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INVARIANT TESTS FOR MEANS WITH COVARIATES[1]

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We consider the problem of testing a hypothesis about the means of a subset of the components of a multivariate normal distribution with unknown covariance matrix, when the means of a second subset (the covariates) are known. Because of the possible correlation between the two subsets, information provided by the second subset can be useful for inferences about the means of the first subset. In this paper attention is restricted to the class of procedures invariant under the largest group of linear transformations which leaves the problem invariant. The family of tests which are admissible within this class is characterized. This family excludes several well-known tests, thereby proving them to be inadmissible (among all tests), while the admissibility (among invariant tests) of other tests is demonstrated. The powers of the likelihood ratio test LRT, the $D_{p+q}^2 - D_q^2$ test, and the overall T^2 test are compared numerically; the LRT is deemed preferable on the basis of power and simplicity.

1. Introduction. Tests for the equality of the mean vectors of two multivariate normal populations when covariates are present have received considerable attention (e.g., Cochran and Bliss (1948), Rao (1946, 1949), Olkin and Shrikhande (1954), Stein (1952), Giri (1961, 1962, 1968), Cochran (1964), Subrahmaniam (1971), Subrahmaniam and Subrahmaniam (1973), Koziol (1978)). By considering the difference of the two sample mean vectors and the pooled covariance matrix, the problem can be studied in the following one-population form: observe (X, S), independently distributed, where X: $(p + q) \times 1$ and S: $(p + q) \times (p + q)$ have the multivariate normal and Wishart distributions, respectively, i.e.,

$$X \sim N_{p+q}(\mu, \Sigma) \qquad \text{and} \qquad S \sim W_{p+q}(n, \Sigma).$$

Assume $n \geqslant p + q$ and Σ nonsingular, insuring that S is nonsingular with probability one. Partition μ and Σ as

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where μ_1: $p \times 1$, μ_2: $q \times 1$, Σ_{11}: $p \times p$, Σ_{22}: $q \times q$, and partition X and S similarly. Consider the problem of testing

$$\text{(1.1)} \quad \begin{aligned} &H_0: \mu_1 = 0, \quad \mu_2 = 0, \quad \Sigma \text{ unspecified} \\ &\text{versus} \\ &H_1: \mu_1 \neq 0, \quad \mu_2 = 0, \quad \Sigma \text{ unspecified}, \end{aligned}$$

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based on (X, S). The components of X_2, having known means, are covariates.

Define the statistics L and M and the parameter Δ by

$$L = \frac{(X_1 - S_{12}S_{22}^{-1}X_2)'(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(X_1 - S_{12}S_{22}^{-1}X_2)}{1 + X_2'S_{22}^{-1}X_2},$$

$$M = X_2'S_{22}^{-1}X_2,$$

$$\Delta = \mu_1'(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\mu_1.$$

The range of the pair (L, M) is $(0, \infty) \times (0, \infty)$, while $0 \leqslant \Delta < \infty$. Since $\mu_2 = 0$, the joint distribution of (L, M) is given by

$$L|M \sim \frac{\chi^2_{\nu_1}\left(\frac{\Delta}{1 + M}\right)}{\chi^2_{\nu_2}}, \qquad M \sim \frac{\chi^2_q}{\chi^2_{n-q+1}}, \tag{1.2}$$

where

$$\nu_1 = p, \qquad \nu_2 = n - p - q + 1, \tag{1.3}$$

$\chi^2_{\nu_1}(\Delta/(1 + M))$ denotes a noncentral chi-squared variable (with noncentrality parameter $\Delta/(1 + M)$) independent of the central chi-squared variable $\chi^2_{\nu_2}$, and χ^2_q and χ^2_{n-q+1} are independent central chi-squared variables. Thus, the joint distribution of (L, M) depends on (μ, Σ) only through the value of Δ, and M is an ancillary statistic. Under H_0, L and M are independent.

For the testing problem (1.1), Rao (1946) and Cochran and Bliss (1948) proposed the "conditional" test, which rejects H_0 when

$$L > \frac{\nu_1}{\nu_2} F^{\alpha}_{\nu_1, \nu_2}, \tag{1.4}$$

where $F^{\alpha}_{a,b}$ is the upper α point of the $F_{a,b}$ distribution. For each *fixed* value of M, this test is the uniformly most powerful conditional level α test for testing $\Delta = 0$ versus $\Delta > 0$ based on the conditional distribution of L given M. In the notation of Rao (1949) and Subrahmaniam and Subrahmaniam (1973), the "conditional" test rejects H_0 for large values of $(D^2_{p+q} - D^2_q)(Q + D^2_q)^{-1}$, where Q is a constant depending on sample size. Rao (1949, pages 352, 357–359) noted that in some situations the estimate of Δ based on $D^2_{p+q} - D^2_q$ is slightly more efficient than that based on $(D^2_{p+q} - D^2_q)(Q + D^2_q)^{-1}$, and therefore suggested that a test of H_0 based on $D^2_{p+q} - D^2_q$ may be more powerful than the "conditional" test. In our notation, the test suggested by Rao rejects H_0 when

$$L(1 + M) > d_{\alpha}, \tag{1.5}$$

where $0 < d_{\alpha} < \infty$ is a constant. The values of the α-level critical points d_{α} have been tabulated by Subrahmaniam and Subrahmaniam (1973). (It is not always the case that if a statistic T_1 is a more efficient *estimator* of a parameter Δ than T_2, then a *test* of $H_0 : \Delta = \Delta_0$ based on T_1 must be more powerful than one based on T_2; for examples, see Sethuraman (1961) and Sundrum (1954). In fact, our results in Sections 3 and 4 indicate that for the present problem, test (1.4) is preferable to (1.5).)

A second alternative is to ignore the information that $\mu_2 = 0$ under the alternative and to treat the variates and covariates alike, i.e., to replace (1.1) by the problem of testing

$$H_0 : \mu = 0, \quad \Sigma \text{ unspecified} \quad \text{versus} \quad H_1' : \mu \neq 0, \quad \Sigma \text{ unspecified}. \tag{1.6}$$

The overall Hotelling T^2 test appropriate for (1.6) rejects H_0 when

$$X'S^{-1}X \equiv (1 + L)(1 + M) - 1 > \frac{p + q}{n - p - q + 1} F^{\alpha}_{p+q,\, n-p-q+1}. \tag{1.7}$$

In the notation of Rao and the Subrahmaniams, $X'S^{-1}X = (\text{constant})\, D^2_{p+q}$.

Olkin and Shrikhande (1954), Stein (1952), and Giri (1961) studied this problem from a decision-theoretic viewpoint, exploiting the invariance of the problem (1.1) under a group G of linear transformations. Let G be the group of $(p + q) \times (p + q)$ nonsingular matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where $A_{11} : p \times p$ and $A_{22} : q \times q$. The group G acts on the sample space via $A : (X, S) \to (AX, ASA')$ and on the parameter space via $A : (\mu, \Sigma) \to (A\mu, A\Sigma A')$. The maximal invariant statistic is the pair (L, M), and the maximal invariant parameter is Δ. Restricting attention to G-invariant tests, the problem (1.1) reduces to that of testing

$$H_0 : \Delta = 0 \quad \text{versus} \quad H_1 : \Delta > 0 \tag{1.8}$$

based on (L, M). Olkin and Shrikhande (1954) and later Giri (1961) showed that the "conditional" test (1.4) is in fact the likelihood ratio test (LRT) for (1.1). (See also Giri (1962).) Giri (1961) also stated, and verified for a few specific values of (n, p, q), that no uniformly most powerful (UMP) invariant level α test exists. (Note that since $q > 0$, the maximal invariant statistic is two-dimensional whereas the maximal invariant parameter is one-dimensional. By contrast, when $q = 0$, the usual Hotelling T^2 problem obtains, in which the maximal invariant statistic and the maximal invariant parameter are both one-dimensional, and a UMP invariant level α test does exist.)

Stein (1952) and Giri (1968) described the locally most powerful (LMP) invariant level α test for alternatives $\Delta \to 0$, which rejects H_0 when

$$\frac{\nu_2 L - \nu_1}{(1 + L)(1 + M)} > c_\alpha, \tag{1.9}$$

where c_α is a constant satisfying $-\nu_2/\nu_1 < c_\alpha < 1$. Stein (1952) and Giri (1968) also stated that the overall T^2 test (1.7) is the asymptotically most powerful (AMP) invariant level α test for alternatives $\Delta \to \infty$ (see Subsection 3.2). Note that the tests (1.4), (1.5), (1.7) and (1.9) are mutually distinct (unless $\alpha = \alpha^*$—see Subsection 3.4).

Cochran (1964) numerically compared the powers of the LRT (1.4) and the overall T^2 test (1.7) using, however, an inaccurate approximation for the power of the former. Subrahmaniam and Subrahmaniam (1973) gave a detailed numerical comparison of the powers of the LRT and the $D_{p+q}^2 - D_q^2$ test (1.5). They compared the *unconditional* power of the test (1.5) to the *conditional* power of the LRT, conditioning on the value of the ancillary statistic M, for various fixed values of M. For all values of M except those in a neighborhood of zero, they found the conditional power of the LRT to be much smaller than the unconditional power of test (1.5), and concluded that "the unconditional procedure (test (1.5)) is vastly superior to the Cochran and Bliss conditional test (LRT)." Unfortunately, this comparison of conditional power to unconditional power is misleading, since M assumes values in the above-mentioned neighborhood of zero with probability greater than $\frac{1}{2}$, usually about 0.7. Furthermore, *conditionally* on M the two tests are based on the *same* test statistic (L), and each rejects H_0 for large values of L (as must any reasonable test for problem (1.8)—see the second sentence following (2.4)). Conditionally, the two tests differ *only* in the way the conditional level depends on the ancillary M. For the LRT, the conditional level is fixed at α regardless of the value of M, while for test (1.5) the conditional level is a strictly increasing function of M whose expectation with respect to M is α. Thus, the two tests have *different* conditional levels for almost every value of M, so it is not appropriate to compare their conditional powers. A power comparison can be based only on unconditional power, i.e., on the expectation (with respect to M) of the conditional powers.

In this paper we present results which clarify and complement the decision-theoretic work of Olkin and Shrikhande, Stein, and Giri and the numerical work of Cochran and the Subrahmaniams. We attack the problem (1.1) by restricting attention to G-invariant tests only, i.e., those based on (L, M), thereby reducing (1.1) to (1.8). By closely examining the likelihood ratio

$$R_\Delta(l, m) \equiv \frac{f_\Delta(l, m)}{f_0(l, m)}, \tag{1.10}$$

where $f_\Delta(l, m)$ is the joint density function of (L, M) when Δ obtains, in Section 2 we present a necessary and sufficient condition (Theorem 2.1) for admissibility of tests for problem (1.8), which then leads to several useful necessary conditions. The latter enable us to conclude in Section 3 that the test (1.5) is inadmissible for problem (1.8), and *a fortiori* inadmissible for the problem (1.1). The overall T^2 test (1.7), being the unique AMP test for (1.8), is admissible for (1.8) (which does not necessarily imply admissibility for (1.1)—however, see subsection 3.6). The behavior of the LRT (1.4) is rather curious. We show in subsection 3.4 that there exists a level α^* ($0 < \alpha^* < 1$) depending on (ν_1, ν_2) such that for $\alpha^* < \alpha < 1$, the level α LRT is inadmissible among invariant tests, whereas for $0 < \alpha \leqslant \alpha^*$, the level α LRT is admissible among invariant tests (in fact, coinciding with the LMP invariant test when $\alpha = \alpha^*$). Specific examples of Bayes tests and other admissible tests are also presented in Section 3.

In Section 4 we present (unconditional) power comparisons for the LRT, $D_{p+q}^2 - D_q^2$ test, and overall T^2 test—(1.4), (1.5), and (1.7), respectively—which improve and clarify the tabulations of Cochran (1964) and Subrahmaniam and Subrahmaniam (1973).

Since the AMP and LMP invariant level α tests are both admissible among invariant tests, and are distinct, it is clear that a UMP invariant level α test cannot exist. In Section 5 of Marden and Perlman (1977), we point out the stronger fact that for all (n, p, q) and for each fixed Δ_1, $0 < \Delta_1 < \infty$, the most powerful invariant test of $\Delta = 0$ versus the simple alternative $\Delta = \Delta_1$ depends nontrivially on Δ_1, which was shown by Giri (1961) only for two specific values of (n, p, q).

Finally, the proof of Theorem 2.1 is given in Section 5 of the present paper. This theorem both relies on and partially extends previous work of Farrell (1968) and Ghia (1976). A brief exposition of Ghia's work is included in this section.

2. Necessary and sufficient conditions for admissibility in problem (1.8).

For the remainder of this paper unless otherwise stated, consideration is restricted to the family of invariant tests for problem (1.1), i.e., the family of tests based on the maximal invariant statistic (L, M). Our results are based on Theorem 2.1, proved and discussed in Section 5, which characterizes the class of admissible tests for (1.8). Recall that a test function $\phi \equiv \phi(l, m)$ is simply a measurable function of (l, m) satisfying $0 \leqslant \phi \leqslant 1$.

THEOREM 2.1. *A test ϕ is admissible for problem* (1.8) *if and only if it is of the form*

$$\begin{aligned}\phi(l, m) &= 1 \quad \text{if} \quad (1 + l)(1 + m) > w_0^{-1} \\ &= 1 \quad \text{if} \quad \textstyle\int_{0-}^{1+}[(R_\Delta - 1)/\Delta]\pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) > c \qquad (2.1)\\ &= 0 \qquad \textit{otherwise}\end{aligned}$$

for a.e. (l, m) [*Lebesgue*], *where* $R_\Delta \equiv R_\Delta(l, m)$ *is given by* (1.10), $|c| < \infty$, $0 \leqslant w_0 \leqslant 1$, π^0 *is a finite measure on* $[0, 1]$, *and* π^1 *is a locally finite measure on* $[1, \infty)$, (i.e., π^1 *assigns finite mass to every compact set*).

REMARK 2.2. It is shown in Lemma 5.3(d) that $(R_\Delta - 1)/\Delta$ is uniformly bounded in (Δ, l, m) for $0 \leqslant \Delta \leqslant 1$, so the first integral in (2.1) is always convergent.

REMARK 2.3. The expression (2.1) can be rewritten as

$$\phi(l, m) = 1 \quad \text{if} \quad (1 + l)(1 + m) > w_0^{-1}$$

(2.2) $$R = 1 \quad \text{if} \quad \gamma \frac{\partial}{\partial \Delta} R_\Delta \Big|_{\Delta=0} + \int_{0+}^{1+} [(R_\Delta - 1)/\Delta] \pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) > c$$

$$= 0 \qquad \textit{otherwise},$$

where $\gamma = \pi^0(\{0\})$. Thus, if $\pi^0((0, 1]) = \pi^1([1, \infty)) = w_0 = 0$, (2.2) reduces to the LMP invariant tests (1.9) (see also subsection 3.1), whereas if $\gamma = \pi^0((0, 1]) = \pi^1([1, \infty)) = 0$ and $0 < w_0 < 1$, (2.2) reduces to the AMP invariant tests (1.7) (see also subsection 3.2).

REMARK 2.4. We adopt the terminology of Ghia (1976) (see also Theorem 7.1 of Farrell (1968), and our Section 5) and refer to tests of the form (2.1) and (2.2) as *truncated generalized Bayes tests.* The measures π^0 and π^1 determine a (possibly improper) prior distribution for Δ over $(0, \infty)$; the value of $\gamma \equiv \pi^0(\{0\})$ determines a prior mass assigned to "local alternatives," i.e., $\Delta \approx 0$. The sets $\{(l, m) | (1 + l)(1 + m) \leq w_0^{-1}\}$ are called *truncation sets;* the value of w_0 determines the (exponential) rate at which the power function of ϕ approaches 1 as $\Delta \to \infty$ (see Subsection 3.2) and may be thought of, roughly, as indicating the magnitude of the prior mass assigned to "distant alternatives," i.e., $\Delta \approx \infty$. (Note: Farrell and Ghia require that the alternative space be topologically separated from the null hypothesis. Their definition of a truncated generalized Bayes test, applied in the present problem, would replace the two integrals in (2.1) by $\int_{0+}^{\infty} R_\Delta \pi(d\Delta)$ for a locally finite measure π. As Theorem 2.1 shows, this definition is too restrictive and would exclude interesting admissible tests—see Section 3 for examples, and also Remark 5.8)

In the remainder of this section, Theorem 2.1 will be used to obtain several useful necessary conditions for admissibility of tests for (1.8)—see Theorem 2.5. Since the integrals appearing in (2.1) are linear in $R_\Delta(l, m)$, monotonicity and convexity properties of $R_\Delta(\cdot\,, \cdot)$ for fixed Δ are inherited by the acceptance regions of admissible tests and thus determine necessary conditions for admissibility (the truncation sets in (2.1) are easily handled separately, although actually they may be ignored—see the proof of Theorem 2.5 and Remark 2.8). We proceed to examine R_Δ.

Since M is an ancillary statistic and is independent of L under H_0, the likelihood ratio R_Δ of (1.10) can be written as

(2.3) $$R_\Delta(l, m) = \frac{f_\Delta(l|m)}{f_0(l)} = \exp\left[-\frac{\Delta}{2(1 + m)}\right] \Sigma_{k=0}^{\infty} \frac{c_k}{k!} \left[\frac{\Delta l}{2(1 + l)(1 + m)}\right]^k,$$

where

$$c_k = \frac{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_1+\nu_2}{2}+k\right)}{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{\nu_1}{2}+k\right)} \tag{2.4}$$

and ν_1, ν_2 are given in (1.3). The infinite sum in (2.3) is a confluent hypergeometric function of type ${}_1F_1$—see Abramowitz and Stegun (1964), Chapter 13. It is apparent that $R_\Delta(l, m)$ is increasing in l for fixed m, so that by Theorem 2.1 (or by a simple direct argument based on the conditional monotone likelihood ratio of L given M), any admissible test for (1.8) must be monotone in l for fixed m, i.e., the acceptance region must be of the form $l \leqslant g(m)$ for some function g. This approach does not seem fruitful, however, since it is difficult to determine necessary conditions which must be satisfied by the function g to yield an admissible test; this difficulty is due to the fact that $R_\Delta(l, m)$ does not admit a factorization in which the variables l and m appear separately. Instead, we shall consider several homeomorphic transformations of the pair (l, m) to new variables, in terms of which R_Δ admits simple factorizations with interesting monotonicity and convexity properties. Although most of the transformations we consider are natural, they were selected in an ad hoc manner—there may well be other transformations which would lead to further necessary conditions for admissibility (see also Remark 2.9). We do not yet have a systematic procedure for selecting such transformations, which would be applicable to other testing problems such as those treated by Marden (1977).

The following four homeomorphic transformations of (l, m) will be considered:

$$\begin{cases} (l, m) \leftrightarrow (v, u) \\ v = \dfrac{l}{(1+l)(1+m)}, \; u = \dfrac{1}{1+m}; \end{cases} \tag{2.5}$$

$$\begin{cases} (l, m) \leftrightarrow (v, w) \\ w = u - v = \dfrac{1}{(1+l)(1+m)}; \end{cases} \tag{2.6}$$

$$\begin{cases} (l, m) \leftrightarrow (v', w) \\ v' = \log v: \end{cases} \tag{2.7}$$

$$\begin{cases} (l, m) \leftrightarrow (v^*, u) \\ v^* = v^{r^*}, \end{cases} \tag{2.8}$$

where the constant $r^* \equiv r^*(\nu_1, \nu_2)$ is defined as follows:

$$r^* = \inf_{z>0}\left[\mathrm{Var}_z(K)/E_z(K)\right], \tag{2.9}$$

where K denotes an integer-valued random variable with probability mass function

$$P_z\{K = k\} \equiv p_z(k) = \frac{\frac{c_k}{k!} z^k}{\sum_{j=0}^{\infty} \frac{c_j}{j!} z^j}, \qquad k = 0, 1, 2, \cdots. \tag{2.10}$$

It will be shown in Lemma 2.11 that $\max\left(\frac{1}{2}, \nu_1/(\nu_1 + \nu_2)\right) \leqslant r^* < 1$. From (2.5)–(2.8), R_Δ in (2.3) can be rewritten in four equivalent ways:

$$R_\Delta(v, u) = \exp\left(-\frac{\Delta u}{2}\right) \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\Delta v}{2}\right)^k; \tag{2.11}$$

$$R_\Delta(v, w) = \exp\left(-\frac{\Delta w}{2}\right) \exp\left(-\frac{\Delta v}{2}\right) \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\Delta v}{2}\right)^k; \tag{2.12}$$

$$R_\Delta(v', w) = \exp\left(-\frac{\Delta w}{2}\right) \exp\left(-\frac{\Delta e^{v'}}{2}\right) \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\Delta e^{v'}}{2}\right)^k; \tag{2.13}$$

$$R_\Delta(v^*, u) = \exp\left(-\frac{\Delta u}{2}\right) \sum_{k=0}^{\infty} \frac{c_k}{k!} \left[\frac{\Delta (v^*)^{1/r^*}}{2}\right]^k. \tag{2.14}$$

Denote the ranges of (v, u), (v, w), (v', w), and (v^*, u) by Ω, Λ, Λ', and Λ^*, respectively:

$$\Omega = \{(v, u) | 0 < v < u < 1\}; \tag{2.15}$$

$$\Lambda = \{(v, w) | w > 0, v > 0, w + v < 1\}; \tag{2.16}$$

$$\Lambda' = \{(v', w) | 0 < w < 1 - e^{v'}\}; \tag{2.17}$$

$$\Lambda^* = \{(v^*, u) | 0 < (v^*)^{1/r^*} < u < 1\}; \tag{2.18}$$

(see Figures 2.1–2.4). For each subset $A \subseteq \Lambda$, let $A'(\subseteq \Lambda')$ and $A^*(\subseteq \Lambda^*)$ be the images of A in Λ' and Λ^* under the transformations $(v, w) \to (v', w)$ and $(v, w) \to (v^*, u)$, respectively, i.e., $A' = \{(v', w) | (e^{v'}, w) \in A\}$, etc. Define the partial ordering $\lesssim$ on Λ as follows: $(v_0, w_0) \lesssim (v, w)$ iff $(v, w) - (v_0, w_0)$ is in the convex cone C determined by the frame vectors $e_1 = (-1, 0)$ and $e_2 = (1, \nu_2/\nu_1)$ (see Figure 2.2). Let $\mathscr{Q}$ denote the collection of all relatively closed subsets $A \subseteq \Lambda$ which satisfy the following three conditions:

(i) A is monotone with respect to $\lesssim$, i.e., if $(v_0, w_0) \in A$ and $(v_0, w_0) \lesssim (v, w)$ $(\in \Lambda)$, then $(v, w) \in A$;

(ii) A' is a convex subset of Λ';

(iii) A^* is a convex subset of Λ^*;

see Figures 2.2–2.4. Condition (i) implies that the "lower boundary" of A is a well-defined function of v, i.e., A is of the form $\{(v, w) \in \Lambda | w \geqslant h(v)\}$ for some

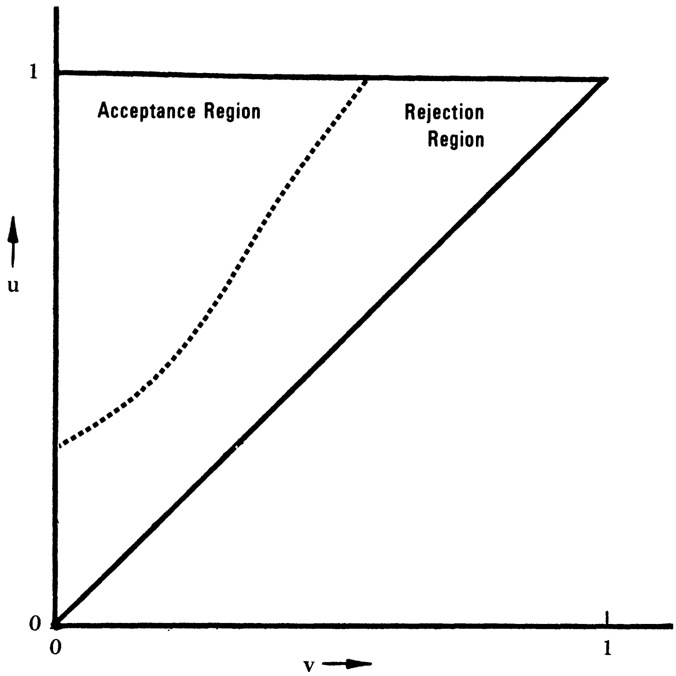

FIG. 2.1. *The space* Ω.

function h. Condition (ii) or (iii) implies that h is continuous (the homeomorphic image of a convex function). Furthermore, (i) implies that h is nondecreasing and can have slope no greater than ν_2/ν_1. Conditions (i)-(iii) do not imply that A or its complement $\Lambda - A$ is convex—see Subsection 3.6.

We remark that in the presence of condition (i), the convexity conditions (ii) and (iii) are independent: there exist acceptance regions A which satisfy (i) and (ii) but not (iii), and others which satisfy (i) and (iii) but not (ii). Examples are given by Marden (1977).

Let Φ_1 be the collection of all test functions $\phi_1 \equiv \phi_1(v, w)$ for problem (1.8) of the form

$$(2.19) \qquad \begin{aligned} \phi_1(v, w) = I_{\Lambda - A}(v, w) &\equiv 0 \quad \text{if} \quad (v, w) \in A \\ &\equiv 1 \quad \text{if} \quad (v, w) \in \Lambda - A \end{aligned}$$

as A ranges over $\mathscr{A}$ (I_B denotes the indicator function of the set B). Let Φ ($\supset \Phi_1$) be the collection of all test functions ϕ on Λ such that $\phi = \phi_1$ a.e. $[\mu]$ for some $\phi_1 \in \Phi_1$, where μ denotes Lebesgue measure on Λ. The next theorem, our second main result, states that conditions (i), (ii), and (iii) are necessary conditions for admissibility of a test with acceptance region A.

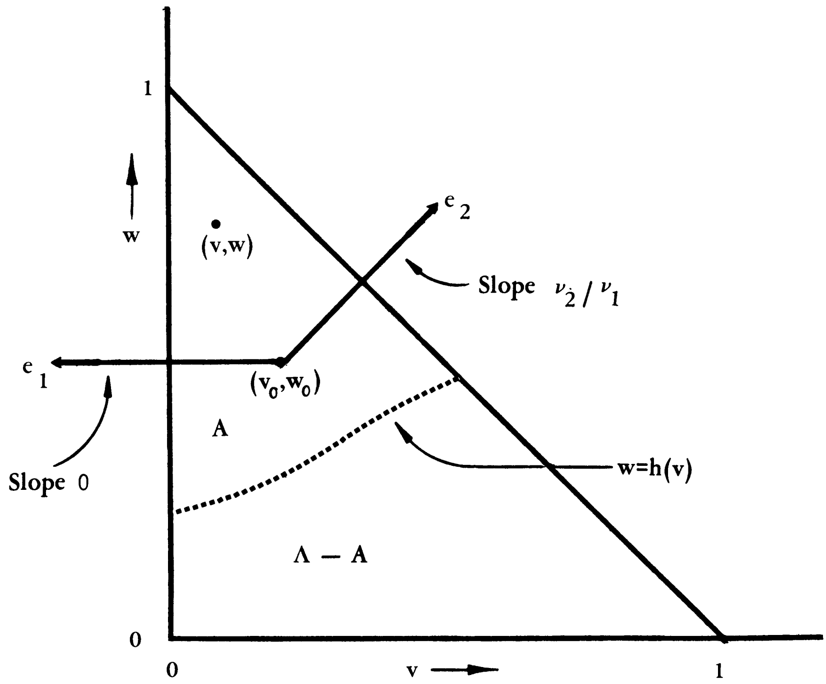

FIG. 2.2. *The space* Λ.

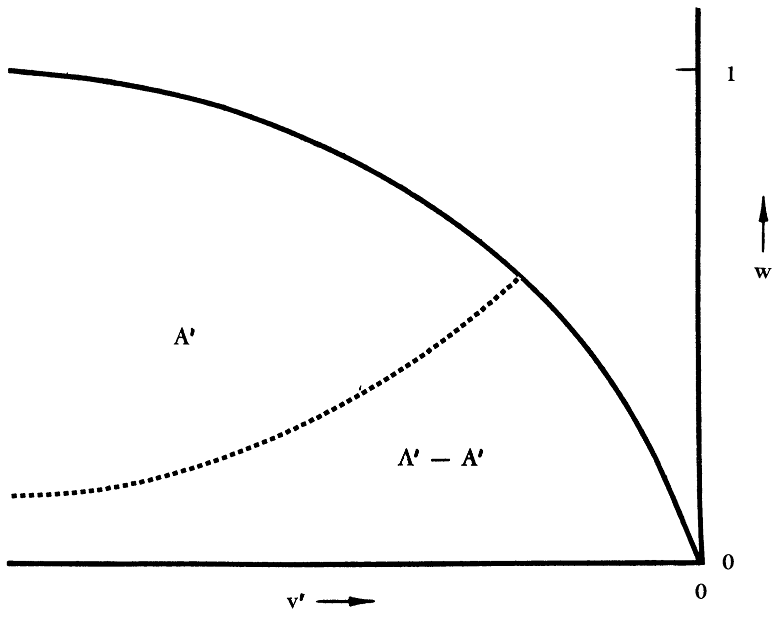

FIG. 2.3. *The space* Λ′.

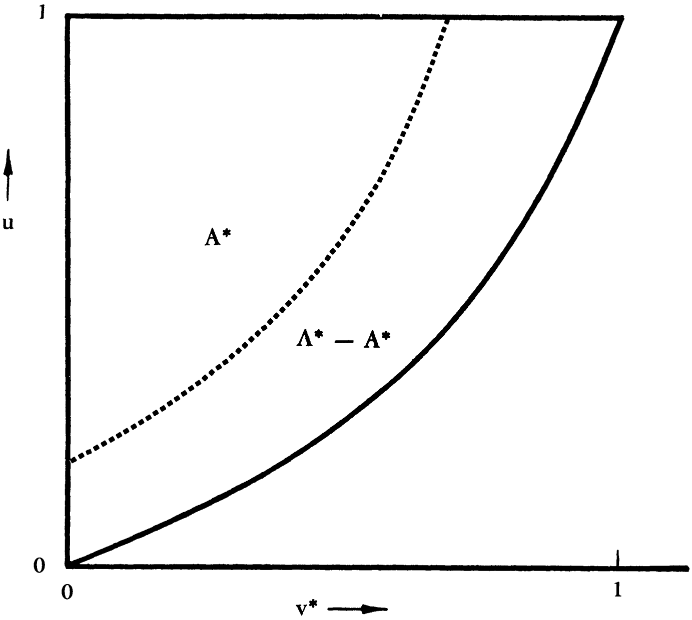

FIG. 2.4. *The space* Λ^*.

THEOREM 2.5. *Φ is a complete class for the testing problem* (1.8). *This result implies that for any test $\psi \notin \Phi$ there exists $\phi \in \Phi$ such that ϕ strictly dominates ψ,* i.e.,

$$r_\Delta(\phi) \leqslant r_\Delta(\psi), \qquad 0 \leqslant \Delta < \infty, \tag{2.20}$$

with strict inequality for at least one value of Δ, where the risk function $r_\Delta(\phi)$ is defined by

$$\begin{aligned} r_\Delta(\phi) &= 1 - E_\Delta \phi \quad \textit{if} \;\; 0 < \Delta < \infty \\ &= E_0 \phi \quad \textit{if} \;\; \Delta = 0. \end{aligned}$$

(Throughout this paper admissibility is defined with respect to the usual $0 - 1$ loss function.) It will be shown in (5.20) that in fact *strict* inequality obtains in (2.20) for all $\Delta > 0$.

The proof of Theorem 2.5 is based on Theorem 2.1 and Lemma 2.6, which presents the needed monotonicity and convexity properties of R_Δ for fixed Δ. Before presenting the lemma, we develop several identities which will be useful for its proof. For $z > 0$ let

$$t(z) = \log\left[\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k\right]. \tag{2.21}$$

Differentiating,

$$t'(z) = \frac{\sum_{k=0}^{\infty} k \frac{c_k}{k!} z^{k-1}}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k} \equiv z^{-1} E_z(K) \tag{2.22}$$

$$= \frac{\sum_{k=0}^{\infty} \frac{c_{k+1}}{k!} z^k}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k} \equiv E_z\left(\frac{c_{K+1}}{c_K}\right)$$

where K is the random variable introduced in (2.9) and (2.10). For future use we note that from (2.4),

$$c_0 = 1 < \frac{c_{k+1}}{c_k} \equiv \frac{\nu_1 + \nu_2 + 2k}{\nu_1 + 2k} \leqslant \frac{\nu_1 + \nu_2}{\nu_1} \equiv c_1 \tag{2.23}$$

for all $k = 0, 1, 2, \cdots$; for $k \geqslant 1$, the last inequality is strict. Therefore, from (2.22),

$$E_z(K) = zE_z\left(\frac{\nu_1 + \nu_2 + 2K}{\nu_1 + 2K}\right). \tag{2.24}$$

Next, differentiating (2.22),

$$t''(z) = \frac{\sum_{k=0}^{\infty} k(k-1)\frac{c_k}{k!} z^{k-2}}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k} - \left[\frac{\sum_{k=0}^{\infty} k\frac{c_k}{k!} z^{k-1}}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k}\right]^2 \equiv z^{-2}\left[\mathrm{Var}_z(K) - E_z(K)\right] \tag{2.25}$$

$$= \frac{\sum_{k=0}^{\infty} \frac{c_{k+2}}{k!} z^k}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k} - \left[\frac{\sum_{k=0}^{\infty} \frac{c_{k+1}}{k!} z^k}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k}\right]^2 \equiv E_z\left(\frac{c_{K+2}}{c_K}\right) - \left[E_z\left(\frac{c_{K+1}}{c_K}\right)\right]^2.$$

LEMMA 2.6. *Fix* $\Delta > 0$.

(a) $R_\Delta(v, w)$ *is strictly monotone decreasing in* (v, w) *on* Λ *with respect to the partial ordering* $\lesssim$.

(b) $R_\Delta(v', w)$ *is a convex function of* (v', w) *on* Λ'.

(c) $R_\Delta(v^*, u)$ *is a convex function of* (v^*, u) *on* Λ^*.

PROOF. (a) It suffices to show that $R_\Delta(v, w)$ is strictly decreasing in the directions of the two frame vectors $(-1, 0)$ and $(1, \nu_2/\nu_1)$ of the convex cone C which determines $\lesssim$. This is equivalent to showing that $R_\Delta(v, w)$ is strictly increasing in v for fixed w, and that $R_\Delta(v,(\nu_2/\nu_1)v + d)$ is strictly decreasing in v for fixed d. By (2.12) and (2.21), the former assertion is equivalent to showing that

$$f(z) \equiv t(z) - z \tag{2.26}$$

is strictly increasing for $z > 0$. However, by (2.22) and (2.23),

$$f'(z) = t'(z) - 1 = E_z\left(\frac{c_{K+1}}{c_K}\right) - 1 > 0 \tag{2.27}$$

as claimed. Next, from (2.12) and (2.21) and the fact that $\nu_2/\nu_1 = c_1 - 1$,

$$\log R_\Delta(v, (\nu_2/\nu_1)v + d) = t\left(\frac{1}{2}\Delta v\right) - \frac{1}{2}\Delta(c_1 v + d),$$

so

$$\begin{aligned}\frac{2}{\Delta}d[\log R_\Delta(v, (\nu_2/\nu_1)v + d)]/dv &= t'\left(\frac{1}{2}\Delta v\right) - c_1\\ &= E_z\left(\frac{c_{K+1}}{c_K}\right) - c_1\\ &< 0\end{aligned}$$

by (2.22) and (2.23), where $z = \dfrac{1}{2}\Delta v > 0$.

(b) It suffices to show that $\log R_\Delta(v', w)$ is convex in (v', w). From (2.13),

$$\log R_\Delta(v', w) = f\left(\tfrac{1}{2}\Delta e^{v'}\right) - \tfrac{1}{2}\Delta w, \tag{2.28}$$

where f is given by (2.26), so it must be shown that $f(e^{v'})$ is convex in v'. By (2.27), (2.22), and (2.23),

$$\begin{aligned}df(e^{v'})/dv' &= z[t'(z) - 1]\\ &= \frac{\sum_{k=0}^{\infty}\left(1 - \dfrac{c_k}{c_{k+1}}\right)\dfrac{c_{k+1}}{k!}z^{k+1}}{\sum_{k=0}^{\infty}\dfrac{c_k}{k!}z^k}\\ &= \frac{\nu_2}{2}\,\frac{\sum_{k=0}^{\infty}\left(\dfrac{k+1}{k+a}\right)\dfrac{c_{k+1}}{(k+1)!}z^{k+1}}{\sum_{k=0}^{\infty}\dfrac{c_k}{k!}z^k}\\ &= \frac{\nu_2}{2}\,\frac{\sum_{k=1}^{\infty}\left(\dfrac{k}{k+a-1}\right)\dfrac{c_k}{k!}z^k}{\sum_{k=0}^{\infty}\dfrac{c_k}{k!}z^k},\end{aligned} \tag{2.29}$$

where now $z = e^{v'}$ and $a = \frac{1}{2}(\nu_1 + \nu_2) \geqslant 1$. If $a = 1$ then (2.29) yields

$$\frac{2}{\nu_2}df(e^{v'})/dv' = 1 - \frac{1}{\sum_{k=0}^{\infty}\dfrac{c_k}{k!}z^k},$$

which is strictly increasing in $z \equiv e^{v'}$. If $a > 1$ then (2.29) yields

$$(2.30)\qquad \frac{2}{\nu_2} df(e^{v'})/dv' = \frac{\sum_{k=0}^{\infty}\left(\frac{k}{k+a-1}\right)\frac{c_k}{k!}z^k}{\sum_{k=0}^{\infty}\frac{c_k}{k!}z^k} \equiv E_z\left(\frac{K}{K+a-1}\right).$$

Since $p_z(k)$ in (2.10) has a strictly monotone likelihood ratio and $k/(k+a-1)$ is a strictly increasing function of k, we conclude that (2.30) is strictly increasing in $z \equiv e^{v'}$ (cf. Lehmann (1959), page 74). Hence $f(e^{v'})$ is strictly convex in v'.

(c) We shall show that $\log R_\Delta(v^*, u)$ is convex in (v^*, u). From (2.14) and (2.21),

$$(2.31)\qquad \log R_\Delta(v^*, u) = t\left(\tfrac{1}{2}\Delta(v^*)^{1/r^*}\right) - \tfrac{1}{2}\Delta u,$$

so we must show that $t((v^*)^{1/r^*})$ is convex in $v^* > 0$. Differentiating,

$$(2.32)\qquad dt\left((v^*)^{1/r^*}\right)/dv^* = (r^*)^{-1}z^{1-r^*}t'(z),$$

where now $z = (v^*)^{1/r^*}$. However, from (2.22) and (2.25),

$$(2.33)\qquad \begin{aligned} d\left[z^{1-r^*}t'(z)\right]/dz &= z^{-r^*}\left[zt''(z) + (1-r^*)t'(z)\right] \\ &= z^{-r^*-1}\left[\operatorname{Var}_z(K) - r^*E_z(K)\right], \end{aligned}$$

which is nonnegative by (2.9), the definition of r^*. Thus $t((v^*)^{1/r^*})$ is convex in v^*.

REMARK 2.7. The proof of Lemma 2.6(c) shows that $t(v) \equiv t((v^r)^{1/r})$ is convex in v^r for every $r \in [0, r^*]$ (for $r = 0$, replace v^r by $\log v \equiv v'$; note that $\log v = \lim_{r\to 0}(v^r - 1)/r$) but is concave in v^r on some nondegenerate interval for every $r > r^*$. (Alternatively, for the convexity part note that if $h(x)$ is nondecreasing and convex in $x > 0$, then $g(x) \equiv h(x^{r^*/r})$ is also convex in x for $r > r^*$.)

PROOF OF THEOREM 2.5. Suppose that $\phi \equiv \phi(v, w)$ is an admissible test function defined on Λ for problem (1.8). By Theorem 2.1, $\phi = I_{\Lambda - A}$ a.e. $[\mu]$ where $A = A_1 \cap A_2$,

$$A_1 = \left\{(v, w) \mid \textstyle\int_{0-}^{1+}\left[(R_\Delta - 1)/\Delta\right]\pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) \leqslant c\right\},$$

$$A_2 = \left\{(v, w) \mid w \geqslant w_0\right\}.$$

Since the integrals defining A_1 are linear in R_Δ, Lemma 2.6 implies that $A_1 \in \mathcal{C}$. It is easy to see that the truncation set $A_2 \in \mathcal{C}$ also. Since $\mathcal{C}$ is closed under intersections, $A \in \mathcal{C}$, so $\phi \in \Phi$ as claimed.

REMARK 2.8. Theorem 2.5 does not require the full power of Theorem 2.1 for its proof. Instead, the technique introduced by Birnbaum (1955) and applied by Matthes and Truax (1967) and Eaton (1970) can be used. First, since any *proper* Bayes test for problem (1.8) has acceptance region of the form $\{(v, w) \mid \int_{0+}^{\infty} R_\Delta \pi(d\Delta) \leqslant c\}$ a.e. $[\mu]$ for some finite measure π, Lemma 2.6 implies that $\mathfrak{B} \subseteq \Phi$, where $\mathfrak{B}$ is the class of all proper Bayes tests. Theorem 5.8 of Wald (1950) guarantees that $\overline{\mathfrak{B}}$ is an essentially complete class, where $\overline{\mathfrak{B}}$ denotes the closure of $\mathfrak{B}$ under weak* convergence (see Definition 5.1). Next, using the techniques of Birnbaum (1955) and Matthes and Truax (1967), it can be shown that $\overline{\Phi}_1 \subseteq \Phi$. However, clearly $\Phi \subseteq \overline{\Phi}_1$, so $\Phi = \overline{\Phi}_1$, hence Φ is closed under weak* convergence. Thus $\overline{\mathfrak{B}} \subseteq \overline{\Phi} = \Phi$, so Φ is an essentially complete class. Since the family of distributions of (L, M) is not complete, further argument is required to show that Φ is a complete class (see Theorem 5.10).

REMARK 2.9. Although Φ is a complete class of tests for (1.8), it is probably larger than the minimal complete class described by Theorem 2.1. Conditions (i)–(iii) impose necessary conditions on the first and second derivatives (if they exist) of the functions $w = h(v)$ which determine the boundaries of admissible acceptance regions. Closer examination of R_Δ, however, might yield necessary conditions on higher derivatives, which would lead to smaller complete classes than Φ, perhaps even to the minimal complete class. Still, Φ is sufficiently small to exclude several popular tests, thereby proving them inadmissible. This is demonstrated in Section 3.

REMARK 2.10. Regarding the necessary condition (iii), Theorem 2.5 and Remark 2.7 show that the acceptance region (AR) of any admissible test for (1.8) is convex in (v^r, u) (up to a null set) for all $r \in [0, r^*]$. Also, if $r > r^*$, there exist admissible tests whose AR's are not convex in (v^r, u): by (2.11) and (2.21) the most powerful (MP) test of $\Delta = 0$ vs. a simple alternative $\Delta = \Delta_1$ has AR $\{t(\frac{1}{2}\Delta_1 v) \leqslant \frac{1}{2}\Delta_1 u + c\}$ which, by Remark 2.7, fails to be convex in (v^r, u) for sufficiently large Δ_1. Next, regarding the necessary condition (ii), we show now that if $r > 0$ then there exist admissible tests whose AR's are not convex in (v^r, w). By (2.12) and (2.26), the MP test of $\Delta = 0$ vs. $\Delta = \Delta_1$ has AR $\{f(\frac{1}{2}\Delta_1 v) \leqslant \frac{1}{2}\Delta_1 w + c\}$. For $r > 0$ let $x = v^r$ and consider the function $f(x^{1/r}) \equiv t(x^{1/r}) - x^{1/r}$. Differentiating,

$$df(x^{1/r})/dx = r^{-1}z^{1-r}[t'(z) - 1]$$

where $z = x^{1/r}$. From (2.27), the monotone likelihood ratio property of $p_z(k)$, and the fact that c_{k+1}/c_k is decreasing in k, we see that $t'(z) - 1$ is decreasing in z. Hence for $r > 1$, $df(x^{1/r})/dx$ is decreasing in x, so $f(\frac{1}{2}\Delta_1 v)$ is *concave* in v^r and the *rejection* region of the MP test above is convex, hence the AR is not convex. For $0 < r < 1$, first note that by the proof of Lemma 2.6(b),

$$\frac{2}{\nu_2} z[t'(z) - 1] = 1 - \frac{1}{\sum_{k=0}^{\infty} \frac{c_k}{k!} z^k} \quad \text{if} \quad a = 1$$

$$= E_z\left(\frac{K}{K + a - 1}\right) \quad \text{if} \quad a > 1$$

so $z[t'(z) - 1] \to \nu_2/2$ as $z \to \infty$ (we use the fact that $K \to \infty$ in probability as $z \to \infty$, which follows from the fact that $p_z(k) \to 0$ as $z \to \infty$ for all fixed k).

Therefore $z^{1-r}[t'(z) - 1]$ must be decreasing for some sufficiently large values of z, hence $f(x^{1/r})$ must be concave on some nondegenerate interval. Therefore, for all $0 < r < 1$ the AR of the MP test above fails to be convex in (v^r, w) for sufficiently large Δ_1.

This section concludes with a discussion of the constants $r^* \equiv r^*(\nu_1, \nu_2)$ defined in (2.9). These constants also appear in problems other than that treated here. For example, Marden (1977) describes a complete class of combination procedures for the problem of combining independent noncentral F-tests; the constants r^* occur in the description of this complete class. The reader may wish to proceed directly to Section 3.

LEMMA 2.11. $\max\{\frac{1}{2}, \nu_1(\nu_1 + \nu_2)^{-1}\} \leqslant r^* < 1$.

PROOF. By (2.22) and (2.25),

$$\frac{\mathrm{Var}_z(K)}{E_z(K)} = 1 - z\left[\frac{\sum_{k=0}^{\infty}\frac{c_{k+1}}{k!}z^k}{\sum_{k=0}^{\infty}\frac{c_k}{k!}z^k} - \frac{\sum_{k=0}^{\infty}\frac{c_{k+2}}{k!}z^k}{\sum_{k=0}^{\infty}\frac{c_{k+1}}{k!}z^k}\right]. \tag{2.34}$$

As $z\downarrow 0$, the term in square brackets approaches $(c_1/c_0) - (c_2/c_1)$, which is strictly positive by (2.23). Hence from (2.9), $r^* < 1$.

Next, in the proof of Lemma 2.6(b) it was shown that $z[t'(z) - 1]$ is strictly increasing in $z > 0$. Therefore, from (2.22) and (2.25),

$$\begin{aligned} 0 &< d\{z[t'(z) - 1]\}/dz \\ &= zt''(z) + t'(z) - 1 \\ &= z^{-1}[\mathrm{Var}_z(K) - E_z(K)] + z^{-1}E_z(K) - 1 \\ &= z^{-1}\,\mathrm{Var}_z(K) - 1. \end{aligned} \tag{2.35}$$

Hence,

$$\frac{\mathrm{Var}_z(K)}{E_z(K)} > \frac{z}{E_z(K)} = \left[E_z\left(\frac{\nu_1 + \nu_2 + 2K}{\nu_1 + 2K}\right)\right]^{-1} > \frac{\nu_1}{\nu_1 + \nu_2}$$

by (2.24) and (2.23). Therefore $r^* \geqslant \nu_1/(\nu_1 + \nu_2)$.

Finally, in view of Remark 2.7, to show that $r^* \geqslant \frac{1}{2}$ it suffices to show that log R_Δ is convex in $(v^{1/2}, u)$. We shall show that

$$\sum_{k=0}^{\infty}\frac{c_k}{k!}\left(\frac{1}{2}z\right)^k = (\text{constant})\int_{-\infty}^{\infty} e^{y_1 z^{1/2}} g(y_1)\,dy_1, \tag{2.36}$$

where

$$g(y_1) = \int \cdots \int R^{\nu_1 - 1}\|y\|^{\nu_2} e^{-\frac{1}{2}\|y\|^2}\,dy_2\,dy_3 \cdots dy_{\nu_1}, \tag{2.37}$$

and $y = (y_1, \cdots, y_{\nu_1})$. The log convexity of R_Δ in $(v^{1/2}, u)$ now follows from

(2.11) by noting that

$$\frac{\partial^2}{\partial(z^{1/2})^2}\log \int_{-\infty}^{\infty} e^{y_1 z^{1/2}} g(y_1)dy_1 = \operatorname{Var}(Y_1) > 0,$$

where Y_1 denotes a random variable with density

$$e^{y_1 z^{1/2}} g(y_1) \Big/ \int_{-\infty}^{\infty} e^{y_1 z^{1/2}} g(y_1)dy_1$$

with respect to Lebesgue measure. We prove (2.36) by finding two equivalent expressions for $E(X^{\nu_2/2})$, where $X \sim \chi^2_{\nu_1}(z)$. Since the distribution of X can be represented as a Poisson mixture of central χ^2's, i.e.,

$$X|K_0 \sim \chi^2_{\nu_1+2K_0}, \qquad K_0 \sim \text{Poisson}(z/2),$$

we have

$$\begin{aligned} E(X^{\nu_2/2}) &= E\left[E\left\{\left(\chi^2_{\nu_1+2K_0}\right)^{\nu_2/2} | K_0\right\}\right] \\ &= E\left[2^{\nu_2/2}\frac{\Gamma\left(\frac{1}{2}(\nu_1+\nu_2+2K_0)\right)}{\Gamma\left(\frac{1}{2}(\nu_1+2K_0)\right)}\right] \\ &= 2^{\nu_2/2}e^{-z/2}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}(\nu_1+\nu_2+2k)\right)}{\Gamma\left(\frac{1}{2}(\nu_1+2k)\right)}\left(\frac{z}{2}\right)^k\frac{1}{k!}. \end{aligned} \tag{2.38}$$

Also, letting $Y \sim N_{\nu_1}((z^{1/2}, 0, \cdots, 0)', I)$,

$$\begin{aligned} E(X^{\nu_2/2}) &= E(\|Y\|^{\nu_2}) \\ &= (2\pi)^{-\nu_1/2}\int\cdots\int_{R^{\nu_1}} e^{-\frac{1}{2}(y_1-z^{1/2})^2} e^{-\frac{1}{2}\sum_{i=2}^{\nu_1} y_i^2}\|y\|^{\nu_2}\,dy_1\cdots dy_{\nu_1} \\ &= (2\pi)^{-\nu_1/2}e^{-z/2}\int_{-\infty}^{\infty} e^{y_1 z^{1/2}} g(y_1)dy_1. \end{aligned} \tag{2.39}$$

Hence, by definition of c_k, (2.36) follows by equating the final expressions in (2.38) and (2.39).

REMARK 2.12. In subsection 3.6 we exhibit specific admissible tests whose acceptance regions are not convex in (v, w), hence not convex in (v, u). In view of Remark 2.10, this provides an alternate proof that $r^* < 1$.

Next, we show that

$$r^*(\nu_1, 2) = \left[\nu_1(\nu_1 + 2)\right]^{1/2} - \nu_1. \tag{2.40}$$

When $\nu_2 = 2$ we have that $c_k = 1 + 2\nu_1^{-1}k$, so by (2.34)

$$\operatorname{Var}_z(K)/E_z(K) = 1 - z\left[(z+b)^{-1} - (z+b+1)^{-1}\right], \tag{2.41}$$

where $b = \nu_1/2$. Thus

$$d[\mathrm{Var}_z(K)/E_z(K)]/dz = (z^2 - b^2 - b)(z + b)^{-2}(z + b + 1)^{-2},$$

which is negative to the left of $[b(b+1)]^{1/2}$ and positive to the right. Hence, (2.41) attains its minimum at $z = [b(b+1)]^{1/2}$, and this minimum is given by (2.40).

In Table 2.1 we have used an iterative routine to calculate r^* for several values of ν_1 and ν_2. As a partial check, the values for $\nu_2 = 2$ agree with those given by (2.40). It is interesting to note that the empirical approximation

$$r^* \doteq \frac{1}{2}\left[1 + \left(\frac{\nu_1}{\nu_1 + \nu_2}\right)^{\frac{1}{2} + \frac{1}{5\nu_1}}\right] \tag{2.42}$$

is accurate to within .004 for all values in the table, while the simpler approximation

$$r^* \doteq \frac{1}{2}\left[1 + \left(\frac{\nu_1}{\nu_1 + \nu_2}\right)^{\frac{1}{2}}\right] \tag{2.43}$$

is accurate to within .017 when $\nu_1 \geqslant 4$. (We thank Kent Bailey for assisting with this observation.)

TABLE 2.1
Values of $r^*(\nu_1, \nu_2)$

	ν_2				
ν_1	2	4	10	20	40
2	.82843	.75647	.66945	.61960	.58366
4	.89898	.84029	.75072	.68784	.63650
10	.95445	.91929	.84828	.78242	.71724
20	.97618	.95544	.90636	.85094	.78559
40	.98780	.97645	.94662	.90731	.85525

3. Admissibility or inadmissibility of specific tests.

3.1 *The locally most powerful (LMP) invariant test.* The level α LMP test (1.9) for problem (1.8) was discussed by Stein (1952) and Giri (1968). It is the essentially unique level α test which maximizes the slope of the power function at $\Delta = 0$, and is easily found by using the Neyman-Pearson lemma to be the test which rejects H_0 for large values of

$$\frac{\partial}{\partial\Delta} R_\Delta \bigg|_{\Delta=0} \equiv \frac{1}{2}(c_1 v - u).$$

In terms of the variables (v, w), the lower boundary in Λ of the acceptance region of the level α LMP test (see (1.9)) is given by the straight line

$$w = \frac{\nu_2}{\nu_1} v - c'_\alpha. \tag{3.1}$$

As it is the essentially unique LMP level α test for (1.8), this test is admissible among invariant tests for the original problem (1.1). Note that the slope ν_2/ν_1 of the boundary line (3.1) is exactly the upper bound imposed by condition (i); any greater slope would imply inadmissibility.

3.2 *The asymptotically most powerful* (AMP) *invariant test.* A level α test ϕ^* is the (essentially unique) AMP level α test for problem (1.8) if for any other essentially different level α test ϕ there exists $\Delta(\phi) < \infty$ such that $E_\Delta\phi^* > E_\Delta\phi$ for all $\Delta > \Delta(\phi)$. An AMP test is admissible for (1.8), hence admissible among invariant tests for (1.1). Stein (1952) and Giri (1968) stated that the overall T^2 test (1.7) is the level α AMP test for (1.8). In terms of the variables (v, w), the lower boundary in Λ of the acceptance region of this test is given by the straight line

$$(3.2) \qquad w = w_0 \equiv \left(\frac{p+q}{n-p-q+1} F^{\alpha}_{p+q,\,n-p-q+1} + 1\right)^{-1},$$

i.e., the acceptance region is just a truncation set. Note that this straight line has slope 0, which is exactly the lower bound imposed by condition (i); any lesser slope would imply inadmissibility.

Starting with expression (2.12) for the likelihood ratio R_Δ, Theorem 1 of Nandi (1963) can be applied to show that the T^2 test (1.7) is indeed AMP. Rather than showing that the conditions of Nandi's theorem are satisfied in our problem, we will give a more informative, direct proof based on the relationship

$$(3.3) \qquad \lim_{\Delta\to\infty} \frac{2}{\Delta}\log(1 - E_\Delta\phi) = -\operatorname{ess\,inf}\{w|\phi(v, w) < 1\},$$

valid for every test function ϕ defined on Λ such that $E_0\phi < 1$. (In (3.3) and elsewhere the essential infimum is taken with respect to Lebesgue measure μ on Λ.) This relationship shows that if the region $\{(v, w)|\phi < 1\}$ is bounded away from the line $\{w = 0\}$ in Λ, then the power function $E_\Delta\phi$ approaches 1 at an exponential rate as $\Delta \to \infty$, i.e.,

$$(3.4) \qquad E_\Delta\phi \sim 1 - \exp\left[-\frac{\Delta}{2}\operatorname{ess\,inf}\{w|\phi(v, w) < 1\}\right].$$

From this it immediately follows that the overall level α T^2 test, with acceptance region determined by (3.2), is the essentially unique AMP level α test for (1.8).

To obtain (3.3), first note from (2.4) that if ν_2 is even then

$$1 \leqslant c_k = \frac{\Gamma(\nu_1/2)}{\Gamma((\nu_1+\nu_2)/2)}\left(\frac{\nu_1}{2} + k\right)\left(\frac{\nu_1}{2} + 1 + k\right)\cdots\left(\frac{\nu_1+\nu_2}{2} - 1 + k\right)$$

for all $k \geqslant 0$, while if ν_2 is odd then

$$1 \leqslant c_k \leqslant \frac{\Gamma(\nu_1/2)}{\Gamma((\nu_1+\nu_2)/2)}\left(\frac{\nu_1}{2} + k\right)\left(\frac{\nu_1}{2} + 1 + k\right)\cdots\left(\frac{\nu_1+\nu_2+1}{2} - 1 + k\right)$$

for all $k \geqslant 1$ (in fact, for all $k \geqslant 0$ unless $\nu_1 = \nu_2 = 1$). Thus there exists a polynomial $F(k)$ of degree $\nu_2/2$ or $(\nu_2 + 1)/2$ such that $1 \leqslant c_k \leqslant F(k)$ for all

$k > 0$. Hence there exists another polynomial $G(x)$ of the same degree as $F(k)$ and with positive coefficients such that

$$e^x \leqslant \Sigma_{k=0}^{\infty} \frac{c_k}{k!} x^k \leqslant G(x)e^x \tag{3.5}$$

for all $x > 0$. (Alternatively, we could appeal to 13.1.4 of Abramowitz and Stegun (1964).) Therefore, from (2.12),

$$e^{-\frac{1}{2}\Delta w} \leqslant R_\Delta(v, w) \leqslant G\left(\frac{1}{2}\Delta v\right)e^{-\frac{1}{2}\Delta w} \leqslant G\left(\frac{1}{2}\Delta\right)e^{-\frac{1}{2}\Delta w} \tag{3.6}$$

since $v < 1$ on Λ. Thus, for any test function $\phi(v, w)$ we have

$$\begin{aligned} \textstyle\int\int_\Lambda (1 - \phi)e^{-\frac{1}{2}\Delta w} f_0 \, dv \, dw &\leqslant \textstyle\int\int_\Lambda (1 - \phi) R_\Delta f_0 \, dv \, dw \\ &\leqslant G\left(\frac{1}{2}\Delta\right) \textstyle\int\int_\Lambda (1 - \phi)e^{-\frac{1}{2}\Delta w} f_0 \, dv \, dw, \end{aligned}$$

where $f_0 = f_0(v, w)$ is the density of (V, W) when $\Delta = 0$. From this and the fact that $f_0 > 0$ on Λ it follows that if

$$1 > E_0\phi \equiv \textstyle\int\int_\Lambda \phi f_0 \, dv \, dw \equiv \alpha,$$

then

$$\begin{aligned} \lim_{\Delta\to\infty} \frac{2}{\Delta}\log(1 - E_\Delta\phi) &= \lim_{\Delta\to\infty} \frac{2}{\Delta}\log\left\{\textstyle\int\int_\Lambda (1 - \phi)e^{-\frac{1}{2}\Delta w} f_0 \, dv \, dw\right\} \\ &= \lim_{\Delta\to\infty} \frac{2}{\Delta}\log\left\{(1 - \alpha)^{-1} \textstyle\int\int_\Lambda (1 - \phi)e^{-\frac{1}{2}\Delta w} f_0 \, dv \, dw\right\} \\ &= \log\left[\lim_{\Delta\to\infty}\left\{(1 - \alpha)^{-1} \textstyle\int\int_\Lambda (1 - \phi)e^{-\frac{1}{2}\Delta w} f_0 \, dv \, dw\right\}^{\frac{2}{\Delta}}\right] \\ &= \log\left[\text{ess sup}\{e^{-w} | \phi(v, w) < 1\}\right] \\ &= -\text{ess inf}\{w | \phi(v, w) < 1\}. \end{aligned}$$

3.3 *The $D_{p+q}^2 - D_q^2$ test.* Let $A_\alpha (\subseteq \Lambda)$ be the acceptance region corresponding to the level α $D_{p+q}^2 - D_q^2$ test (1.5). The lower boundary of A_α is determined by the equation

$$w = \frac{-v + (v^2 + 4vd_\alpha^{-1})^{\frac{1}{2}}}{2},$$

where $0 < v < d_\alpha(1 + d_\alpha)^{-1}$, (which restriction guarantees that $(v, w) \in \Lambda$), and has slope

$$\frac{\partial w}{\partial v} = \frac{v + 2d_\alpha^{-1}}{2\left[(v + 2d_\alpha^{-1})^2 - 4d_\alpha^{-2}\right]^{\frac{1}{2}}} - \frac{1}{2} > 0.$$

From this, it is seen that the lower boundary is a concave increasing function of v whose slope approaches ∞ as $v \downarrow 0$, so that A_α violates condition (i) for all

$0 < \alpha < 1$. Thus test (1.5) is not a member of Φ, hence, by Theorem 2.5, is inadmissible among invariant tests for problem (1.1). In addition, this test violates condition (ii) for certain values of α—see Marden and Perlman (1977).

3.4 *The likelihood ratio test* (LRT). From (1.4), the level α LRT is given by

$$\begin{aligned} \phi_\alpha(v, w) &= 1 \quad \text{if} \quad \frac{w}{v} < \frac{\nu_2}{\nu_1 F^\alpha_{\nu_1, \nu_2}} \equiv \beta(\alpha), \\ &= 0 \quad \text{if} \quad \frac{w}{v} \geqslant \beta(\alpha). \end{aligned} \tag{3.7}$$

The lower boundary of $A_\alpha (\subseteq \Lambda)$, the level α acceptance region corresponding to ϕ_α, is given by the straight line $w = \beta(\alpha)v$. For levels α such that $\beta(\alpha) > (\nu_2/\nu_1)$, i.e., $F^\alpha_{\nu_1, \nu_2} < 1$, the slope of the line exceeds ν_2/ν_1, so that A_α violates condition (i). Hence, by Theorem 2.5 the LRT is inadmissible among invariant tests for (1.1) whenever $\alpha^* < \alpha < 1$, where $\alpha^* \equiv \alpha^*(\nu_1, \nu_2)$ is determined by

$$F^{\alpha^*}_{\nu_1, \nu_2} = 1. \tag{3.8}$$

(Representative values of α^* are given in Table 3.1.) Therefore, when $\alpha^* < \alpha < 1$, even though the LRT is conditionally UMP level α among invariant tests for each fixed value of the ancillary M, unconditionally it is inadmissible.

When $\alpha = \alpha^*$, by comparing (3.1) and (3.7) it can be seen that $c'_{\alpha^*} = 0$ and the level α^* LRT coincides with the level α^* LMP invariant test, implying the former is admissible among invariant tests.

For $0 < \alpha < \alpha^*$ we will apply Theorem 2.1 to show that the level α LRT is admissible among invariant tests for (1.1). It is convenient to work with the variables $(v, u) \equiv (v, w + v)$, so rewrite (3.7) to express the level α LRT as

$$\begin{aligned} \phi^*_\alpha(v, u) &= 1 \quad \text{if} \quad \frac{v}{u} > \frac{1}{1 + \beta(\alpha)} \equiv x(\alpha) \\ &= 0 \quad \text{if} \quad \frac{v}{u} \leqslant x(\alpha). \end{aligned} \tag{3.9}$$

As α increases from 0 to α^*, $\beta(\alpha)$ increases from 0 to ν_2/ν_1, so $x(\alpha)$ decreases from 1 to $\nu_1/(\nu_1 + \nu_2) = c_1^{-1}$. We shall show that for each $x \in (c_1^{-1}, 1)$, there exists at

TABLE 3.1
Level α^* below which the LRT is admissible and above which it is inadmissible

	ν_1				
ν_2	1	2	4	8	20
2	.42	.50	.56	.59	.61
4	.37	.44	.50	.54	.57
8	.35	.41	.46	.50	.53
16	.33	.39	.44	.47	.51
50	.32	.38	.42	.45	.48
∞	.32	.37	.41	.43	.46

least one point $\gamma \equiv \gamma(x) \in (0, 1)$ such that

$$\int_0^\infty [(R_\Delta - 1)/\Delta] \frac{d\Delta}{\Delta^\gamma} > 0 \quad \text{if } \frac{v}{u} > x \qquad (3.10)$$
$$\leqslant 0 \quad \text{if } \frac{v}{u} \leqslant x.$$

This shows that for $0 < \alpha < \alpha^*$ the level α LRT is of the form (2.1) with

$$c = \int_1^\infty \Delta^{-1-\gamma}\, d\Delta = \gamma^{-1}, \quad w_0 = 0, \quad \pi^0(d\Delta) = \Delta^{-\gamma} I_{(0,1)}(\Delta) d\Delta,$$
$$\pi^1(d\Delta) = \Delta^{-\gamma-1} I_{[1,\infty)}(\Delta) d\Delta;$$

hence, is admissible for (1.8).

To deduce (3.10), start from (2.11) and use an integration by parts to obtain that for any $\gamma \in (0, 1)$,

(3.11)

$$\int_0^\infty [(R_\Delta - 1)/\Delta] \frac{d\Delta}{\Delta^\gamma} = \Sigma_{k=1}^\infty \frac{c_k}{k!} v^k \int_0^\infty \left(\frac{\Delta}{2}\right)^k e^{-\frac{1}{2}\Delta u} \frac{d\Delta}{\Delta^{1+\gamma}} - \int_0^\infty \left(1 - e^{-\frac{1}{2}\Delta u}\right) \frac{d\Delta}{\Delta^{1+\gamma}}$$
$$= \left(\frac{u}{2}\right)^\gamma \Gamma(1-\gamma) \left[f_\gamma\left(\frac{v}{u}\right) - \frac{1}{\gamma} \right],$$

where

$$f_\gamma(x) \equiv \Sigma_{k=1}^\infty \frac{c_k}{k!} \frac{\Gamma(k-\gamma)}{\Gamma(1-\gamma)} x^k \qquad (3.12)$$

is a power series in x whose radius of convergence is $|x| = 1$. The function $f_\gamma(x)$ is jointly continuous in (γ, x) for $0 < \gamma$, $x < 1$ and is strictly increasing in x. Furthermore, since

$$1 < c_1 < c_k < c_1^k \qquad k \geqslant 2$$

and

$$\Sigma_{k=1}^\infty \frac{\Gamma(k-\gamma)}{k!\,\Gamma(1-\gamma)} = \frac{1}{\gamma}$$

(expand $[1 - (1 - x)^\gamma]/\gamma$ in a Taylor series about $x = 0$ and evaluate at $x = 1$) we have that

$$f_\gamma(c_1^{-1}) < \frac{1}{\gamma} < \frac{c_1}{\gamma} f_\gamma(1) \leqslant \infty.$$

Therefore, for each $\gamma \in (0, 1)$ there exists a unique point $x \equiv x(\gamma) \in (c_1^{-1}, 1)$ such that

$$f_\gamma(x(\gamma)) = \frac{1}{\gamma}. \qquad (3.13)$$

The function $x(\cdot)$ is continuous for $\gamma \in (0, 1)$. Also (3.12) and (3.13) imply that

$$\frac{1}{c_1} < x(\gamma) \leqslant \frac{1}{c_1\gamma},$$

so

$$\lim_{\gamma\uparrow 1} x(\gamma) = c_1^{-1}. \tag{3.14}$$

Lastly, suppose that $x(\gamma)$ were bounded away from 1 in a neighborhood of $\gamma = 0$, say $x(\gamma) \leqslant y < 1$. Then from (3.13),

$$\infty = \lim_{\gamma\downarrow 0} f_\gamma(x(\gamma)) \leqslant \lim_{\gamma\downarrow 0}\Sigma_{k=1}^{\infty}\frac{c_k}{k!}\frac{\Gamma(k-\gamma)}{\Gamma(1-\gamma)}y^k = \Sigma_{k=1}^{\infty}\frac{c_k}{k}y^k < \infty,$$

a contradiction, so we deduce that

$$\limsup_{\gamma\downarrow 0} x(\gamma) = 1. \tag{3.15}$$

From (3.14), (3.15), and the continuity of $x(\cdot)$, it follows that $x(\gamma)$ assumes every value in $(c_1^{-1}, 1)$ as γ ranges over (0, 1). Thus, given $x \in (c_1^{-1}, 1)$, there exists at least one point $\gamma \equiv \gamma(x) \in (0, 1)$ such that

$$f_{\gamma(x)}(x) = \frac{1}{\gamma(x)}. \tag{3.16}$$

From (3.11) and (3.16) we conclude that (3.10) holds with $\gamma = \gamma(x)$, as claimed.

REMARK 3.1. From (3.10), we may think of the level α LRT for $0 < \alpha < \alpha^*$ as essentially a Bayes test with respect to the improper prior $\pi(d\Delta) = \Delta^{-1-\gamma}\, d\Delta$. Since this prior assigns infinite mass to every neighborhood of $\Delta = 0$, it might be expected that the LRT is relatively powerful against local alternatives. Indeed, the power tables in Section 4 indicate that the LRT is more powerful than the $D_{p+q}^2 - D_q^2$ test against alternatives Δ in a quite wide interval of the form $(0, \Delta^*)$ (depending on n, p, q, α).

To illustrate the scope of Theorem 2.1 it is of interest to point out that because of the presence of local alternatives in (1.8), the level α LRT $(0 < \alpha < \alpha^*)$ cannot be expressed as a generalized Bayes test in the sense of Farrell (1968) and Ghia (1976) (cf. also Remark 2.4). That is, for $0 < \alpha < \alpha^*$ there exist no locally finite nonzero measure π on $(0, \infty)$ and no constant $c \leqslant \infty$ such that

$$\frac{w}{v} > \beta(\alpha) \Leftrightarrow \int_0^\infty R_\Delta(v, w)\pi(d\Delta) < c \qquad \text{for a.e. } (v, w) \in \Lambda,$$

where $\beta(\alpha)$ is defined in (3.7). This is demonstrated on pages 30–32 of Marden and Perlman (1977) by using the uniqueness of the Laplace transform. Hence, Theorem 5.1 of Farrell (1968) cannot be used to prove admissibility (among invariant tests) of the level α LRT $(0 < \alpha < \alpha^*)$.

3.5 *Some further admissible Bayes tests for problem (1.8).* Proper Bayes tests corresponding to the gamma priors

$$\pi(d\Delta) = e^{-s\Delta}\Delta^{t-1}d\Delta \qquad s, t > 0 \tag{3.17}$$

are easily obtained. From (2.11), these tests reject H_0 when

$$\left(\frac{2}{2s+u}\right)^t \sum_{k=0}^{\infty} \frac{c_k}{k!}\Gamma(t+k)\left(\frac{v}{2s+u}\right)^k > b_\alpha. \tag{3.18}$$

If we take $t = \nu_1/2$, then (3.18) reduces to the simple form

$$(2s+w+v)^{\nu_2/2}(2s+w)^{-(\nu_1+\nu_2)/2} > b'_\alpha. \tag{3.19}$$

A family of improper Bayes tests can be obtained by taking $s = 0$ in (3.17) and (3.18). If in addition we set $t = \nu_1/2$, then from (3.19) we obtain the improper Bayes test which rejects H_0 when

$$(w+v)^{\nu_2/2}w^{-(\nu_1+\nu_2)/2} > b'_\alpha, \tag{3.20}$$

or equivalently, when

$$(1+l) > b''_\alpha(1+m)^{-\nu_1/(\nu_1+\nu_2)}. \tag{3.21}$$

By Theorem 2.1, or by Theorem 5.1 of Farrell (1968), these improper ($\equiv$ generalized) Bayes tests are admissible for problem (1.8). By comparing (3.21) and (1.7), it is seen that these tests are similar in form to the AMP test, which is to be expected since the improper prior distribution obtained by setting $s = 0$ and $t = \nu_1/2$ in (3.17) assigns infinite mass to any neighborhood of ∞.

Another improper prior distribution which leads to an admissible test for (1.8) is given by $\pi(d\Delta) = d\Delta/\Delta$ on $(0, \infty)$, obtained by setting $s = t = 0$ in (3.17) or by setting $\gamma = 0$ in Remark 3.1. Since this prior assigns infinite mass to neighborhoods of both 0 and ∞, it may be expected to yield a test that is relatively powerful against a broad range of alternatives; however, we have not yet carried out the necessary numerical calculations. To derive the test corresponding to this prior, we cannot simply set $s = t = 0$ in (3.18), since the left-hand side becomes identically $+\infty$, as $\Gamma(0) = \infty$. Put another way, $\int_0^\infty R_\Delta d\Delta/\Delta \equiv \infty$, as $R_0 \equiv 1$, so the prior $d\Delta/\Delta$ does not lead to a generalized Bayes test in Farrell's sense. Nor can we substitute $\gamma = 0$ in (3.11), since $\int_1^\infty d\Delta/\Delta = \infty$. Instead, we compute the test obtained by setting $w_0 = 0$, $\pi^0(d\Delta) = I_{(0,1)}(\Delta)d\Delta$, and $\pi^1(\Delta) = \Delta^{-1}I_{[1,\infty)}(\Delta)d\Delta$ in (2.1). Since

$$\begin{aligned}
&\int_0^1[(R_\Delta - 1)/\Delta]d\Delta + \int_1^\infty R_\Delta \frac{d\Delta}{\Delta} \\
&= \sum_{k=1}^{\infty}\frac{c_k}{k!}v^k\int_0^\infty\left(\frac{\Delta}{2}\right)^k e^{-\frac{1}{2}\Delta u}\frac{d\Delta}{\Delta} + \int_0^1(e^{-\frac{1}{2}\Delta u}-1)\frac{d\Delta}{\Delta} + \int_1^\infty e^{-\frac{1}{2}\Delta u}\frac{d\Delta}{\Delta} \\
&= \sum_{k=1}^{\infty}\frac{c_k}{k}\left(\frac{v}{u}\right)^k + \int_0^{u/2}(e^{-y}-1)\frac{dy}{y} + \int_{u/2}^\infty e^{-y}\frac{dy}{y} \qquad (3.22)\\
&= \sum_{k=1}^{\infty}\frac{c_k}{k}\left(\frac{v}{u}\right)^k + \int_{u/2}^{1/2}\frac{dy}{y} + \int_0^{1/2}(e^{-y}-1)\frac{dy}{y} + \int_{1/2}^\infty e^{-y}\frac{dy}{y} \\
&= \sum_{k=1}^{\infty}\frac{c_k}{k}\left(\frac{v}{u}\right)^k - \log u + (\text{constant}),
\end{aligned}$$

this test rejects H_0 for large values of

$$\Sigma_{k=1}^{\infty}\frac{c_k}{k}\left(\frac{l}{1+l}\right)^k + \log(1+m), \tag{3.23}$$

hence is similar to the LRT which rejects H_0 for large values of l. It would be of interest to compare the powers of these two tests.

3.6 *Remarks.* Thus far we have shown that the following tests are admissible for (1.8): the LMP and AMP tests, the level α LRT for $0 < \alpha < \alpha^*$, the proper Bayes tests (3.18) and (3.19), the improper Bayes tests obtained by setting $s = 0$ in (3.18) (which includes (3.20)), the test based on (3.23), and the most powerful (MP) test for testing H_0: $\Delta = 0$ vs. a simple alternative $\Delta = \Delta_1 > 0$ (see Remark 2.10). Each of these tests has the property that its *rejection* region is convex in (v, w), so one might ask whether all admissible tests have this property. It is easy to see that the answer is no: truncate any of the above tests (except the AMP) with a truncation set $\{w \geq w_0\}$ where $0 < w_0 < 1$.

An important open question, beyond the scope of the present paper, is to determine the class of invariant tests which are admissible within the class of *all* procedures for problem (1.1). In particular, is the level α LRT $(0 < \alpha < \alpha^*)$ admissible for problem (1.1)? Only one such admissible test, the overall T^2 test, is known at present—see the following paragraph. Kiefer and Schwartz ((1965), page 762–763) point out that the noninvariant test which rejects H_0 for large values of $X_1'S_{11}^{-1}X_1$ is admissible for problem (1.1), since it is proper Bayes with respect to a prior for (μ, Σ) under which $\Sigma_{12} = 0$ with probability 1. The statistic $X_1'S_{11}^{-1}X_1$ is simply the T^2 statistic based on the first p variables, so

$$X_1'S_{11}^{-1}X_1 \sim \chi_p^2(\Delta^*)/\chi_{n-p+1}^2$$

where $\Delta^* = \mu_1'\Sigma_{11}^{-1}\mu_1 \leq \Delta$. For fixed μ_1, Σ_{11}, and Σ_{22}, however, Δ^*/Δ can be arbitrarily small if $\Sigma_{12} \neq 0$, so this test can be substantially less powerful than those we have considered (e.g., the LRT and overall T^2 test). In applications, it might be wise to obtain a preliminary estimate of the magnitude of the canonical correlations between X_1 and X_2, and apply the test based on $X_1'S_{11}^{-1}X_1$ only if these are sufficiently small.

By utilizing the exponential structure of the distribution of (X, S), the method of Stein (1956) and Schwartz (1967) can be applied to show that the overall T^2 test is admissible for problem (1.1). The joint density of (X, S), expressed as an exponential family, is of the form

$$f_{\xi,\Gamma}(x, s) = \beta(\xi, \Gamma)h(s)\exp\left\{-\frac{1}{2}\operatorname{tr}\Gamma\, t + x'\xi\right\},$$

where $\Gamma = \Sigma^{-1}$, $\xi = \Sigma^{-1}\mu$, and $t = s + xx'$. The testing problem (1.1) can be restated as

$$H_0 : (\xi, \Gamma) \in \Theta_0 \quad \text{vs.} \quad H_1 : (\xi, \Gamma) \in \Theta_1,$$

where

$$\Theta_0 = \{(0, \Gamma)|\Gamma \text{ positive definite}\},$$

$$\Theta_1 = \{(\xi, \Gamma)|\Gamma \text{ positive definite, } (\Gamma^{-1}\xi)_1 \neq 0, (\Gamma^{-1}\xi)_2 = 0\},$$

and where $(\Gamma^{-1}\xi)_1 = \mu_1$, $(\Gamma^{-1}\xi)_2 = \mu_2$. Since $(\xi, \Gamma) \in \Theta_1 \Rightarrow (\lambda\xi, \lambda\Gamma) \in \Theta_1$ for all $\lambda > 0$, the theorem of Stein (1956) implies that for any subset $\Theta_2 \subset \Theta_1$,

$$\{(x, s)|\sup_{(\xi, \Gamma) \in \Theta_2}(-\tfrac{1}{2}\text{tr}\Gamma\, t + x'\xi) \leq c\}$$

is an admissible acceptance region (AAR) for problem (1.1). In particular, if G is the group of $(p + q) \times (p + q)$ matrices introduced above (1.8) in Section 1, then

$$\{(x, s)|\sup_{A \in G}(-\tfrac{1}{2}\text{tr}A'At + x'A'e) \leq c\} \tag{3.24}$$

is an invariant AAR for (1.1), where $e = (1, 0, \cdots, 0)' : (p + q) \times 1$. Now, we can choose $A_0 \in G$ such that $A_0 t A_0' = I$ and $A_0 x = (v^{1/2}e_1', (1 - u)^{1/2}e_2')'$, where $e_1 = (1, 0, \cdots, 0)' : p \times 1$, $e_2 = (1, 0, \cdots, 0)' : q \times 1$, and u, v are defined in (2.5) (note that $x't^{-1}x = v + (1 - u)$). Since G is a group,

$$\begin{aligned}\sup(-\tfrac{1}{2}\text{tr}\, A'At + x'A'e) &= \sup_{A \in G}(-\tfrac{1}{2}\text{tr}\, A_0'A'AA_0t + x'A_0'A'e)\\ &= \sup_{A \in G}[-\tfrac{1}{2}\text{tr}\, A'A + (v^{1/2}e_1', (1 - u)^{1/2}e_2')A'e]\\ &= \sup_{A \in G}[-\tfrac{1}{2}\Sigma_{i,j=1}^{p+q}a_{ij}^2 + a_{11}v^{1/2} + a_{1,p+1}(1 - u)^{1/2}]\\ &= \sup_{\sigma, \tau \text{ real}}[-\tfrac{1}{2}\sigma^2 - \tfrac{1}{2}\tau^2 + \sigma v^{1/2} + \tau(1 - u)^{1/2}]\\ &= \tfrac{1}{2}[v + (1 - u)].\end{aligned}$$

Since $u - v = w$, the AAR (3.24) is thus equivalent to $\{(x, s)|w \geq c_1\}$, which by (2.6) and (1.7) is exactly the acceptance region of the overall T^2 test. (Although this test has now been shown to be admissible for problem (1.1), the numerical results in Section 4 show that its power tends to be low for most alternatives, even in comparison to other invariant tests.)

4. Power comparisons. Table 4.1 compares the (unconditional) powers of the LRT (1.4), the $D_{p+q}^2 - D_q^2$ test (1.5), and the overall T^2 test (1.7) for $\alpha = .10$; $n = 18, 38$; $p = 1, 5, 11$; and $q = 2, 6$. The relative performances of the three tests are summarized in Tables 4.2-4.4. Initially, the power of the LRT exceeds that of the $D_{p+q}^2 - D_q^2$ test, but then eventually drops below as Δ increases. Table 4.2 gives the power value at which the crossover occurs. Tables 4.3 and 4.4 were obtained by examining the differences in powers of the tests for the values of Δ considered in Table 4.1, except for the three asterisked numbers, which are based on additional computations. The dashes in Table 4.4 represent cases for which the LRT has greater power than the T^2 test for all values of Δ considered.

From these tables and similar computations carried out for $\alpha = .05$, we find that among the three tests the LRT is most powerful for small or moderate values of the alternative Δ, while the $D_{p+q}^2 - D_q^2$ test is most powerful for large values of Δ. Although the T^2 test is AMP, its asymptotic optimality is obviously of no practical significance. From Tables 4.2 and 4.3, it appears that the performance of the LRT relative to the $D_{p+q}^2 - D_q^2$ test tends to improve as p increases. However, the differences in power between these two tests are small. Therefore, the LRT is preferable since its critical points are simply those of the standard F distribution (see(1.4)), whereas those of the $D_{p+q}^2 - D_q^2$ test are more difficult to obtain. The overall T^2 test is substantially less powerful than the other two tests (confirming

TABLE 4.1a
Powers of three invariant tests ($p = 1, \alpha = .10$)

		Δ										
(n, q)	Test	.5	1.0	2.0	3.0	4.0	7.0	10.0	16.0	20.0	25.0	35.0
	LRT	.1764	.2500	.3859	.5044	.6048	.8107	.9148	.9847	.9954	.9990	1.0000
(38, 2)	$D_{p+q}^2 - D_q^2$	.1761	.2495	.3853	.5038	.6043	.8107	.9151	.9850	.9956	.9991	1.0000
	T^2	.1430	.1876	.2788	.3690	.4550	.6707	.8157	.9511	.9815	.9949	.9997
	LRT	.1679	.2337	.3567	.4661	.5612	.7672	.8824	.9726	.9900	.9972	.9998
(38, 6)	$D_{p+q}^2 - D_q^2$	.1671	.2323	.3546	.4639	.5593	.7668	.8831	.9736	.9907	.9976	.9998
	T^2	.1244	.1503	.2054	.2636	.3232	.4981	.6493	.8513	.9229	.9685	.9957
	LRT	.1688	.2353	.3594	.4695	.5650	.7704	.8843	.9727	.9898	.9970	.9997
(18, 2)	$D_{p+q}^2 - D_q^2$	.1677	.2334	.3566	.4665	.5623	.7697	.8852	.9742	.9909	.9976	.9998
	T^2	.1388	.1789	.2609	.3427	.4216	.6263	.7736	.9277	.9687	.9897	.9991
	LRT	.1513	.2013	.2967	.3845	.4642	.6543	.7816	.9146	.9540	.9782	.9945
(18, 6)	$D_{p+q}^2 - D_q^2$	.1480	.1954	.2870	.3730	.4523	.6462	.7793	.9194	.9599	.9833	.9970
	T^2	.1192	.1392	.1814	.2257	.2712	.4091	.5379	.7413	.8333	.9081	.9753

TABLE 4.1b
Powers of three invariant tests ($p = 5, \alpha = .10$)

		Δ										
(n, q)	Test	1.0	4.0	8.0	12.0	16.0	20.0	24.0	30.0	40.0	50.0	100.0
	LRT	.1596	.3577	.6013	.7766	.8841	.9434	.9737	.9922	.9991	.9999	1.0000
(38, 2)	$D_{p+q}^2 - D_q^2$	.1590	.3558	.5993	.7755	.8839	.9436	.9740	.9925	.9992	.9999	1.0000
	T^2	.1503	.3232	.5519	.7315	.8513	.9229	.9621	.9880	.9985	.9998	1.0000
	LRT	.1519	.3243	.5453	.7178	.8352	.9081	.9505	.9814	.9966	.9994	1.0000
(38, 6)	$D_{p+q}^2 - D_q^2$	.1502	.3185	.5383	.7129	.8330	.9080	.9515	.9826	.9972	.9996	1.0000
	T^2	.1354	.2595	.4404	.6070	.7406	.8376	.9028	.9582	.9912	.9984	1.0000
	LRT	.1456	.2954	.4921	.6556	.7769	.8604	.9149	.9608	.9897	.9973	1.0000
(18, 2)	$D_{p+q}^2 - D_q^2$	.1438	.2894	.4838	.6484	.7724	.8585	.9150	.9624	.9913	.9981	1.0000
	T^2	.1392	.2712	.4535	.6146	.7413	.8333	.8961	.9515	.9878	.9972	1.0000
	LRT	.1299	.2265	.3589	.4829	.5913	.6817	.7548	.8363	.9177	.9585	.9979
(18, 6)	$D_{p+q}^2 - D_q^2$	.1263	.2125	.3351	.4549	.5640	.6584	.7371	.8272	.9190	.9640	.9996
	T^2	.1223	.1958	.3027	.4110	.5134	.6057	.6858	.7825	.8888	.9464	.9992

TABLE 4.1c
Powers of three invariant tests ($p = 11$, $\alpha = .10$)

(n, q)	Test	Δ 3.0	6.0	12.0	20.0	30.0	40.0	50.0	60.0	100.0	350.0	1000.0
	LRT	.2072	.3312	.5746	.8068	.9422	.9854	.9967	.9993	1.0000	1.0000	1.0000
(38, 2)	$D^2_{p+q} - D^2_q$	.2058	.3285	.5712	.8049	.9420	.9857	.9969	.9994	1.0000	1.0000	1.0000
	T^2	.1999	.3167	.5526	.7884	.9338	.9829	.9961	.9992	1.0000	1.0000	1.0000
	LRT	.1902	.2940	.5062	.7332	.8949	.9634	.9882	.9964	1.0000	1.0000	1.0000
(38, 6)	$D^2_{p+q} - D^2_q$	.1862	.2865	.4952	.7248	.8925	.9639	.9892	.9970	1.0000	1.0000	1.0000
	T^2	.1759	.2650	.4570	.6826	.8636	.9496	.9834	.9950	1.0000	1.0000	1.0000
	LRT	.1528	.2100	.3297	.4833	.6447	.7652	.8492	.9051	.9865	1.0000	1.0000
(18, 2)	$D^2_{p+q} - D^2_q$	.1511	.2067	.3238	.4756	.6375	.7602	.8466	.9044	.9880	1.0000	1.0000
	T^2	.1505	.2055	.3214	.4722	.6336	.7565	.8435	.9020	.9874	1.0000	1.0000
	LRT	.1173	.1343	.1673	.2093	.2586	.3048	.3479	.3883	.5253	.8953	.9961
(18, 6)	$D^2_{p+q} - D^2_q$	.1165	.1327	.1642	.2045	.2520	.2968	.3388	.3784	.5143	.8961	.9981
	T^2	.1165	.1327	.1641	.2043	.2518	.2965	.3386	.3781	.5139	.8958	.9981

TABLE 4.2
Power value at which the power functions of the LRT *and* $D^2_{p+q} - D^2_q$ *test cross* ($\alpha = .10$)

(n, q)	p = 1	5	11
(38, 2)	.81	.94	.96
(38, 6)	.81	.91	.95
(18, 2)	.82	.91	.92
(18, 6)	.82	.91	.88

TABLE 4.3
($\alpha = .10$)
Maximum [*power of* LRT—*power of* ($D^2_{p+q} - D^2_q$) *test*]
(*Maximum* [*power of* ($D^2_{p+q} - D^2_q$) *test—power of* LRT])

(n, q)	p = 1	5	11
(38, 2)	.0006 (.0003)	.0020 (.0003)	.0034 (.0003)
(38, 6)	.0022 (.0010)	.0070 (.0012)	.0104* (.0010)
(18, 2)	.0030 (.0015)	.0083 (.0016)	.0077 (.0015)
(18, 6)	.0119 (.0059)	.0280 (.0055)	.0110 (.0042)*

Table 4.4
($\alpha = .10$)
Maximum [power of LRT*—power of T^2 test]*
(Maximum [power of T^2 test—power of LRT])

(n, q)	p = 1	5	11
(38, 2)	.1498 (---)	.0494 (---)	.0220 (.0000)
(38, 6)	.2691 (---)	.1108 (.0000)	.0506 (.0000)
(18, 2)	.1441 (---)	.0410 (.0000)	.0111 (.0009)
(18, 6)	.1930 (---)	.0779 (.0013)	.0114 (.0041)*

Cochran's (1964) findings) except when p is large and n and q are small (see Table 4.4), in which case all three tests have small differences in power. Thus, on the basis of these computations and the results of Section 3, use of the LRT is recommended in all cases.

5. Proof of Theorem 2.1; truncated generalized Bayes tests. Our starting point is Theorem 5.8 of Wald (1950), which implies that the set of all weak* limits of sequences $\{\phi_n\}$ of proper Bayes tests for problem (1.8) is an essentially complete class of tests. We follow the approach of Ghia (1976, Chapter 3) to show in Theorem 5.7 that any such weak* limit must be of the form (2.1). Further argument is required to show that tests of the form (2.1) form a complete class (Theorem 5.10) and are admissible (Theorem 5.11).

Theorem 5.7 goes farther than the comparable result in Chapter 3 of Ghia (1976), for his Assumption 2.5 applied to our problem (1.8) would require that the null and alternative hypotheses be topologically separated in the parameter space. This is obviously not the case in (1.8) since "local alternatives" ($\Delta \to 0$) are present —see Remark 5.8. (Chapter 3 of Ghia (1976) treats one-parameter families of distributions, whereas Chapter 4 deals with certain multiparameter families; in each case local alternatives are not permitted. Marden (1977) has extended all of these results to testing problems where local alternatives may be present.)

To begin our development toward Theorem 5.7, suppose that ϕ_n, ϕ are test functions for problem (1.8) defined on Λ, i.e., measurable functions of (v, w) such that $0 \leqslant \phi_n, \phi \leqslant 1$.

DEFINITION 5.1. We say ϕ_n converges to ϕ in the weak* sense, written $\phi_n \to_{w^*} \phi$, if

$$\int_\Lambda f\phi_n d\mu \to \int_\Lambda f\phi \, d\mu$$

for all bounded measurable functions f on Λ, where μ is Lebesgue measure on Λ.

Any proper Bayes test ϕ_n is of the form

$$\phi_n = \phi_n(v, w) = 1 \quad \text{if } \textstyle\int_0^\infty R_\Delta(v, w)\pi_n(d\Delta) > 1 \qquad (5.1)$$
$$= 0 \quad \textit{otherwise} \qquad \text{a.e.}[\mu],$$

where π_n is a finite measure on $(0, \infty)$ and R_Δ is given by (1.10) and (2.12). Note that

$$\mu\{(v, w) | \textstyle\int_{0-}^{\infty} R_\Delta \pi(d\Delta) = d\} = 0 \qquad (5.2)$$

for any $0 \leqslant d < \infty$ and any nonnegative measure π on $[0, \infty)$ which assigns positive mass to $(0, \infty)$. This follows from the fact that for $\Delta > 0$, $R_\Delta(v, w)$ is strictly increasing in v for each fixed w (see the proof of Lemma 2.6(a)), and an application of Fubini's theorem; alternatively, (5.2) follows from (2.11) and the uniqueness of the coefficients of a power series.

The next lemma together with (5.1) shows that in order to determine the form of the weak* limit of a sequence $\{\phi_n\}$ of proper Bayes tests, it suffices to study the limiting behavior of $\int R_\Delta \pi_n(d\Delta)$.

LEMMA 5.2 (Ghia (1976), Theorem 2.1). *Let ψ_n, ψ be test functions on Λ with ψ_n of the form*

$$\psi_n(v, w) = 1 \quad \textit{if } H_n(v, w) > 1$$
$$= 0 \quad \textit{otherwise},$$

where $\{H_n\}$ is a sequence of real-valued measurable functions on Λ. If $\psi_n \to_{w^*} \psi$ and $H_n \to H$ a.e. $[\mu]$, *where H is an extended real-valued measurable function on Λ, then ψ is of the form*

$$\psi(v, w) = 1 \quad \textit{if } H(v, w) > 1$$
$$= \chi(v, w) \quad \textit{if } H(v, w) = 1$$
$$= 0 \quad \textit{if } H(v, w) < 1 \qquad \text{a.e. } [\mu]$$

for some measurable function χ with $0 \leqslant \chi \leqslant 1$.

PROOF. Let $B = \{H < 1\} \subseteq \Lambda$. Clearly $\psi_n \to 0$ on B, so $\int_B \psi_n \, d\mu \to 0$. By Definition 5.1, however, $\int_B \psi_n \, d\mu \to \int_B \psi \, d\mu$. Thus, since $0 \leqslant \psi \leqslant 1$, $\psi = 0$ a.e. on B. Similarly, $\psi = 1$ a.e. on $\{H > 1\}$.

The following elementary properties of the functions R_Δ will be applied often in subsequent arguments without explicit references.

LEMMA 5.3.

(a) *$R_\Delta(v, w)$ is strictly positive and is jointly continuous in (Δ, v, w) for $0 \leqslant \Delta < \infty$, $(v, w) \in \Lambda$.*

(b) *For each fixed $(v, w) \in \Lambda$,*

(1) $R_0(v, w) = 1$,

(2) $R_\Delta(v, w) \to 0$ iff $\Delta \to \infty$,

(3) $\sup\{R_\Delta(v, w) | 0 \leqslant \Delta < \infty\} < \infty$,

(4) $\{\Delta | R_\Delta(v, w) \geqslant a\}$ *is compact for each* $a > 0$.

(c) *For each fixed* $(v_1, w_1), (v_2, w_2) \in \Lambda$,

(1) $\lim_{\Delta\to\infty}[R_\Delta(v_1, w_1)/R_\Delta(v_2, w_2)] = 0$ *if* $w_1 > w_2$,

(2) $0 < \lim_{\Delta\to\infty}[R_\Delta(v_1, w_1)/R_\Delta(v_2, w_2)] < \infty$ *if* $w_1 = w_2$,

(3) $S(v_1, w_1;\ v_2, w_2) \equiv \sup\{R_\Delta(v_1, w_1)/R_\Delta(v_2, w_2) | 0 \leqslant \Delta < \infty\} < \infty$ *if* $w_1 \geqslant w_2$.

(d) *Extend the definition of* $(R_\Delta - 1)/\Delta$ *to* $\Delta = 0$ *by continuity*, i.e.,

$$\left[(R_\Delta(v, w) - 1)\Delta\right]_{\Delta=0} = \left.\frac{\partial R_\Delta}{\partial \Delta}\right|_{\Delta=0} = \frac{1}{2}\left(\frac{\nu_2}{\nu_1}v - w\right).$$

Then for any $a < \infty$, $(R_\Delta(v, w) - 1)/\Delta$ *is uniformly bounded for* $(v, w) \in \Lambda$, $0 \leqslant \Delta \leqslant a$.

PROOF. Parts (a) and (b)(1) follow directly from (2.12), while (b)(2) follows from (3.6). These in turn imply (b)(3) and (b)(4). Parts (c)(1) and (c)(2) follow from (2.12) and (3.6); more precisely, either 13.1.4 of Abramowitz and Stegun (1964) or (2.36) and (2.39) show that the limit in (c)(2) equals $(v_1/v_2)^{\nu_2/2}$. Part (c)(3) follows from the preceding and the fact that for all $N < \infty$ and all $(v, w) \in \Lambda$,

$$\inf\{R_\Delta(v, w) | 0 \leqslant \Delta \leqslant N\} > 0. \tag{5.3}$$

To prove part (d), we have from (2.12) that for all $(v, w) \in \Lambda$ and $0 < \Delta \leqslant a$,

$$\begin{aligned}\left|\frac{R_\Delta(v, w) - 1}{\Delta}\right| &\leqslant \left|\frac{e^{-\frac{1}{2}\Delta(v+w)} - 1}{\Delta}\right| + \Sigma_{k=1}^{\infty}\frac{c_k}{k!}\left(\frac{v}{2}\right)^k\Delta^{k-1}\\ &\leqslant \frac{1}{2} + \Sigma_{k=1}^{\infty}\frac{c_k}{k!}\left(\frac{1}{2}\right)^k a^{k-1}\\ &< \infty,\end{aligned}$$

since $0 < v + w \equiv u < 1$.

DEFINITION 5.4. A nonnegative measure π on $[0, \infty)$ is said to be locally finite if $\int_{0-}^{\infty} g\,d\pi < \infty$ for all continuous functions g on $[0, \infty)$ with compact support. If $\{\pi_n\}$ is a sequence of nonnegative measures on $[0, \infty)$ and π is locally finite, we say that π_n converges vaguely to π, written $\pi_n \to_v \pi$, if

$$\int_{0-}^{\infty} g\,d\pi_n \to \int_{0-}^{\infty} g\,d\pi$$

for all such g.

LEMMA 5.5 (Ghia (1976), Lemma 2.2). *Let* $\{\pi_n\}$ *be a sequence of nonnegative measures on* $[0, \infty)$. *Then either*

$$\limsup_{n\to\infty}\pi_n([0, N]) = \infty \tag{5.4}$$

for some $N < \infty$, *or there exists a locally finite measure* π *(possibly the zero measure) on* $[0, \infty)$ *and an increasing subsequence* $\{j(n)\} \subseteq \{n\}$ *of the positive integers such that* $\pi_{j(n)} \to_v \pi$ *(or both). In the former case, there exists a subsequence*

$\{i(n)\} \subseteq \{n\}$ *such that*

(5.5) $$\int_{0-}^{\infty} R_\Delta(v, w)\pi_{i(n)}(d\Delta) \to \infty \qquad \forall (v, w) \in \Lambda;$$

the subsequence $\{i(n)\}$ *does not depend on* (v, w).

PROOF. If the limit superior in (5.4) is finite for all $N < \infty$, then we can apply the Cantor diagonalization process to find a subsequence $\{j(n)\}$ such that $\pi_{j(n)}([0, N])$ converges to a finite limit for all rational $N \in [0, \infty)$. It follows from the standard theory of weak convergence of distribution functions on the real line (cf. Loeve (1963), pages 179–180) that $\pi_n \to_v \pi$ for some locally finite measure π. Finally, if (5.4) holds, then (5.5) follows from (5.3).

LEMMA 5.6 (Ghia (1976), Lemmas 2.3, 3.1, 3.2, 3.3). *Let* π_n, π *be nonnegative measures on* $[0, \infty)$, *with* π *locally finite, such that* $\pi_n \to_v \pi$. *Let* (v, w), (v_1, w_1), (v_2, w_2) *denote points in* Λ, *and let* $\{k(n)\} \subseteq \{n\}$ *denote an increasing subsequence of the positive integers.*

(a) *If* $w_1 \geq w_2$ *then*

$$\lim_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v_1, w_1)\pi_{k(n)}(d\Delta) = \infty \Rightarrow \lim_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v_2, w_2)\pi_{k(n)}(d\Delta) = \infty.$$

In particular, for each $w \in (0, 1)$ *either*

$$\limsup_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) = \infty$$

for all v *or for no* v.

(b) *There exists* $w_0 \in [0, 1]$ *and an increasing subsequence* $\{l(n)\} \subseteq \{n\}$ *such that*

(5.6) $$\lim_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v, w)\pi_{l(n)}(d\Delta) = \infty \qquad \text{if } w < w_0$$

(5.7) $$\lim_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) = \int_{0-}^{\infty} R_\Delta(v, w)\pi(d\Delta) < \infty \qquad \text{if } w > w_0;$$

the subsequence $\{l(n)\}$ *does not depend on* (v, w).

PROOF.

(a) Immediate from Lemma (5.3)(c)(3) and the inequality

(5.8) $$\int R_\Delta(v_1, w_1)\pi(d\Delta) \leq S(v_1, w_1; v_2, w_2) \int R_\Delta(v_2, w_2)\pi(d\Delta).$$

(b) By (a),

$$\sup\{w | \limsup_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) = \infty\}$$
$$= \inf\{w | \limsup_{n\to\infty} \int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) < \infty\} \equiv w_0;$$

clearly $0 \leq w_0 \leq 1$. To prove (5.6), assume that $w_0 > 0$ (otherwise (5.6) is vacuous), choose a sequence $\{w_n\}$ such that $0 < w_n \uparrow w_0$, and let v_n be any point such that $(v_n, w_n) \in \Lambda$. For each n, we may choose an integer $l(n)$ large enough that

(5.9)

$$\int_{0-}^{\infty} R_\Delta(v_n, w_n)\pi_{l(n)}(d\Delta) \geq n \cdot \max\{1, S(v_n, w_n; v_{n-1}, w_{n-1}), \cdots, S(v_n, w_n; v_1, w_1)\}.$$

From (5.8) and (5.9),

$$\int_{0-}^{\infty} R_\Delta(v_r, w_r)\pi_{l(n)}(d\Delta) \geq n, \qquad 1 \leq r \leq n,$$

so for each fixed r,

(5.10) $$\lim_{n\to\infty}\int_{0-}^{\infty} R_\Delta(v_r, w_r)\pi_{l(n)}(d\Delta) = \infty.$$

Since $w_r \uparrow w_0$, (5.6) follows from (5.10) and part (a).

To prove (5.7), assume that $w_0 < 1$ and fix $(v, w) \in \Lambda$ with $w > w_0$. First, if N is a continuity point of π (i.e., $\pi(\{N\}) = 0$) then

$$\int_{0-}^{N} R_\Delta(v, w)\pi(d\Delta) = \lim_{n\to\infty}\int_{0-}^{N} R_\Delta(v, w)\pi_n(d\Delta) \leqslant \liminf_{n\to\infty}\int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta),$$

which is finite. Let $N \to \infty$ to obtain

(5.11) $$\int_{0-}^{\infty} R_\Delta(v, w)\pi(d\Delta) < \infty.$$

Next, choose $(v_1, w_1) \in \Lambda$ such that $w_0 < w_1 < w$. Then

$$\int_{N}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) \leqslant S_N \int_{0-}^{\infty} R_\Delta(v_1, w_1)\pi_n(d\Delta)$$

where

$$S_N = \sup\{R_\Delta(v, w)/R_\Delta(v_1, w_1) | N \leqslant \Delta < \infty\}.$$

Thus,

(5.12) $$\limsup_{n\to\infty}\int_{N}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) \leqslant S_N \limsup_{n\to\infty}\int_{0-}^{\infty} R_\Delta(v_1, w_1)\pi_n(d\Delta) \to 0$$

as $N \to \infty$, by Lemma 5.3(c)(1). Finally,

$$\begin{aligned}&\left|\int_{0-}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) - \int_{0-}^{\infty} R_\Delta(v, w)\pi(d\Delta)\right| \\ &\quad\leqslant \left|\int_{0-}^{N} R_\Delta(v, w)\pi_n(d\Delta) - \int_{0-}^{N} R_\Delta(v, w)\pi(d\Delta)\right| \\ &\quad\quad + \int_{N}^{\infty} R_\Delta(v, w)\pi_n(d\Delta) + \int_{N}^{\infty} R_\Delta(v, w)\pi(d\Delta).\end{aligned}$$

Let $n \to \infty$, then $N \to \infty$, and apply (5.11) and (5.12) to obtain (5.7). (Note: it could happen that $\int_{0-}^{\infty} R_\Delta(v, w)\pi(d\Delta) < \infty$ for some $w < w_0$.)

THEOREM 5.7. *If ϕ is the weak* limit of a sequence of proper Bayes tests $\{\phi_n\}$ defined on Λ, then ϕ is of the form* (2.1). *Thus, the set of all tests of the form* (2.1) *is an essentially complete class for problem* (1.8).

PROOF. Each ϕ_n is of the form (5.1) for some finite measure π_n on $(0, \infty)$. If we were to apply Lemmas 5.5 and 5.6 directly to the sequence $\{\pi_n\}$, thereby obtaining (in one case) a subsequence $\{\pi_{k(n)}\}$ converging vaguely to some locally finite measure π on $[0, \infty)$ and satisfying $\int R_\Delta \pi_{k(n)}(d\Delta) \to \int R_\Delta \pi(d\Delta)$, it could happen that π is the probability measure degenerate at $\Delta = 0$. In that case, $\int R_\Delta \pi(d\Delta) \equiv 1$ for all (v, w), so Lemma 5.2 would provide no information about the form of ϕ (however, see Remark 5.8). To circumvent this difficulty, rewrite the inequality in (5.1) so that $\phi_n = 1$ if

(5.13) $$\int_{0}^{1-}(R_\Delta - 1)\pi_n(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi_n(d\Delta) > 1 - \int_{0}^{1-}\pi_n(d\Delta).$$

Define

$$s_n = \int_{0}^{1-}\Delta\pi_n(d\Delta) + \left|1 - \int_{0}^{1-}\pi_n(d\Delta)\right|,$$

and note that $0 < s_n < \infty$ for all n. Divide both sides of (5.13) by s_n, so that $\phi_n = 1$ if

$$a_n \int_0^{1-} [(R_\Delta - 1)/\Delta] \pi_n^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi_n^1(d\Delta) > c_n, \tag{5.14}$$

where

$$\begin{aligned} a_n &= s_n^{-1} \int_0^{1-} \Delta \pi_n(d\Delta) && \text{if} \quad \pi_n((0, 1)) > 0 \\ &= 0 && \text{if} \quad \pi_n((0, 1)) = 0, \\ \pi_n^0(d\Delta) &= a_n^{-1} s_n^{-1} \Delta I_{(0,1)}(\Delta) \pi_n(d\Delta) && \text{if} \quad \pi_n((0, 1)) > 0 \\ &= \pi^*(d\Delta) && \text{if} \quad \pi_n((0, 1)) = 0, \\ \pi_n^1(d\Delta) &= s_n^{-1} I_{[1, \infty)}(\Delta) \pi_n(d\Delta), \\ c_n &= s_n^{-1}\left(1 - \int_0^{1-} \pi_n(d\Delta)\right), \end{aligned} \tag{5.15}$$

and where π^* is an arbitrary, fixed probability measure on (0, 1). Note that $a_n \geqslant 0$, $a_n + |c_n| = 1$, and π_n^0 is a probability measure on (0, 1).

Apply Lemma 5.5 to the sequence $\{\pi_n^1\}$. Either (a) there exists a subsequence $\{i(n)\} \subseteq \{n\}$ such that (5.5) holds with $\pi_{i(n)}$ replaced by $\pi_{i(n)}^1$, or (b) there exists a subsequence $\{j(n)\} \subseteq \{n\}$ and a locally finite measure π^1 (possibly the zero measure) on $[1, \infty)$ such that $\pi_{j(n)}^1 \to_v \pi^1$. In case (a), the first integral in (5.14) is bounded (see Lemma 5.3(d)) and the second integral (with n replaced by $i(n)$) approaches ∞, while c_n is bounded. Therefore, Lemma 5.2 implies that $\phi = 1$ a.e. $[\mu]$, which is of the form (2.1) with $w_0 = 1$.

In case (b), Lemma 5.6 states that there exists $w_0 \in [0, 1]$ and a subsequence $\{l(n)\} \subseteq \{j(n)\}$ such that (5.6) and (5.7) hold with $\pi_{l(n)}$, π_n, and π replaced by $\pi_{l(n)}^1$, $\pi_{j(n)}^1$, and π^1, respectively. Furthermore, since $\{a_n\}$ and $\{c_n\}$ are bounded real sequences, and since $\{\pi_n^0\}$ is a sequence of probability measures on a compact set, there exist a subsequence $\{k(n)\} \subseteq \{l(n)\}$, real numbers a and c with $a \geqslant 0$ and $a + |c| = 1$, and a probability measure π^0 on [0, 1], such that $a_{k(n)} \to a$, $c_{k(n)} \to c$, and $\pi_{k(n)}^0 \to \pi^0$ weakly. Return to (5.14) with n replaced by $k(n)$, and invoke Lemma 5.2 to conclude that in this case,

$$\begin{aligned} \phi &= 1 && \text{if} \quad w < w_0 \\ &= 1 && \text{if} \quad a \int_{0-}^{1+} [(R_\Delta - 1)/\Delta] \pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) > c \\ &= \chi(v, w) && \text{if} \quad a \int_{0-}^{1+} [(R_\Delta - 1)/\Delta] \pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) = c \\ &= 0 && \text{otherwise} \qquad \text{a.e. } [\mu], \end{aligned} \tag{5.16}$$

for some measurable function χ with $0 \leqslant \chi \leqslant 1$. However, the functions $(R_\Delta(v, w) - 1)/\Delta$, $\Delta \in [0, 1]$, and $R_\Delta(v, w)$, $\Delta \in [1, \infty)$, are strictly increasing in v for fixed w (see Lemma 2.6(a) and 5.3(d)). Thus, unless $a = 0$ and $\pi^1 = 0$, an argument similar to that leading to (5.2) yields

(5.17) $\mu\{(v, w)|a\int_{0-}^{1+}[(R_\Delta - 1)/\Delta]\pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta) = c\} = 0,$

so that (5.16) reduces to (2.1). If $a = 0$ and $\pi^1 = 0$, then either $c = 1$ and $\phi = I_{\{w<w_0\}}$a.e. $[\mu]$, or $c = -1$ and $\phi = 1$ a.e. $[\mu]$. In either case, ϕ is of the form (2.1). This completes the proof of Theorem 5.7.

REMARK 5.8. If "local alternatives" were not permitted in problem (1.8), e.g., if the alternative hypothesis $\Delta > 0$ were changed to $\Delta \geqslant a$ for some $a > 0$, then the measure π mentioned in the second sentence of the preceding proof would assign all its mass to $[a, \infty)$, and so could not degenerate at $\Delta = 0$. In that case π would satisfy (5.2) with $d = 1$, so the most general form for a weak* limit ϕ of a sequence of proper Bayes tests would be

$$
\begin{aligned}
\phi &= 1 \quad \text{if} \quad w < w_0 \\
&= 1 \quad \text{if} \quad \int_{a-}^{\infty} R_\Delta \pi(d\Delta) > 1, \qquad w > w_0 \qquad (5.18)\\
&= 0 \quad \text{otherwise} \qquad\qquad \text{a.e. } [\mu];
\end{aligned}
$$

in fact, Theorem 2.1 would hold with (2.1) replaced by (5.18). This is the type of result obtained by Ghia (1976) and Farrell (1968); Ghia refers to (5.18) as a truncated generalized Bayes test. The presence of local alternatives in our problem (1.8) opens the possibility of admissible tests more general than (5.18), namely (2.1).

Since the family of distributions of (L, M) in problem (1.8) is not complete (in the sense of Lehmann (1959), page 131) as M is ancillary, we cannot immediately conclude that the set of all tests of the form (2.1) is a complete class, i.e., that any test not of the form (2.1) is inadmissible. This conclusion will be reached in Theorem 5.10, however, by appealing to the strict monotone likelihood ratio of the conditional density of L given M to show that any test randomized on a set of positive measure is inadmissible for (1.8). This argument, similar to several in Brown, Cohen, and Strawderman (1976), was suggested to us by L. D. Brown. We shall need the following lemma, which is a modification of Lemma 2(iii) and Problem 10(iii) in Lehmann (1959), pages 74–75 and page 112 (see also Brown, Cohen, and Strawderman (1976)).

LEMMA 5.9. *Let Y be a positive random variable with density $f_\theta(y)$ with respect to $\nu \equiv$ Lebesgue measure on $(0, \infty)$, where θ is a real parameter. Assume that $0 < f_\theta(y) < \infty$ for all $y > 0$, $\theta \geqslant 0$, and that $f_\theta(y)$ has a strictly monotone likelihood ratio. Suppose that $\bar{\phi}(y)$ is a test function for testing $\theta = 0$ versus $\theta > 0$. Choose y_0 $(0 \leqslant y_0 \leqslant \infty)$ such that the test function $\hat{\phi}$ defined by*

$$
\begin{aligned}
\hat{\phi}(y) &= 0 \quad \text{if} \quad y \leqslant y_0 \\
&= 1 \quad \text{if} \quad y > y_0
\end{aligned}
$$

has the same level as $\bar{\phi}$, i.e., $E_0\hat{\phi}(Y) = E_0\bar{\phi}(Y)$. If $\{y|\hat{\phi}(y) \neq \bar{\phi}(y)\}$ has positive Lebesgue measure, then $E_\theta\hat{\phi}(Y) > E_\theta\bar{\phi}(Y)$ for all $0 < \theta < \infty$.

THEOREM 5.10. *The set of all tests of the form* (2.1) *is a complete class for problem* (1.8).

PROOF. It must be shown that if ϕ_1 is not of the form (2.1) then there exists some ϕ of the form (2.1) such that ϕ *strictly* dominates ϕ_1. By Theorem 5.7, there exists ϕ_2 of the form (2.1) such that ϕ_2 dominates ϕ_1, i.e., $r_\Delta(\phi_2) \leq r_\Delta(\phi_1)$ for all $0 \leq \Delta < \infty$, but not necessarily strictly. Let $\bar{\phi} = \frac{1}{2}(\phi_1 + \phi_2)$. Since the risk function $r_\Delta(\cdot)$ is linear, $r_\Delta(\bar{\phi}) \leq r(\phi_1)$ for all $0 \leq \Delta < \infty$. Also, ϕ_1 and ϕ_2 must differ on a set D of positive Lebesgue measure. Hence $0 < \bar{\phi} < 1$ on this set, so that $\bar{\phi}$ is a randomized test function.

For each $0 < m < \infty$, define

$$\alpha(m) = E_0[\bar{\phi}(L, M)|M = m];$$

$\alpha(m)$ is the conditional level of the test $\bar{\phi}$ given $M = m$. (In this proof, we express all test functions in terms of (l, m).) Define the test function $\hat{\phi}$ by

$$\begin{aligned} \hat{\phi}(l, m) &= 0 \quad \text{if} \quad l \leq F_0^{-1}(1 - \alpha(m)) \\ &= 1 \quad \text{if} \quad l > F_0^{-1}(1 - \alpha(m)), \end{aligned}$$

where F_0 is the cumulative distribution function of L when $\Delta = 0$. From (1.2), F_0 is the cumulative distribution function of a (nonnormalized) central F variate, so F_0^{-1} is well defined and continuous. Hence $\hat{\phi}$ is measurable, and

$$E_0[\hat{\phi}(L, M)|M = m] = \alpha(m)$$

for each $0 < m < \infty$, i.e., $\hat{\phi}$ and $\bar{\phi}$ have the same conditional levels. However, the conditional density of L given M, being a noncentral F density, has a strictly monotone likelihood ratio in (l, Δ). Therefore, Lemma 5.9 implies that

$$E_\Delta[\hat{\phi}(L, M)|M = m] > E_\Delta[\bar{\phi}(L, M)|M = m], \qquad 0 < \Delta < \infty, \tag{5.19}$$

for all points $m \in (0, \infty)$ such that $\{l|\hat{\phi}(l, m) \neq \bar{\phi}(l, m)\}$ has positive Lebesgue measure. The set of all such points m must itself have positive Lebesgue measure, since $\hat{\phi}$ is nonrandomized whereas $\bar{\phi}$ is randomized on the nonnull set D (apply Fubini's theorem). Therefore, from (5.19), unconditionally

$$E_\Delta\hat{\phi}' > E_\Delta\bar{\phi}, \qquad 0 < \Delta < \infty, \tag{5.20}$$

while $E_0\hat{\phi} = E_0\bar{\phi}$. Thus $\hat{\phi}$ *strictly* dominates $\bar{\phi}$, hence strictly dominates ϕ_1. Finally, Theorem 5.7 implies that there exists a test ϕ of the form (2.1) which dominates $\hat{\phi}$, hence *strictly* dominates ϕ_1.

To complete the proof of Theorem 2.1 it remains to show that all tests of the form (2.1) are admissible, hence comprise the minimal complete class for problem (1.8).

THEOREM 5.11. *Any test of the form* (2.1) *is admissible for* (1.8).

PROOF. It suffices to show that if ψ is any test such that

$$\int\int_\Lambda \psi f_0 d\mu = \int\int_\Lambda \phi f_0 d\mu \equiv \alpha, \tag{5.21}$$
$$\mu(\{\psi \neq \phi\}) > 0,$$

then

$$r_\Delta(\phi) < r_\Delta(\psi) \qquad \text{for some} \quad \Delta \in (0, \infty). \tag{5.22}$$

Here $f_0 \equiv f_0(v, w)$ denotes the density of (V, W) when $\Delta = 0$. Two cases must be considered. First, if

$$\mu(\{\psi < 1\} \cap \{w < w_0\}) > 0,$$

where w_0 is the constant determining the truncation set for ϕ in (2.1), then (3.3) or (3.4) implies that $r_\Delta(\phi) < r_\Delta(\psi)$ or all sufficiently large Δ.

Second, suppose that

$$\mu(\{\psi < 1\} \cap \{w < w_0\}) = 0. \tag{5.23}$$

For $0 < \varepsilon < 1$ define the measures π and π_ε on $(0, \infty)$ as follows:

$$\pi(d\Delta) = \Delta^{-1}I_{(0,1)}(\Delta)\pi^0(d\Delta) + I_{[1,\infty)}(\Delta)\pi^1(d\Delta) + \rho\delta_1(d\Delta),$$
$$\pi_\varepsilon(d\Delta) = \gamma\varepsilon^{-1}\delta_\varepsilon(d\Delta) + I_{(\varepsilon,\varepsilon^{-1}]}(\Delta)\pi(d\Delta),$$

where $\rho = \pi^0(\{1\})$, $\gamma = \pi^0(\{0\})$, and δ_x is the probability measure degenerate at x. We shall show that for any level α test $\psi = \psi(v, w)$,

$$\begin{aligned}(5.24)\quad -\infty &< \lim_{\varepsilon\to 0}\left[\int r_\Delta(\psi)\pi_\varepsilon(d\Delta) - (1-\alpha)\int_{\varepsilon-}^{1-}\pi_\varepsilon(d\Delta)\right] \\ &= \int\int_\Lambda (1-\psi)\left\{\int_{0-}^{1+}\left[(R_\Delta - 1)/\Delta\right]\pi^0(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi^1(d\Delta)\right\} f_0 d\mu \\ &\leqslant \infty.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}\int r_\Delta(\psi)\pi_\varepsilon(d\Delta) &= \int\left[\int\int_\Lambda (1-\psi)R_\Delta f_0\, d\mu\right]\pi_\varepsilon(d\Delta) \\ &= \int\int_\Lambda (1-\psi)\left[\int R_\Delta \pi_\varepsilon(d\Delta)\right] f_0\, d\mu,\end{aligned}$$

so the left side of (5.24) may be written as

$$\begin{aligned}(5.25)\quad &\lim_{\varepsilon\to 0}\int\int_\Lambda (1-\psi)\left[\int_{\varepsilon-}^{1-}(R_\Delta - 1)\pi_\varepsilon(d\Delta) + \int_{1-}^{\infty} R_\Delta \pi_\varepsilon(d\Delta)\right] f_0\, d\mu \\ &= \lim_{\varepsilon\to 0}\int\int_\Lambda (1-\psi)\left\{\gamma\left[(R_\varepsilon - 1)/\varepsilon\right] + \int_{\varepsilon+}^{1-}\left[(R_\Delta - 1)/\Delta\right]\pi^0(d\Delta)\right. \\ &\qquad \left. + \int_{1-}^{\varepsilon^{-1}} R_\Delta \pi(d\Delta)\right\} f_0\, d\mu.\end{aligned}$$

By the monotone convergence theorem,

$$\begin{aligned}(5.26)\quad &\lim_{\varepsilon\to 0}\int\int_\Lambda (1-\psi)\left[\int_{1-}^{\varepsilon^{-1}} R_\Delta \pi(d\Delta)\right] f_0(d\mu) \\ &\qquad = \int\int_\Lambda (1-\psi)\left[\int_{1-}^{\infty} R_\Delta \pi(d\Delta)\right] f_0\, d\mu \leqslant \infty.\end{aligned}$$

Furthermore, by Lemma 5.3(d) the dominated convergence theorem may be applied to show that

(5.27) $\lim_{\varepsilon\to 0}\int\int_\Lambda(1-\psi)\{\gamma[(R_\varepsilon - 1)/\varepsilon] + \int_{\varepsilon+}^{1-}[(R_\Delta - 1)/\Delta]\pi^0(d\Delta)\}f_0\,d\mu$
$= \int\int_\Lambda(1-\psi)\{\int_{0-}^{1-}[(R_\Delta - 1)/\Delta]\pi^0(d\Delta)\}f_0\,d\mu$

which is finite. Together, (5.25)–(5.27) yield (5.24). Now, note that when $\psi = \phi$, (2.1) implies that the right side of (5.24) is $< \infty$, hence is finite. Therefore, for any test ψ of the same level as ϕ

(5.28) $-\infty < \lim_{\varepsilon\to 0}\int[r_\Delta(\psi) - r_\Delta(\phi)]\pi_\varepsilon(d\Delta)$
$= \int\int_\Lambda(\phi-\psi)\{\int_{0-}^{1+}[(R_\Delta - 1)/\Delta]\pi^0(d\Delta) + \int_{1-}^{\infty}R_\Delta\pi^1(d\Delta)\}f_0\,d\mu$
$\leqslant \infty.$

If this expression $= \infty$, then (5.22) is immediate. If it is finite, then (2.1), (5.21), and (5.23) imply (5.22).

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