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ON ASYMPTOTIC OPTIMALITY OF LIKELIHOOD RATIO TESTS FOR MULTIVARIATE NORMAL DISTRIBUTIONS

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In multivariate analysis under normality assumptions, many likelihood ratio criteria ($\lambda^{(n)}$) are distributed as

$$k\Pi_{i=1}^{m} Z_{1i}^{a_i}(1 - Z_{1i})^{b_i}\Pi_{j=1}^{m'} Z_{2j}^{c_j}$$

for some constants, k, m, m', a_i, b_i, and c_j when their associated null hypotheses are true, where Z_{ij} are independently distributed beta variates. Let $T^{(n)} = -n^{-1} \ln \lambda^{(n)}$. This paper shows that a sequence $\{T^{(n)}\}$ of this kind is asymptotically optimal in the sense of exact slopes. Explicit forms of the exact slopes are obtained.

1. Introduction. It is shown in [2] (see also [3], [4], [5]) under certain conditions that likelihood ratio (LR) statistics are asymptotically optimal in the sense of exact slopes. It is believed (see, e.g., [5], page 140) that these conditions are satisfied by a great variety of examples. Nevertheless, for a specified problem verification of the required conditions and evaluation of the optimal slopes are still indispensable before one can conclude the asymptotic optimality of the LR statistic. This paper is devoted to clarifying the asymptotic optimality (in the sense of exact slopes) of a class of LR statistics in multivariate analysis under normality assumptions.

The test statistics considered include likelihood ratios (or their modifications) for testing (1) the general linear hypothesis, (2) equality of covariance matrices, (3) equality of both mean vectors and covariance matrices, (4) independence of sets of variates, and (5) sphericity of a covariance matrix (see, e.g., [1]). We establish the asymptotic optimality of these tests by verifying refined versions of Conditions 1 and 2 stated in [5]. Explicit form of the exact slopes for each case is also obtained.

We note here that although the main results of this paper are obtained through Conditions 1 and 2 of [5] which are known to be less restrictive than those stated in Theorem 10.1 of [4], they may have been covered implicitly by the latter theorem. However, the verification of the conditions stated in [4] seems much more intricate and difficult to carry out.

2. Main results. We consider the following q-population ($q \geqslant 1$) problem. Let $X_k^{(i)}$ ($k = 1, \cdots, n_i$; $i = 1, \cdots, q$) be independent random vectors, each with p components. Denote $x^{(n)} = (X_k^{(i)}: k = 1, \cdots, n_i;\ i = 1, \cdots, q)$ with $n = n_1 + \cdots + n_q$. Suppose that for each i, $X_k^{(i)}$ have common continuous probability

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density function, say $f_i(x; \theta_i)$, where θ_i is a parameter. Suppose also that $\theta = (\theta_1, \cdots, \theta_q)$ is a point in certain space Ω. Let Ω_0 be a proper subset of Ω and let $\Omega_0' = \Omega - \Omega_0$. We wish to test the hypothesis $H : \theta \in \Omega_0$ against the alternative $H' : \theta \in \Omega_0'$. For large sample theory, we assume that as $n \to \infty$, $n_i/n \to \gamma_i$ such that $0 < \gamma_i < 1$ $(i = 1, \cdots, q)$ and $\gamma_1 + \cdots + \gamma_q = 1$.

For fixed $\theta = (\theta_1, \cdots, \theta_q) \in \Omega$ and fixed $\theta_0 = (\theta_1^0, \cdots, \theta_q^0) \in \Omega_0$, we define (as in [5])

$$K_n(x^{(n)}; \theta, \theta_0) = n^{-1}\Sigma_{i=1}^q \Sigma_{k=1}^{n_i} \ln[f_i(X_k^{(i)}; \theta_i)/f_i(X_k^{(i)}; \theta_i^0)]. \tag{2.1}$$

For each i, $i = 1, \cdots, q$, let

$$I_i(\theta_i, \theta_i^0) = E_{\theta_i}[f_i(X_k^{(i)}; \theta_i)/f_i(X_k^{(i)}; \theta_i^0)]. \tag{2.2}$$

Then $I_i(\cdot, \cdot)$ is the Kullback-Leibler information number for the ith population, which is well defined and $0 \leqslant I_i \leqslant \infty$ (see, e.g., [4], Theorem 4.1). Consequently by the law of large numbers, as $n \to \infty$, the limit of (2.1) exists and equals

$$I(\theta, \theta_0) = \Sigma_{i=1}^q \gamma_i I_i(\theta_i, \theta_i^0), \tag{2.3}$$

where $0 \leqslant I \leqslant \infty$, and I is known as the generalized Kullback-Leibler information number according to [5]. Let

$$J(\theta) = \inf\{I(\theta, \theta_0) : \theta_0 \in \Omega_0\}. \tag{2.4}$$

We shall call $J(\theta)$ the J-value of the associated problem at $\theta \in \Omega$. In many examples the following lemma is useful in computing J-values.

LEMMA 2.1. *In the framework of the problem stated above, suppose that the maximum likelihood estimate* (MLE) *of the parameter* θ *based on* $x^{(n)}$ *over* Ω_0, *say* $\hat{\theta}_0^{(n)}$, *exists and is such that* $\hat{\theta}_0^{(n)} \to \theta_0^*$ a.e. P_θ *as* $n \to \infty$, *where* $\theta_0^* \in \Omega_0$ *and* θ_0^* *may depend on* $\theta \in \Omega$. *If*

$$\lim_{n\to\infty} K_n(x^{(n)}; \theta, \hat{\theta}_0^{(n)}) = I(\theta, \theta_0^*) \qquad \text{a.e. } P_\theta,$$

then

$$J(\theta) = I(\theta, \theta_0^*).$$

PROOF. The lemma follows from $J(\theta) \leqslant I(\theta, \theta_0^*)$ and

$$K_n(x^{(n)}; \theta, \hat{\theta}_0^{(n)}) \leqslant K_n(x^{(n)}; \theta, \theta_0)$$

for all n, $\theta \in \Omega$ and $\theta_0 \in \Omega_0$. □

Let $T^{(n)}$ be a statistic based on $x^{(n)}$ for testing the hypothesis H. Suppose that $T^{(n)}$ is of the form

$$T^{(n)} = -n^{-1} \ln \lambda^{(n)}, \tag{2.5}$$

where $\lambda^{(n)}$ is some statistic based on $x^{(n)}$. Consider the following conditions.

CONDITION A. For each θ in Ω_0',

$$\liminf_{n\to\infty} T^{(n)} \geqslant J(\theta) \qquad \text{a.e. } P_\theta. \tag{2.6}$$

Condition B. When the hypothesis H is true, i.e., $\theta \in \Omega_0$, $\lambda^{(n)}$ is distributed as

(2.7) $$W^{(n)} = [k^{(n)}]^{-1}\left[\prod_{i=1}^{m} Z_{1i}^{a_i(n)}(1 - Z_{1i})^{b_i(n)}\right]\prod_{j=1}^{m'} Z_{2j}^{c_j(n)},$$

where Z_{ij} are independently distributed beta variates such that Z_{1i} ($i = 1, \cdots, m$) has density $\beta[z; a_i^{(n)} + a_i, b_i^{(n)} + b_i]$ and Z_{2j} ($j = 1, \cdots, m'$) has density $\beta[z; c_j^{(n)} + c_j]$; $m, m', a_i, b_i, c_j, d_j$ are fixed constants independent of n; $a_i^{(n)}, b_i^{(n)}, c_j^{(n)}$ are positive and depend on n such that as $n \to \infty$, $a_i^{(n)}/n \to \alpha_i$, $b_i^{(n)}/n \to \beta_i$, $c_j^{(n)}/n \to \delta_j$, $0 \leqslant \alpha_i \leqslant 1$, $0 \leqslant \beta_i \leqslant i$, $0 \leqslant \delta_j \leqslant 1$; and

(2.8) $$k^{(n)} = \prod_{i=1}^{m}\left[a_i^{(n)}/(a_i^{(n)} + b_i^{(n)})\right]^{a_i^{(n)}}\left[b_i^{(n)}/(a_i^{(n)} + b_i^{(n)})\right]^{b_i^{(n)}}.$$

Here we adopt the convention that when $m = 0$ (or $m' = 0$) the associated product in (2.7) is defined to be one.

Lemma 2.2. *Let $T^{(n)}$ and $\lambda^{(n)}$ be defined in* (2.5). *If $\lambda^{(n)}$ satisfies Condition B, then*

(2.9) $$\limsup_{n\to\infty} n^{-1}\ln P_{\theta_0}[T^{(n)} \geqslant t] \leqslant -t$$

holds for any $t > 0$ and $\theta_0 \in \Omega_0$.

Proof. By Markov's inequality (see, e.g., [4], Theorem 2.1), for fixed $t > 0$ and $\theta \in \Omega_0$,

(2.10) $$P_{\theta_0}[T^{(n)} \geqslant t] = P_{\theta_0}[-nt + \ln(\lambda^{(n)})^{-1} \geqslant 0] \leqslant e^{-nht}E_{\theta_0}[\lambda^{(n)}]^{-h}$$

holds for all $h > 0$. Now suppose that $\lambda^{(n)}$ satisfies Condition B, then using (2.7) and Stirling's approximation to gamma functions (e.g., [6], page 66), it can be verified that for $0 < h < 1$,

(2.11) $$\lim_{n\to\infty} n^{-1}\ln E_{\theta_0}[\lambda^{(n)}]^{-h} = 0.$$

The values of h are so restricted to ensure that the gamma functions in the expression of $E_{\theta_0}[\lambda^{(n)}]^{-h}$ are defined for positive values. Applying (2.11) to (2.10) gives

(2.12) $$\limsup_{n\to\infty} n^{-1}\ln P_{\theta_0}[T^{(n)} \geqslant t] \leqslant -ht.$$

Since (2.12) holds for all h between 0 and 1, (2.9) is valid. □

Note that (2.6) and (2.9) are essentially equivalent to Conditions 1 and 2 stated in [5]. As a consequence of that corollary and the lemma just proved, we establish the following theorem.

Theorem 2.1. *Let $\{T^{(n)}\}$ be a sequence of test statistics defined by* (2.5). *If $\{T^{(n)}\}$ satisfies Conditions* A *and* B, *then it is an asymptotically optimal sequence with exact slope equaling*

$$C(\theta) = 2J(\theta),$$

for all $\theta \in \Omega_0'$.

3. Examples of asymptotically optimal sequences. In this section we use Theorem 2.1 to conclude the asymptotic optimality of the LR statistics for the problems mentioned in Section 1. For simplicity, some notation is adopted directly from [1]. We assume that all covariance matrices used below are positive definite and hence have inverses. The null hypothesis for the ith problem is denoted by H_i ($i = 1, \cdots, 5$) and the alternative hypothesis corresponding to H_i is the complement of H_i. However, the same notation Ω is used to represent the unrestricted parameter space for each of the five problems. Similarly, Ω_0 is used to represent the restricted space corresponding to each of the five null hypotheses. As in Section 2, we denote $\Omega_0' = \Omega - \Omega_0$. When q ($q \geqslant 2$) populations are involved in a problem with sample sizes $N_1, \cdots, N_q$ respectively, we assume that there exist constants γ_i ($i = 1, \cdots, q$), $0 < \gamma_i < 1$, $\gamma_1 + \cdots + \gamma_q = 1$ such that as $N = N_1 + \cdots + N_q \to \infty$, $N_i/N \to \gamma_i$ ($i = 1, \cdots, q$).

3.1. *Testing the general linear hypothesis.* We discuss here only the one-way layout MANOVA problem (see [7] for the general case). Let $X_k^{(i)}$ be independent observations from $N_p(\mu_i, \Sigma)$, $k = 1, \cdots, N_i$; $i = 1, \cdots, q$. The null hypothesis is $H_1 : \mu_1 = \cdots = \mu_q$. For fixed $\theta = (\mu_1, \cdots, \mu_q, \Sigma) \in \Omega$ and $\theta_0 = (\mu_0, \cdots, \mu_0, \Sigma_0) \in \Omega_0$, the Kullback-Leibler information number is found to be

$$I_1(\theta, \theta_0) = -\tfrac{1}{2}\ln(|\Sigma|/|\Sigma_0|) - \tfrac{1}{2}p + \tfrac{1}{2}\operatorname{tr}\{[\Sigma + \Sigma_{i=1}^q \gamma_i(\mu_i - \mu_0)(\mu_i - \mu_0)']\Sigma_0^{-1}\}.$$

Applying results of [1, page 213] and Lemma 2.1 gives

$$J_1(\theta) = \tfrac{1}{2}\ln|I + \Sigma_{i=1}^q \gamma_i(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})'\Sigma^{-1}|, \tag{3.1.1}$$

where $\bar{\mu} = \Sigma_{i=1}^q \gamma_i \mu_i$. Consider the likelihood ratio criterion [1],

$$\lambda_1^{(N)} = |N\hat{\Sigma}_\Omega|^{\frac{1}{2}N}/|N\hat{\Sigma}_\omega|^{\frac{1}{2}N} \tag{3.1.2}$$

where $\hat{\Sigma}_\Omega$ and $\hat{\Sigma}_\omega$ are statistics defined by (11) and (13) of Anderson [1, page 214] with $Y_\alpha^{(i)}$ replaced by $X_k^{(i)}$. Define

$$T_1^{(N)} = -N^{-1}\ln\lambda_1^{(N)}. \tag{3.1.3}$$

Then direct computation shows $\lim_{n\to\infty} T_1^{(N)} = J_1(\theta)$ a.e. P_θ. Thus $\{T_1^{(N)}\}$ satisfies Condition A. Further, by Theorem 8.5.1 of [1], $\{T_1^{(N)}\}$ also satisfies Condition B. Hence by Theorem 2.1 the sequence $\{T_1^{(N)}\}$ is asymptotically optimal for testing H_1 and it has exact slope $C_1(\theta) = 2J_1(\theta)$ for $\theta \in \Omega_0'$. This result is conformable with that of Bahadur [4, Example 5.1] and also that of [8].

3.2. *Testing equality of covariance matrices.* Let $X_k^{(i)}$ ($k = 1, \cdots, N_i$; $i = 1, \cdots, q$) be independent observations from $N_p(\mu_i, \Sigma_i)$. Denote $n_i = N_i - 1$ ($i = 1, \cdots, q$) and $n = N - q$. The null hypothesis is $H_2 : \Sigma_1 = \cdots = \Sigma_q$. Let $\theta = (\mu_1, \cdots, \mu_q, \Sigma_1, \cdots, \Sigma_q) \in \Omega$. Then by Lemma 2.1,

$$J_2(\theta) = \tfrac{1}{2}\ln[|\Sigma_{i=1}^q \gamma_i \Sigma_i|/\Pi_{i=1}^q |\Sigma_i|^{\gamma_i}]. \tag{3.2.1}$$

Consider the modified likelihood ratio criterion

$$(3.2.2) \qquad \lambda_2^{(n)} = \left[k_2^{(n)}\right]^{-1} V_1$$

where V_1 is the statistic defined in page 249 of [1], and

$$(3.2.3) \qquad k_2^{(n)} = \Pi_{i=1}^q (n_i/n)^{\frac{1}{2}pn_i}.$$

Let $T_2^{(n)} = -n^{-1} \ln \lambda_2^{(n)}$, then it follows $\lim_{n\to\infty} T_2^{(n)} = J_2(\theta)$ a.e. P_θ.

Next, from the hth moment of V_1 (see, e.g., [1], page 253) and using arguments similar to those for Theorem 8.5.1 of [1], it can be verified under the hypothesis H_2, $\lambda_2^{(n)}$ is distributed as

$$W_2^{(n)} = \left[k_2^{(n)}\right]^{-1}\left\{\Pi_{i=1}^p \Pi_{j=1}^{q-1} Z_{ij}^{\frac{1}{2}\bar{n}_j}(1 - Z_{ij})^{\frac{1}{2}n_{j+1}}\right\}\left\{\Pi_{i=1}^p Z_{i_q}^{\frac{1}{2}n}\right\},$$

where Z_{ij} are independently distributed beta variates, Z_{ij} has density $\beta[z; \frac{1}{2}(\bar{n}_j + j(1-i)), \frac{1}{2}(n_{j+1} + 1 - i)]$ with $\bar{n}_j = n_1 + \cdots + n_j$ for $i = 1, \cdots, p$; $j = 1, \cdots, q-1$, and Z_{iq} has density $\beta[z; \frac{1}{2}(n + q(1-i)), \frac{1}{2}(q-1)(i-1)]$ for $i = 2, \cdots, p$. Condition B is thus satisfied by the sequence $\{T_2^{(n)}\}$ noting

$$\Pi_{i=1}^q (n_i/n)^{\frac{1}{2}pn_i} = \Pi_{i=1}^q \Pi_{j=1}^{q-1} (\bar{n}_j/\bar{n}_{j+1})^{\frac{1}{2}\bar{n}_j} (n_{j+1}/\bar{n}_{j+1})^{\frac{1}{2}n_{j+1}}.$$

Therefore, $\{T_2^{(n)}\}$ is asymptotically optimal with slope $C_2(\theta) = 2J_2(\theta)$ for all $\theta \in \Omega_0'$.

3.3. *Testing equality of mean vectors and covariance matrices.* Let $X_k^{(i)}$ ($k = 1, \cdots, N_i$; $i = 1, \cdots, q$) be independent observation, $X_k^{(i)} \sim N_p(\mu_i, \Sigma_i)$. Again let $n_i = N_i - 1$, $n = N - q$. The null hypothesis is $H_3 : \mu_1 = \cdots = \mu_q$ and $\Sigma_1 = \cdots = \Sigma_q$. Let $\theta = (\mu_1, \cdots, \mu_q, \Sigma_1, \cdots, \Sigma_q) \in \Omega$. The J-value for this problem is

$$(3.3.1) \qquad J_3(\theta) = \tfrac{1}{2}\ln\left\{\left|\Sigma_{i=1}^q \gamma_i\left[\Sigma_i + (\mu_i - \bar{\mu})(\mu_i - \bar{\mu})'\right]\right| / \Pi_{i=1}^q |\Sigma_i|^{\gamma_i}\right\},$$

where $\bar{\mu} = \Sigma_{i=1}^q \gamma_i \mu_i$. Let

$$(3.3.2) \qquad \lambda_3^{(n)} = \left[k_2^{(n)}\right]^{-1} V,$$

where V is the statistic defined on page 251 of [1] and $k_2^{(n)}$ is the constant defined by (3.2.3). Let

$$(3.3.3) \qquad T_3^{(n)} = -n^{-1} \ln \lambda_3^{(n)}.$$

Then it follows $\lim_{n\to\infty} T_3^{(n)} = J_3(\theta)$ a.e. P_θ. Further, using the hth moment of V (see, [1], page 253), it is seen that under the null hypothesis H_3, $\lambda_3^{(n)}$ is distributed as

$$W_3 = \Pi_{i=1}^q (n_i/n)^{-\frac{1}{2}pn_i} \Pi_{i=1}^p \left[\Pi_{j=1}^{q-1} Z_{ij}^{\frac{1}{2}\bar{n}_j}(1 - Z_{ij})^{\frac{1}{2}n_{j+1}}\right] Z_{iq}^{\frac{1}{2}n},$$

where Z_{ij} are independently distributed beta variates, Z_{ij} has density $\beta[z; \frac{1}{2}(\bar{n}_j + j(1-i)), \frac{1}{2}(n_{j+1} + 1 - i)]$ with $\bar{n}_j = n_1 + \cdots + n_j$ for $i = 1, \cdots, p$; $j = 1, \cdots, q-1$, and Z_{iq} has density $\beta[z; \frac{1}{2}(n + q(1-i)), \frac{1}{2}(q-1)i]$ for $i = 1, \cdots, p$. Therefore, Conditions A and B are satisfied by $\{T_3^{(n)}\}$. By Theorem 2.1,

$\{T_3^{(n)}\}$ is an asymptotically optimal sequence for testing H_3, with exact slope $C_3(\theta) = 2J_3(\theta)$ for all $\theta \in \Omega_0'$. When $p = 1$, $C_3(\theta)$ reduces to the optimal slope obtained in [9].

3.4. *Testing independence of sets of variates.* Let $\{X_k\}$ be a sequence of independent random vectors, $X_k \sim N_p(\mu, \Sigma)$. Suppose that X_k is partitioned into q parts, say $X_k^{(1)}, \cdots, X_k^{(q)}$ with $p_1, \cdots, p_q$ components respectively, $p = p_1 + \cdots + p_q$. Let $\mu^{(i)}$ denote the mean vector of $X_k^{(i)}$ and Σ_{ij} denote the covariance matrix between $X_k^{(i)}$ and $X_k^{(j)}$ (see, Section 9.2 of [1]). The null hypothesis is $H_4 : \Sigma_{ij} = 0$ matrix, $i \neq j$. Let $\theta = (\mu, \Sigma) \in \Omega$. Then using Lemma 2.1 the J-value is found to be

$$J_4(\theta) = \tfrac{1}{2}\ln[\Pi_{i=1}^q|\Sigma_{ii}|/|\Sigma|].$$

Let $\lambda_4^{(N)}$ be the LR criterion (based on a sample of size N) defined in (16) of [1, page 236], and let $T_4^{(N)} = -N^{-1}\ln\lambda_4^{(N)}$. It follows $\lim_{n\to\infty} T_4^{(N)} = J_4(\theta)$ a.e. P_θ. Further, it is known [1, page 236] that under H_4, $\lambda_4^{(N)}$ is distributed as a random variable having form (2.7). Hence by Theorem 2.1 the sequence $\{T_4^{(N)}\}$ is asymptotically optimal with exact slope $C_4(\theta) = 2J_4(\theta)$ for all $\theta \in \Omega_0'$.

3.5. *Testing sphericity of a covariance matrix.* Let $X_1, X_2, \cdots$ be a sequence of independent random vectors, $X_k \sim N_p(\mu, \Sigma)$. We wish to test the hypothesis H_5: $\Sigma = \sigma^2 I$, where I is the identity matrix of order p and σ^2 is an unspecified positive real number. Using Lemma 2.1 we obtain for $\theta = (\mu, \Sigma) \in \Omega$,

$$J_5(\theta) = \tfrac{1}{2}\ln[(\mathrm{tr}\,\Sigma/p)^p/|\Sigma|].$$

Based a sample of size N, say $X_1, \cdots, X_N$, the LR criterion (see, e.g., [1], page 261) is

$$\lambda_5^{(N)} = |A|^{\frac{1}{2}N}/[\mathrm{tr}(A/p)]^{\frac{1}{2}pN},$$

where $A = \Sigma_{k=1}^N(X_k - \bar{X})(X_k - \bar{X})'$ with $\bar{X} = N^{-1}\Sigma_{k=1}^N X_k$. Define $T_5^{(N)} = -N^{-1}\ln\lambda_5^{(N)}$, then $\lim_{n\to\infty} T_5^{(N)} = J_5(\theta)$ a.e. P_θ. Further, from the hth moment of $W = [\lambda_5^{(N)}]^{2/N}$ (see, [1], page 262), it can be shown that under H_5, $\lambda_5^{(N)}$ is distributed as

$$W_5 = p^{\frac{1}{2}Np}\left\{\Pi_{i=1}^{p-1} Z_i^{\frac{1}{2}iN}(1 - Z_i)^{\frac{1}{2}N}\right\} Z_p^{\frac{1}{2}pN},$$

where Z_i are independent beta variates, Z_i has density $\beta[z; \frac{1}{2}(iN - \frac{1}{2}i(i+1)), \frac{1}{2}(N - 1 - i)]$ for $i = 1, \cdots, p-1$, and Z_p has density $\beta[z; \frac{1}{2}(pN - \frac{1}{2}p(p+1)), \frac{1}{4}p(p-1)]$. Noting

$$p^{-\frac{1}{2}Np} = \Pi_{i=1}^{p-1}\left[\frac{i}{i+1}\right]^{\frac{1}{2}iN}\left[\frac{1}{i+1}\right]^{\frac{1}{2}N}$$

we see that $\lambda_5^{(N)}$ satisfies Condition B. Hence by Theorem 2.1 the sequence $\{T_5^{(N)}\}$ is asymptotically optimal with exact slope $C_5(\theta) = 2J_5(\theta)$ for all $\theta \in \Omega_0'$.

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