ \frac{it}{2n^{3/2}\sigma_n} \rho^{n-1}\left(\frac{t}{n^{1/2}\sigma_n}\right) n\,\mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1)\right) \hat{h}_2(U_1, U_1)$$
$$+ \frac{it}{6n^{5/2}\sigma_n} \rho^{n-3}\left(\frac{t}{n^{1/2}\sigma_n}\right) n(n-1)(n-2)$$
$$\cdot\,\mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2) + h_1(U_3))\right) \cdot h_3(U_1, U_2, U_3)$$
$$+ \frac{it}{6n^{5/2}\sigma_n} \rho^{n-2}\left(\frac{t}{n^{1/2}\sigma_n}\right) 3n(n-1)$$

$$
\begin{aligned}
&\cdot \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2))\right)\cdot h_3(U_1, U_1, U_2)\\
&+\frac{it}{6n^{5/2}\sigma_n}\rho^{n-1}\left(\frac{t}{n^{1/2}\sigma_n}\right) n\,\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}h_1(U_1)\right)\hat{h}_3(U_1, U_1, U_1)\\
&+\frac{it}{n^{3/2}\sigma_n}\rho^{n-1}\left(\frac{t}{n^{1/2}\sigma_n}\right) n\,\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}h_1(U_1)\right)h_4(U_1)\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-4}\left(\frac{t}{n^{1/2}\sigma_n}\right) n(n-1)(n-2)(n-3)\\
&\cdot\left(\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2))\right)h_2(U_1, U_2)\right)^2\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-3}\left(\frac{t}{n^{1/2}\sigma_n}\right)4n(n-1)(n-2)\\
&\cdot\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))\right)h_2(U_1, U_2)h_2(U_1, U_3)\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-3}\left(\frac{t}{n^{1/2}\sigma_n}\right)2n(n-1)(n-2)\\
&\cdot\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))\right)\hat{h}_2(U_1, U_1)h_2(U_2, U_3)\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-2}\left(\frac{t}{n^{1/2}\sigma_n}\right)4n(n-1)\\
&\cdot\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2))\right)\hat{h}_2(U_1, U_1)\cdot h_2(U_1, U_2)\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-2}\left(\frac{t}{n^{1/2}\sigma_n}\right)2n(n-1)\,\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1)+h_1(U_2))\right)\\
&\cdot(h_2(U_1, U_2))^2\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-2}\left(\frac{t}{n^{1/2}\sigma_n}\right)n(n-1)\left(\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}h_1(U_1)\right)\hat{h}_2(U_1, U_1)\right)^2\\
&+\frac{(it)^2}{8n^3\sigma_n^2}\rho^{n-1}\left(\frac{t}{n^{1/2}\sigma_n}\right)n\,\mathscr{E}\exp\left(\frac{it}{n^{1/2}\sigma_n}h_1(U_1)\right)(\hat{h}_2(U_1, U_1))^2.
\end{aligned}
\tag{4.23}
$$

To proceed we first derive an asymptotic expansion for the factors $\rho^{n-m}(t/n^{1/2}\sigma_n)$ appearing in the terms on the right of (4.23). Since $\sigma^{-1}(n-m)^{-1/2}\sum_{i=1}^{n-m}h_1(U_i)$ is a properly standardized sum of independent identically distributed rv's with expectation zero, variance one, and finite fourth moment, an expansion with remainder $o(n^{-1})$ for $\rho^{n-m}(t/(n-m)^{1/2}\sigma)$ for $|t|\le an^{1/2}$ for some $a>0$ follows directly from the classical theory of Edgeworth expansions for such sums (see, e.g., Gnedenko-Kolmogorov (1954), Section 41, Theorem 2.1, inequality (b)). We perform a change of variables $t_n = tn^{1/2}\sigma_n/((n-m)^{1/2}\sigma)$. It follows after expanding $e^{-t_n^2/2}$ around t and using (4.14) that we obtain for some $a>0$ and all $n\ge 1$, uniformly for all $|t|\le an^{1/2}$

$$
\begin{aligned}
&\left|\rho^{n-m}\left(\frac{t}{n^{1/2}\sigma_n}\right) - e^{-t^2/2}\left(1-\frac{(it)^2}{n}\left(\frac{m}{2}+b\right)\right.\right.\\
&\left.\left.+\frac{(it)^3\int_0^1 h_1^3(u)\,du}{6n^{1/2}\sigma^3}+\frac{(it)^4\left(\int_0^1 h_1^4(u)\,du-3\sigma^4\right)}{24n\sigma^4}+\frac{(it)^6\left(\int_0^1 h_1^3(u)\,du\right)^2}{72n\sigma^6}\right)\right|\\
&= o(n^{-1}|t|\,P(t)e^{-t^2/4}),\qquad \text{as}\quad n\to\infty.
\end{aligned}
\tag{4.24}
$$

where P is a fixed polynomial in t, $\sigma^2 = \sigma^2(J, F)$ is as in (2.7) and $b = b(J, F)$ as in (2.13).

Next we expand the expectations appearing on the right of (4.23). We shall show that uniformly for all t:

$$\begin{aligned}&\Bigg| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right) h_2(U_1, U_2) \\ &\quad - \frac{(it)^2}{n\sigma^2} \int_0^1 \int_0^1 h_1(u)h_1(v)h_2(u, v)\, du\, dv \\ &\quad - \frac{(it)^3}{n^{3/2}\sigma^3} \int_0^1 \int_0^1 h_1^2(u)h_1(v)h_2(u, v)\, du\, dv \Bigg| = O(n^{-2}(t^2 + t^4) + n^{-5/2} |t|^3).\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1)\right) \hat{h}_2(U_1, U_1) - \frac{it}{n^{1/2}\sigma} \int_0^1 h_1(u)h_2(u, u)\, du \right| \\ &\qquad = O(n^{-1}t^2 + n^{-3/2}|t|).\end{aligned} \tag{4.26}$$

$$\begin{aligned}&\Bigg| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2) + h_1(U_3))\right) h_3(U_1, U_2, U_3) \\ &\quad - \frac{(it)^3}{n^{3/2}\sigma^3} \int_0^1 \int_0^1 \int_0^1 h_1(u)h_1(v)h_1(w)h_3(u, v, w)\, du\, dv\, dw \Bigg| = O(n^{-2}t^4 + n^{-5/2}|t|^3).\end{aligned} \tag{4.27}$$

$$\begin{aligned}&\Bigg| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right) h_3(U_1, U_1, U_2) \\ &\quad - \frac{it}{n^{1/2}\sigma} \int_0^1 \int_0^1 h_1(u)h_3(u, v, v)\, du\, dv \Bigg| = O(n^{-1}t^2 + n^{-3/2}|t|).\end{aligned} \tag{4.28}$$

$$\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1)\right) \hat{h}_3(U_1, U_1, U_1) \right| = O(n^{-1/2}|t|). \tag{4.29}$$

$$\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1)\right) h_4(U_1) - \frac{it}{n^{1/2}\sigma} \int_0^1 h_1(u)h_4(u)\, du \right| = O(n^{-1}t^2 + n^{-3/2}|t|). \tag{4.30}$$

$$\begin{aligned}&\Bigg| \left(\mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right) h_2(U_1, U_2) \right)^2 \\ &\quad - \frac{(it)^4}{n^2\sigma^4} \left(\int_0^1 \int_0^1 h_1(u)h_1(v)h_2(u, v)\, du\, dv \right)^2 \Bigg| = O(n^{-5/2}|t|^5 + n^{-3}t^4).\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\Bigg| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2) + h_1(U_3))\right) h_2(U_1, U_2)h_2(U_1, U_3) \\ &\quad - \frac{(it)^2}{n\sigma^2} \int_0^1 \int_0^1 \int_0^1 h_1(u)h_1(v)h_2(u, w)h_2(v, w)\, du\, dv\, dw \Bigg| = O(n^{-3/2}|t|^3 + n^{-2}t^2).\end{aligned} \tag{4.32}$$

$$\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2) + h_1(U_3))\right) \hat{h}_2(U_1, U_1)h_2(U_2, U_3) \right| = O(n^{-3/2}|t|^3). \tag{4.33}$$

$$\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right) \hat{h}_2(U_1, U_1)h_2(U_1, U_2) \right| = O(n^{-1/2}|t|). \tag{4.34}$$

$$\begin{aligned}&\Bigg| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n}(h_1(U_1) + h_1(U_2))\right) (h_2(U_1, U_2))^2 \\ &\quad - \int_0^1 \int_0^1 h_2^2(u, v)\, du\, dv \Bigg| = O(n^{-1/2}|t|).\end{aligned} \tag{4.35}$$

$$\left| \left(\mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1) \right) \hat{h}_2(U_1, U_1) \right)^2 \right| = O(n^{-1}t^2). \tag{4.36}$$

$$\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} h_1(U_1) \right) (\hat{h}_2(U_1, U_1))^2 \right| = O(1), \quad \text{as} \quad n \to \infty. \tag{4.37}$$

Because the statements (4.25)—(4.37) are all proved in essentially the same manner we shall only prove, by way of an example, (4.25). Expanding $\exp((it/n^{1/2}\sigma_n)(h_1(U_1) + h_1(U_2)))$ around $t = 0$ and applying Lemma 3.3 we find for all t

$$\begin{aligned} &\left| \mathscr{E} \exp\left(\frac{it}{n^{1/2}\sigma_n} (h_1(U_1) + h_1(U_2)) \right) h_2(U_1, U_2) - \frac{(it)^2}{n\sigma_n^2} \int_0^1 \int_0^1 h_1(u)h_1(v)h_2(u, v)\, du\, dv \right. \\ &\left. - \frac{(it)^3}{n^{3/2}\sigma_n^3} \int_0^1 \int_0^1 h_1^2(u)h_1(v)h_2(u, v)\, du\, dv \right| \le \frac{t^4}{n^2\sigma_n^4} \mathscr{E} \,|h_1(U_1) + h_1(U_2)|^2 |h_2(U_1, U_2)|. \end{aligned} \tag{4.38}$$

Application of Lemma 3.3. (a) (note that $|h_2(U_1, U_2)| \le \| J' \| \cdot H(U_i)$ for $i = 1, 2$) shows that the term on the right of (4.38) is $O(n^{-2}\sigma_n^{-4}t^4)$. Since (4.15) implies that $\sigma_n^{-1} = \sigma^{-1} + O(n^{-1})$ we have proved (4.25).

We are now in a position to prove (4.1). We first apply (4.16) with $0 < \varepsilon < \frac{1}{2}$ to see that the integral on the left of (4.16) is $o(n^{-1})$, as $n \to \infty$. Secondly we use (4.22) with $0 < \varepsilon < \frac{1}{6}$ to find that the integral on the left of (4.22) is also $o(n^{-1})$. To proceed let us note that we can write down $\tilde{\rho}_n(t)$ explicitly as

$$\tilde{\rho}_n(t) = e^{-t^2/2}\left(1 - \frac{it^3\kappa_3}{6n^{1/2}} + \frac{3\kappa_4 t^4 - \kappa_3^2 t^6}{72n} \right). \tag{4.39}$$

Next we apply (4.39) and (4.23) – (4.37) to check that

$$\int_{|t| \le an^{1/2}} |\tau_{1n}(t) - \tilde{\rho}_n(t)| \, |t|^{-1}\, dt = o(n^{-1}) \tag{4.40}$$

with a as in (4.24). Hence we can conclude that (4.1) holds for $0 < \varepsilon < \frac{1}{6}$.

Next we consider (4.2) and (4.3). To prove (4.2) we remark first that application of Theorem 4.1 of van Zwet (1977) shows that his bound applies to our situation, under the conditions of Theorem 2.1. It is also clear from van Zwet (1977) that the only missing ingredient to complete the proof of (4.2) is the requirement that there exist positive numbers e and E such that $e \le n^{1/2}\sigma(T_n) \le E$ for all $n \ge 1$. To see this we first use (4.15) and the positivity of $\sigma^2 = \sigma^2(J, F)$ to see that $n^{1/2}\sigma(S_n)$ is bounded away from zero and infinity and then apply (4.20). This proves (4.2). To see that (4.3) is also true we simply use (4.40) and the fact that κ_3 and κ_4 are finite under the assumptions of Theorem 2.1. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2. To start with we remark that for each $n \ge 1$ and real x

$$G_n(x) = F_n^*(x\sigma n^{-1/2}\sigma^{-1}(T_n) + (\mu - \mathscr{E}(T_n))\sigma^{-1}(T_n)). \tag{5.1}$$

Using this identity and applying Theorem 2.1 we find that

$$\sup_x |G_n(x) - K_n(x\sigma n^{-1/2}\sigma^{-1}(T_n) + (\mu - \mathscr{E}(T_n))\sigma^{-1}(T_n))| = o(n^{-1}) \quad \text{as} \quad n \to \infty. \tag{5.2}$$

To proceed we need expansions for $\sigma n^{-1/2}\sigma^{-1}(T_n)$ and $(\mu - \mathscr{E}(T_n))\sigma^{-1}(T_n)$. We shall first prove that

$$|\sigma n^{-1/2}\sigma^{-1}(T_n) - 1 + bn^{-1}| = O(n^{-3/2}) \quad \text{as} \quad n \to \infty \tag{5.3}$$

with $b = b(J, F)$ as in (2.13). Application of (4.15), (4.20) and the Cauchy-Schwarz inequality yields

$$\frac{\sigma^2}{n\sigma^2(T_n)} = \frac{\sigma^2}{n\sigma^2(S_n)} (1 + O(n^{-3/2})) \quad \text{as} \quad n \to \infty. \tag{5.4}$$

Relation (4.14) implies that

$$\frac{\sigma^2}{n\sigma^2(S_n)} = 1 - 2\frac{b}{n} + O(n^{-3/2}) \quad \text{as} \quad n \to \infty. \tag{5.5}$$

Combining (5.4) and (5.5) we find

$$\frac{\sigma^2}{n\sigma^2(T_n)} = 1 - 2\frac{b}{n} + O(n^{-3/2}) \quad \text{as} \quad n \to \infty, \tag{5.6}$$

from which (5.3) is immediate. Next we shall show that

$$|(\mu - \mathscr{E}(T_n))\sigma^{-1}(T_n) - an^{-1/2}| = O(n^{-3/2}) \quad \text{as} \quad n \to \infty, \tag{5.7}$$

with $a = a(J, F)$ as in (2.12). We first use (4.5), (4.6) and Taylor's theorem to find that

$$\begin{aligned} \mathscr{E}|T_n - S_n| = O\Big(\mathscr{E}\int_0^1 |\Gamma_n(s) - s|^4\, dF^{-1}(s) \\ + n^{-1}\mathscr{E}\int_0^1 |\Gamma_n(s) - s|^2\, dF^{-1}(s) + n^{-2}\mathscr{E}|X_1|\Big) \end{aligned} \tag{5.8}$$

as $n \to \infty$. Application of Lemma 3.2 implies that the first term on the right of (5.8) is $O(n^{-2})$ as $n \to \infty$. To treat the second term on the right of (5.8) we first note that this term is at most $n^{-1}(\mathscr{E}(\int_0^1 |\Gamma_n(s) - s^2|\, dF^{-1}(s))^2)^{1/2}$ and then we apply Lemma 3.2 once more to find that this term is $O(n^{-2})$ as $n \to \infty$. Combining these results we obtain

$$\mathscr{E}T_n = \mathscr{E}S_n + O(\mathscr{E}|T_n - S_n|) = \mathscr{E}S_n + O(n^{-2}). \tag{5.9}$$

Using now the definition of S_n (cf. (4.6)) and noting that $\mathscr{E}(\Gamma(s) - s)^3 = n^{-2}s(1 - s)(1 - 2s)$ we can easily check that

$$\mathscr{E}S_n = \mu - a\sigma n^{-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty \tag{5.10}$$

so that (5.9) implies that

$$\mu - \mathscr{E}T_n = a\sigma n^{-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Because (5.6) directly implies that

$$\sigma^{-1}(T_n) = n^{1/2}\sigma^{-1} + O(n^{-1/2}) \quad \text{as} \quad n \to \infty \tag{5.12}$$

we have proved (5.7). To complete now the proof of Theorem 2.2 we use (2.4), (2.14), (5.3), (5.7) and apply a Taylor expansion argument to find that

$$K_n(xn^{-1/2}\sigma^{-1}(T_n)\sigma + (\mu - \mathscr{E}(T_n))\sigma^{-1}(T_n)) = L_n(x) + O(n^{-3/2}) \quad \text{as} \quad n \to \infty \tag{5.13}$$

uniformly in x. Combining this with (5.2) completes the proof of Theorem 2.2.

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