The Annals of Statistics
1978, Vol. 6, No. 2, 377-416

MINIMAX ESTIMATION OF LOCATION PARAMETERS FOR SPHERICALLY SYMMETRIC UNIMODAL DISTRIBUTIONS UNDER QUADRATIC LOSS

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Families of minimax estimators are found for the location parameter of a p-variate ($p \geqq 3$) spherically symmetric unimodal distribution with respect to general quadratic loss. The estimators of James and Stein, Baranchik, Bock and Strawderman are all considered for this general problem. Specifically, when the loss is general quadratic loss given by $L(\delta, \theta) = (\delta - \theta)'D(\delta - \theta)$ where D is a known $p \times p$ positive definite matrix, one main result, for one observation, X, on a multivariate s.s.u. distribution about θ, presents a class of minimax estimators whose risks dominate the risk of X, provided $p \geqq 3$ and trace $D \geqq 2d_L$ where d_L is the maximum eigenvalue of D. This class is given by $\delta_{a,r}(X) = (1 - a(r(||X||^2)/||X||^2))X$ where $0 \leqq r(\cdot) \leqq 1$, $r(||X||^2)$ is nondecreasing, $r(||X||^2)/||X||^2$ is nonincreasing, and $0 \leqq a \leqq (c_0/E_0(||X||^{-2}))((\text{trace } D/d_L) - 2)$, where $c_0 = 2p/((p+2)(p-2))$ when $p \geqq 4$ and $c_0 \approx .96$ when $p = 3$.

1. Introduction. When sampling from a p-dimensional spherically symmetric unimodal (s.s.u.) distribution about an unknown parameter θ, with invariant loss $L(\delta, \theta)$, the usual estimator of θ is the best invariant procedure. For $p \geqq 3$ it has long been known that the best invariant procedure is inadmissible with respect to a large class of loss functions (Brown [8]). However, except in certain cases explicit estimators which are better have not been found. In this paper we will present explicit minimax estimators which are better than the best invariant procedure when the loss is one of the following:

Quadratic (sum of squared errors) loss

$$L(\delta, \theta) = ||\delta - \theta||^2 = \sum_{i=1}^{p} (\delta_i - \theta_i)^2 \quad \text{where} \quad \delta = [\delta_1, \delta_2, \cdots, \delta_p]' \quad \text{and} \quad \theta = [\theta_1, \theta_2, \cdots, \theta_p]' ; \tag{1.1}$$

or

General quadratic loss

$$L(\delta, \theta) = (\delta - \theta)'D(\delta - \theta) \quad \text{where } D \text{ is a known } p \times p \text{ positive definite symmetric matrix.} \tag{1.2}$$

Received November 1975; revised May 1977.

[1] This work was completed while the author was at the University of Minnesota.

[2] Research supported by National Science Foundation Grant MCS 75-23343 A01.

AMS 1970 *subject classifications.* Primary 62C99; Secondary 62F10, 62H99.

Key words and phrases. Minimax estimation, spherically symmetric, unimodal, multivariate, location parameter.

The best known spherically symmetric unimodal distribution about θ is the p-variate normal distribution with mean vector θ and covariance matrix the identity (MVN (θ, I)). Stein [15] in 1955 investigated the question of the admissibility of the best invariant procedure, $\bar{X}$, when sampling from a MVN (θ, I) distribution with respect to quadratic loss (1.1). The admissibility of $\bar{X}$ for the univariate normal distribution had been shown [5, 9, 10] some time prior to the publication of Stein's celebrated work and Stein's objective had been to prove the admissibility of $\bar{X}$ for $p \geqq 2$; his results showed, however, that $\bar{X}$ is admissible for $p = 2$ and inadmissible for $p \geqq 3$. Furthermore, in 1961 James and Stein [11] proved the inadmissibility of $\bar{X}$ was not exclusive to the case of quadratic loss but was true for any loss function $L(\delta, \theta) = F(\|\delta - \theta\|)$ where F has a bounded derivative, is continuously differentiable and concave. In [11] James and Stein stated that these results not only apply to the normal distribution but are true for other location parameter family distributions when certain fourth moment conditions are satisfied. One might infer as well from this work that the inadmissibility of the best invariant procedure when $p \geqq 3$ could be generalized to a larger class of distributions.

Brown [7, 8], in 1965 and 1966 respectively, considered the general location parameter estimation problem and proved that under mild assumptions about the loss function, the best invariant procedure is admissible in one and two dimensions and inadmissible in three or more dimensions.

The work of Stein and Brown suggests a new problem, namely that of finding explicit estimators which are better than the best invariant procedure when sampling from a location parameter family. Heretofore, such estimators have only been found for the normal and certain s.s.u. nonnormal distributions.

In their 1961 paper, James and Stein presented, in addition to the admissibility results stated, explicit estimators for the mean vector of a MVN (θ, I) distribution which are better than X, where X is one observation on the distribution. They considered estimators of the form

$$\delta_a(X) = (1 - a/\|X\|^2)X \tag{1.3}$$

and showed that for $p \geqq 3$ and $0 \leqq a \leqq 2(p-2) = 2/E_0(\|X\|^{-2})$ estimators of the form (1.3) are minimax with respect to the loss given by (1.1). These estimators are at least as good as the best invariant procedure X which is known to be minimax. In addition, when $0 < a < 2/E_0(\|X\|^{-2})$, δ_a is in fact better than X.

In 1964, Baranchik [2] found a class of minimax estimators for θ which includes the James–Stein class, when X is one observation on a MVN (θ, I) distribution and the loss is given by (1.1). He proved that $\delta_r(X) = (1 - (p - 2)(r(\|X\|^2/2)/\|X\|^2))X$ is minimax provided $p \geqq 3$, $r(\|X\|^2/2)$ is a nondecreasing function of $\|X\|$ and $0 \leqq r(\cdot) \leqq 2$.

After this, until 1974, explicit estimators which were better than the best invariant procedure were only available for the mean vector of a multivariate normal distribution. Then, in 1974, Strawderman [16] and in 1975, Berger [3] found minimax estimators which are better than the best invariant procedure when sampling from certain s.s.u. distributions about θ. Strawderman's paper treats the problem of estimating the location parameter θ when X is a $p \times 1$ random vector with a density, a variance mixture of normals (with respect to Lebesgue measure) of the form

$$\int (2\pi\sigma^2)^{-p/2} \exp(-(\|X - \theta\|^2/2\sigma^2))\, dG(\sigma)$$

where $G(\cdot)$ is a known cdf on $(0, \infty)$. Although this class is not the whole class of s.s.u. location parameter families it does contain "thick" tailed as well as "thin" tailed distributions by a suitable choice of $G(\cdot)$. He proved that when $p \geqq 3$, $\delta_{a,r}(X) = (1 - a(r(\|X\|^2)/\|X\|^2))X$, is minimax provided $r(\|X\|^2)$ is nondecreasing, $r(\|X\|^2)/\|X\|^2$ is nonincreasing, $0 \leqq a \leqq 2/E_0(\|X\|^{-2})$ and the loss is the sum of squared errors (1.1). As in previous cases, the minimaxity of these estimators was proven by showing they are at least as good or better than X.

P. K. Bhattacharya [4] and later Bock [6] considered estimation problems for the MVN (θ, I) distribution when the loss is general quadratic loss (1.2). In particular, Bock in 1975 extended the results of James and Stein and Baranchik to this case. Provided trace $D > 2d_L$, where d_L = maximum eigenvalue of D, she was able to find explicit minimax estimators of the mean vector θ.

When X is one observation from a p-dimensional spherically symmetric unimodal distribution about θ we produce analogous results to those of James and Stein, Baranchik, Strawderman and Bock.

We begin with one observation X on a p-dimensional uniform distribution over a sphere, $(\|X - \theta\|^2 \leqq R^2)$, with known radius R.

DEFINITION 1.1. A $p \times 1$ random vector X is said to have a p-dimensional spherical uniform distribution with location parameter $\theta (X \sim U\{\|X - \theta\|^2 \leqq R^2\})$ if the density of X with respect to Lebesgue measure is

$$f_\theta(x) = c(R) I_S(x, R)$$

$$\text{where} \quad S = ((x, R) : \|x - \theta\|^2 \leqq R^2), \quad R \text{ is known},$$

$$(1.4) \qquad I_S(x, R) = 1 \quad \text{if} \quad (x, R) \in S$$

$$= 0 \quad \text{if} \quad (x, R) \notin S$$

and

$$c(R) = 1/\int I_S(x, R)\, dx .$$

By first considering the estimators of James and Stein given by (1.3) and later those of Baranchik and Bock, we explicitly find classes of minimax estimators with respect to losses (1.1) and (1.2).

As in previous works, we show that the risk of the best invariant procedure X (which is minimax) is dominated by the risks of these new procedures. In addition, all estimators in our classes of minimax estimators except X are better than X.

We, like Strawderman [16], broaden this problem by extending it to that of estimating the location parameter when the distribution under consideration has a density which is a "mixture" of uniforms, i.e., X is one observation on a distribution which has a density (with respect to Lebesgue measure) of the form

$$g(\|x - \theta\|) = \int c(R) I_S(x, R)\, dF(R)$$

where $c(R)$, S and $I_S(x, R)$ are defined by (1.4) and $F(\cdot)$ is a known cdf on $(0, \infty)$. We find explicit minimax estimators which are better than X when $p \geqq 3$.

Specifically, for $p \geqq 4$ we prove that with respect to loss (1.1), $\delta_{a,r}(X) = (1 - a(r(\|X\|^2)/\|X\|^2))X$ is better than X for $0 < a \leqq (2p/(p+2))/E_0(\|X\|^{-2})$ when the distribution is a "mixture" of spherical uniforms, provided $0 \leqq r(\cdot) \leqq 1$, $r(\|X\|^2)$ is nondecreasing and $r(\|X\|^2)/\|X\|^2$ is nonincreasing. When $p = 3$, the results are slightly different from those we obtain for $p \geqq 4$. However, minimax estimators of the form given for $p \geqq 4$ exist when $p = 3$ by adjusting the constant a (the upper limit on a).

A $p \times 1$ random vector X is said to have a s.s.u. distribution about θ if the density g of X with respect to Lebesgue measure is a nonincreasing function of $\|X - \theta\|$. It is well known that such a density can be written as a "mixture" of uniform distributions. Hence, by mixing normals Strawderman obtains explicit minimax estimators for some s.s.u. distributions about θ and we, by mixing uniforms, obtain explicit minimax estimators for all s.s.u. distributions about θ.

We close this introduction by presenting an ordered outline of this paper. Section 2 contains results analogous to those of James and Stein and Baranchik for the spherical uniform distribution when the loss is (1.1). Section 3 is an extension of these results to one observation on a s.s.u. distribution about θ. An extension of the results of Sections 2 and 3 for general quadratic loss (1.2) is given in Section 4. In Section 5, we make some statements about the case of multiple observations as well as the usefulness and benefits of using the improved estimators. Lastly, Section 6 is an appendix containing integral expressions as well as other facts used throughout this paper.

2. Minimax estimators of the location parameter of a p-dimensional spherical uniform distribution with respect to quadratic loss. We consider the problem of estimating the location parameter θ of a p-dimentional ($p \geqq 3$) spherical uniform distribution. For the problem of estimating the location parameter θ from $X = [X_1, X_2, \cdots, X_p]' \sim U\{\|X - \theta\|^2 \leqq R^2\}$ (see Definition 1.1), X is the best invariant procedure with respect to quadratic loss (1.1) and is therefore a minimax estimator of θ. This follows from the results of Kiefer [13]. In this section, we will find classes of minimax estimators which are better than X when the loss is the sum of squared errors (1.1) and $p \geqq 3$.

2.1. *Minimax estimators for dimension $p \geqq 4$.* Consider $X = [X_1, X_2, \cdots, X_p]'$, where X is one observation on a spherical uniform distribution with location parameter θ. When the loss is (1.1), we will prove that $\delta_a(X)$ given by

$$\delta_a(X) = (1 - (a/\|X\|^2))X \tag{2.1.1}$$

is a minimax estimator of θ when $0 \leqq a \leqq 2c_0R^2$, where

$$c_0 = (p-2)/(p+2) \quad \text{when} \quad p \geqq 4 \qquad (2.1.2)$$
$$= .125 \quad \text{when} \quad p = 3\,.$$

When $p \geqq 4$ the results proved differ somewhat from those for $p = 3$. Hence, only dimensions $p \geqq 4$ are considered in this section and dimension $p = 3$ will be dealt with in Section 2.2.

Clearly $\delta_a(X)$ will be minimax, with respect to quadratic loss (1.1), if the risk of $\delta_a(X)$, $R(\delta_a(X), \theta)$, dominates (is less than or equal to for all θ) the risk, $R(X, \theta)$, of the best invariant procedure X.

In order to prove this, it will be shown that, for all θ,

$$R(X,\theta) - R(\delta_a(X),\theta) = E_\theta\|X-\theta\|^2 - E_\theta\|(1-a\|X\|^{-2})X-\theta\|^2 \qquad (2.1.3)$$
$$= aE_\theta[2 - 2(\theta'X)\|X\|^{-2} - a\|X\|^{-2}]$$

is nonnegative.

Many of the calculations required to obtain expressions for the difference in risks have been deferred to the appendix in Section 6 to enable a smoother presentation of the proofs in this section.

It is straightforward to use (2.1.3) and Lemma 6.1.2, to obtain the following expressions for the difference in risks:

$$\begin{aligned} R(X,\theta) &- R(\delta_a(X),\theta) \\ &= a[2 - 2E_{\|\theta\|}((\|\theta\|X_1)\|X\|^{-2}) - aE_{\|\theta\|}(\|X\|^{-2})] \\ &= a[2 - E_0 2\|\theta\|(X_1+\|\theta\|)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1} \\ &\quad - aE_0((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}] \end{aligned} \qquad (2.1.4)$$

where $\|Y\|^2 = \|X\|^2 - X_1^2 = \sum_{i=2}^p X_i^2$ and $E_{\|\theta\|}$ is the expected value when $\theta = [\|\theta\|, 0, 0, \cdots, 0]'$. As (2.1.4) indicates, the difference in risks only depends on $\|\theta\|$. Substituting in (2.1.4), the expressions for the expected values given by (6.1.6) and (6.1.7), we obtain the following:

$$\begin{aligned} D(a, &\|\theta\|) \\ &= (R(X,\theta) - R(\delta_a(X),\theta))/aM \\ &= \frac{4}{(p-1)}\int_0^R \frac{(R^2-y^2)^{(p-1)/2}[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2(R^2-y^2)]\,dy}{d_{R,\|\theta\|}(y)} \\ &\quad - \frac{2a}{(p-2)}\int_0^R \frac{(R^2-y^2)^{(p-3)/2}[R^4 - \|\theta\|^2R^2 + 2\|\theta\|^2(R^2-y^2)]\,dy}{d_{R,\|\theta\|}(y)} \end{aligned} \qquad (2.1.5)$$

where

$$d_{R,\|\theta\|}(y) = (R^2 - \|\theta\|^2)^2 + 4\|\theta\|^2(R^2-y^2) \qquad (2.1.6)$$

and

$$M = \left[\frac{2}{p}R^2\int_0^R (R^2-y^2)^{(p-3)/2}\,dy\right]^{-1}. \qquad (2.1.7)$$

Clearly, $R(X, \theta) - R(\delta_a(X), \theta) \geqq 0$ if $0 \leqq a \leqq b(\|\theta\|)$ where

$$(2.1.8) \qquad b(\|\theta\|) = \frac{\frac{2}{(p-1)} \int_0^R \frac{(R^2 - y^2)^{(p-1)/2}[R^4 - 3\|\theta\|^2 R^2 + 4\|\theta\|^2(R^2 - y^2)]\, dy}{d_{R,\|\theta\|}(y)}}{\frac{1}{(p-2)} \int_0^R \frac{(R^2 - y^2)^{(p-3)/2}[R^4 - \|\theta\|^2 R^2 + 2\|\theta\|^2(R^2 - y^2)]\, dy}{d_{R,\|\theta\|}(y)}} .$$

With respect to the density

$$(2.1.9) \qquad \begin{aligned} g_{p,\|\theta\|}(y) &= ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))/(\textstyle\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy) \\ &\qquad \text{for} \quad 0 \leqq y \leqq R \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

(this is the density given by (6.2.1) when $q = (p - 1)/2$),

$$(2.1.10) \qquad b(\|\theta\|) = \frac{2(p-2)}{(p-1)} \left[\frac{R^4 - 3\|\theta\|^2 R^2 + 4\|\theta\|^2 E_{\|\theta\|}(R^2 - Y^2)}{(R^4 - \|\theta\|^2 R^2) E_{\|\theta\|}(R^2 - Y^2)^{-1} + 2\|\theta\|^2} \right] .$$

Using (2.1.9), (2.1.10) and (6.1.1) we easily obtain the following:

$$(2.1.11) \qquad b(0) = (2(p-2)/p)R^2$$

$$(2.1.12) \qquad b(R) = (2(p-2)^2/p(p-1))R^2$$

and

$$(2.1.13) \qquad \lim_{\|\theta\| \to \infty} b(\|\theta\|) = (2(p-2)/(p+2))R^2 .$$

Note that when $p \geqq 4$, $b(0) > b(R) \geqq \lim_{\|\theta\| \to \infty} b(\|\theta\|)$. It will be shown in Theorem 2.1.1 that $\lim_{\|\theta\| \to \infty} b(\|\theta\|)$, in fact, is less than or equal to $b(\|\theta\|)$ for all $\|\theta\|$. Note also, $\lim_{\|\theta\| \to \infty} b(\|\theta\|) = (2(p-2)/(p+2))R^2 = (2p/(p+2))(E_0(\|X\|^{-2}))$ as stated in Lemma 6.1.1.

THEOREM 2.1.1. *If X is one observation on a p-dimensional family of the form* (1.4), *then $\delta_a(X)$ given by* (2.1.1) *is minimax provided* $0 \leqq a \leqq (2(p-2)/(p+2))R^2$, $p \geqq 4$ *and the loss is sum of squared errors* (1.1). *Furthermore, $\delta_a(X)$ is actually better than X for* $0 < a \leqq (2p/(p+2))(1/E_0(\|X\|^{-2}))$.

PROOF. Clearly by (2.1.4) and (2.1.5)

$$R(X, \theta) - R(\delta_a(X), \theta) = aMD(a, \|\theta\|) \geqq aMD((2(p-2)/(p+2))R^2, \|\theta\|)$$

provided

$$0 \leqq a \leqq (2(p-2)/(p+2))R^2 = (2p/(p+2))(1/E_0(\|X\|^{-2})) .$$

To prove $\delta_a(X)$ is minimax we will show $D((2(p-2)/(p+2))R^2, \|\theta\|) \geqq 0$. To do this, consider the following 3 cases: $\|\theta\|^2 \leqq R^2$, $R^2 < \|\theta\|^2 \leqq (p/4)R^2$ and $(p/4)R^2 < \|\theta\|^2$.

CASE 1. $\|\theta\|^2 \leqq R^2$.

With respect to the density given by (2.1.9), using D as defined in (2.1.5),

$$\begin{aligned}(2.1.14)\quad & D_1((2(p-2)/(p+2))R^2, \|\theta\|) \\ &= D((2(p-2)/(p+2))R^2, \|\theta\|)/\textstyle\int_0^R ((R^2-y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\,dy \\ &= (4/(p-1))[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2E_{\|\theta\|}(R^2-Y^2)] \\ &\quad - (4R^2/(p+2))[(R^4-\|\theta\|^2R^2)E_{\|\theta\|}(R^2-Y^2)^{-1} + 2\|\theta\|^2]\,.\end{aligned}$$

By Lemma 6.2.1, when $\|\theta\|^2 \leqq R^2$, $g_{p,\|\theta\|}(y)$ has monotone likelihood ratio (MLR) nondecreasing in y. By Lehmann [14], page 74, this implies $E_{\|\theta\|}(R^2 - Y^2) \geqq E_R(R^2 - Y^2) = ((p-1)/p)R^2$ and $E_{\|\theta\|}(R^2 - Y^2)^{-1} \leqq E_R(R^2 - Y^2)^{-1} = ((p-2)/(p-3))R^{-2}$. Therefore,

$$\begin{aligned}& D_1((2(p-2)/(p+2))R^2, \|\theta\|) \\ &\quad \geqq (4/(p-1))[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2R^2((p-1)/p)] \\ &\qquad - (4/(p+2))R^2[(R^2-\|\theta\|^2)((p-2)/(p-3)) + 2\|\theta\|^2] \\ &\quad = (8(p-4)/(p(p-1)(p+2)(p-3)))R^2[pR^2 - 3\|\theta\|^2] \geqq 0\,,\end{aligned}$$

when $p \geqq 4$. Hence, $D((2(p-2)/(p+2))R^2, \|\theta\|) \geqq 0$.

CASE 2. $R^2 < \|\theta\|^2 \leqq (p/4)R^2$.

(Note that when $p = 4$, this case is vacuous.)

When $R^2 < \|\theta\|^2 \leqq (p/4)R^2$, by Lemma 6.2.1, $g_{p,\|\theta\|}(y)$ given by (2.1.9) has MLR nonincreasing in y. Therefore, $E_{\|\theta\|}(R^2-Y^2) \geqq E_R(R^2-Y^2) = ((p-1)/p)R^2$ and $E_{\|\theta\|}(R^2 - Y^2)^{-1} \geqq \lim_{\|\theta\|\to\infty} E_{\|\theta\|}(R^2 - Y^2)^{-1} = (p/(p-1))R^{-2}$. Hence, for $\|\theta\|^2 \leqq (p/4)R^2$,

$$\begin{aligned}D_1((2(p-2)/(p+2))R^2, \|\theta\|) &\geqq (4/(p-1))[R^4 - 3\|\theta\|^2R^2 + 4((p-1)/p)\|\theta\|^2R^2] \\ &\quad - (4R^2/(p+2))[(p/(p-1))(R^2-\|\theta\|^2) + 2\|\theta\|^2] \\ &= (8R^2/(p(p-1)(p+2)))[pR^2 - 4\|\theta\|^2] \geqq 0\,.\end{aligned}$$

Clearly, by (2.1.14), $D((2(p-2)/(p+2))R^2, \|\theta\|) \geqq 0$.

CASE 3. $\|\theta\|^2 \geqq (p/4)R^2$.

In order to show that $D((2(p-2)/(p+2))R^2, \|\theta\|) \geqq 0$, when $\|\theta\|^2 > (p/4)R^2$, we first obtain a new expression for $D((2(p-2)/(p+2))R^2, \|\theta\|)$. By applying (6.1.3) to (2.1.5), simple calculations lead to

$$\begin{aligned}& D((2(p-2)/(p+2))R^2, \|\theta\|) \\ &\quad = (1/(p(p-1)(p+2)))[[(p^2+4p-8)R^2 - p(p+2)\|\theta\|^2]\textstyle\int_0^R (R^2-y^2)^{(p-3)/2}\,dy \\ &\qquad + (\|\theta\|^4 - R^4)p[(p-4)R^2 + (p+2)\|\theta\|^2]\textstyle\int_0^R ((R^2-y^2)^{(p-3)/2}/d_{R,\|\theta\|}(y))\,dy]\,.\end{aligned}$$

Hence, clearly by Lemma 6.1.5,

$$\begin{aligned}& D((2(p-2)/(p+2))R^2, \|\theta\|) \\ &\quad = (\textstyle\int_0^R (R^2-y^2)^{(p-3)/2}\,dy/(p(p-1)(p+2)))[(p^2+4p-8)R^2 - p(p+2)\|\theta\|^2 \\ &\qquad + (\|\theta\|^4 - R^4)p[(p-4)R^2 + (p+2)\|\theta\|^2][h(\|\theta\|, R)]_p] \\ &\quad = (\textstyle\int_0^R (R^2-y^2)^{(p-3)/2}\,dy/(p(p-1)(p+2)))D_2((2(p-2)/(p+2))R^2, \|\theta\|)\,,\end{aligned}$$

where $[h(\|\theta\|, R)]_p = (1/(\|\theta\|^2(R^2 + \|\theta\|^2))) \sum_{i=0}^{\infty} (-1)^i a_i (R^2/\|\theta\|^2)^i$, $a_0 = 1$, and $a_i = [(p - 2(i+1))/(p + 2(i-1))]a_{i-1}$ for $i = 1, 2, \cdots$.

Notice that $D_2((2(p-2)/(p+2))R^2, \|\theta\|) \propto D((2(p-2)/(p+2))R^2, \|\theta\|)$. By the definition of $[h(\|\theta\|, R)]_p$,

$$\begin{aligned}
&\|\theta\|^2 D_2((2(p-2)/(p+2))R^2, \|\theta\|) \\
&\quad = \|\theta\|^4[-p(p+2) + p(p+2)] + \|\theta\|^2 R^2[(p^2 + 4p - 8) - p(p+2)a_1 - 6p] \\
&\qquad + R^4[p(p+2)a_2 + 6pa_1 - p(p-4)] + p(p-4) \textstyle\sum_{i=1}^{\infty} (-1)^{i-1} a_i R^{2i+4}\|\theta\|^{-2i} \\
&\qquad + 6p \textstyle\sum_{i=2}^{\infty} (-1)^{i-1} a_i R^{2i+2}\|\theta\|^{-2i+2} + p(p+2) \textstyle\sum_{i=3}^{\infty} (-1)^i a_i R^{2i}\|\theta\|^{-2i+4} \\
&\quad = (pR^6/\|\theta\|^2) \textstyle\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i
\end{aligned}$$

where

$$\begin{aligned}
c_i &= (p-4)a_{i+1} - 6a_{i+2} - (p+2)a_{i+3} \\
&= [8(i+1)(p^2 - 3p + 2)/((p + 2(i+2))(p + 2(i+1)))]a_{i+1} \qquad i = 0, 1, 2, \cdots
\end{aligned}$$

and so,

$$c_{i+1} = [((i+2)(p - 2(i+3)))/((i+1)(p + 2(i+3)))]c_i \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

Note, if p is an even number, then a_{i+1}, and thus c_i, is zero for $i \geqq (p-4)/2$. For p even and $i \leqq (p-6)/2$, c_i is positive. When $p = 4j + 1$ for $j = 1, 2, \cdots$ then $(-1)^i a_{i+1}$, and hence $(-1)^i c_i$, is positive for $i \geqq (p-5)/2$ and c_i is positive for $i \leqq (p-7)/2$. Similarly, if $p = 4j + 3$ for $j = 1, 2, \cdots$ then $(-1)^i c_i$ is negative for $i \geqq (p-5)/2$ and c_i positive for $i \leqq (p-7)/2$. So,

$$|c_{i+1}| = [((i+2)(p - 2(i+3)))/((i+1)(p + 2(i+3)))]|c_i| \quad \text{when} \quad i \leqq (p-6)/2 \tag{2.1.15}$$

and

$$|c_{i+1}| = [((i+2)(2(i+3) - p))/((i+1)(p + 2(i+3)))]|c_i| \quad \text{when} \quad i \geqq (p-5)/2\,. \tag{2.1.16}$$

Note too, that from (2.1.15) and (2.1.16)

$$\begin{aligned}
&|c_{i+1}| \leqq (p/4)|c_i| \quad \text{for} \quad i \leqq (p-6)/2 \quad \text{and} \\
&|c_{i+1}| \leqq |c_i| \quad \text{for} \quad i \geqq (p-5)/2\,.
\end{aligned} \tag{2.1.17}$$

For p an even number,

$$\textstyle\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i = \sum_{i=0}^{(p-6)/2} (-1)^i |c_i| (R^2/\|\theta\|^2)^i\,.$$

By (2.1.17), $|c_{i+1}| \leqq (p/4)|c_i|$ and since

$$(R^2/\|\theta\|^2) < 4/p\,, \qquad (R^2/\|\theta\|^2)|c_{i+1}| \leqq |c_i|$$

implying

$$\textstyle\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i = \sum_{i=0}^{(p-6)/2} (-1)^i |c_i| (R^2/\|\theta\|^2)^i \geqq 0\,.$$

For $p = 4j + 1, j = 1, 2, \cdots$

$$\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i \geqq \sum_{i=0}^{(p-7)/2} (-1)^i |c_i| (R^2/\|\theta\|^2)^i .$$

This sum is nonnegative for the same reasons as in the case when p is even. In addition, for $p = 4j + 3, j = 1, 2, \cdots$ by similar reasoning,

$$\sum_{i=0}^{(p-9)/2} (-1)^i c_i (R^2/\|\theta\|^2)^i \geqq 0 .$$

So,

$$\begin{aligned}
&\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i \\
&\quad \geqq \sum_{i=(p-7)/2}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i \\
&\quad = |c_{(p-7)/2}| (R^2/\|\theta\|^2)^{(p-7)/2} - \sum_{i=(p-5)/2}^{\infty} |c_i| (R^2/\|\theta\|^2)^i \\
&\quad \geqq |c_{(p-7)/2}| (R^2/\|\theta\|^2)^{(p-7)/2} - |c_{(p-5)/2}| \sum_{i=(p-5)/2}^{\infty} (R^2/\|\theta\|^2)^i \\
&\quad = |c_{(p-7)/2}| (R^2/\|\theta\|^2)^{(p-7)/2} - |c_{(p-5)/2}| (R^2/\|\theta\|^2)^{(p-5)/2} (\|\theta\|^2/(\|\theta\|^2 - R^2)) \\
&\quad = |c_{(p-7)/2}| (R^2/\|\theta\|^2)^{(p-7)/2} [1 - ((p-3)/((p-5)(2p-1)))(R^2/(\|\theta\|^2 - R^2))] .
\end{aligned}$$

The last three statements follow from (2.1.17) and (2.1.15).

Since $\|\theta\|^2 > (p/4)R^2$, $R^2/(\|\theta\|^2 - R^2) \leqq 4/(p-4)$, and so, since $p \geqq 7$,

$$\begin{aligned}
((p-3)/((p-5)(2p-1)))(R^2/(\|\theta\|^2 - R^2)) \\
&\leqq 4(p-3)/((p-5)(2p-1)(p-4)) \\
&\leqq (\tfrac{2}{3})((p-3)/(2p-1)) \\
&< 1
\end{aligned}$$

implying

$$\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i \geqq 0 .$$

Clearly, $D((2(p-2)/(p+2))R^2, \|\theta\|) \geqq 0$, if

$$\sum_{i=0}^{\infty} (-1)^i c_i (R^2/\|\theta\|^2)^i \quad \text{is nonnegative.}$$

Combining these 3 cases, $R(X, \theta) - R(\delta_a(X), \theta) \geqq 0$ for all θ when $0 \leqq a \leqq (2(p-2)/(p+2))R^2$ and so $\delta_a(X)$ is at least as good as X for these a's. However, when $0 < a \leqq (2(p-2)/(p+2))R^2$

$$\begin{aligned}
R(X, 0) - R(\delta_a(X), 0) &= a[2 - aE_0(\|X\|^{-2})] \quad \text{by (2.1.4)} \\
&= a[2 - (ap/((p-2)R^2))] \quad \text{by Lemma 6.1.1} \\
&\geqq a[2 - 2p/(p+2)] \\
&= 4a/(p+2) > 0 ,
\end{aligned}$$

implying $\delta_a(X)$ is better than X. This completes the proof of Theorem 2.1.1. □

2.2. *Minimax estimators for $p = 3$.* We saw in Section 2.1 that with respect to quadratic loss (1.1), $R(X, \theta) - R(\delta_a(X), \theta)$ is nonnegative, provided $0 \leqq a \leqq b(\|\theta\|)$ where $b(\|\theta\|)$ is defined by (2.1.8). For $p = 3$, by (2.1.11)—(2.1.13), $b(0) = (\frac{2}{3})R^2$, $b(R) = (\frac{1}{3})R^2$ and $\lim_{\|\theta\| \to \infty} b(\|\theta\|) = (\frac{2}{5})R^2$. Hence, the best possible result would be to show $\delta_a(X)$ is minimax for $0 \leqq a \leqq (R^2/3)$. It will become clear that this is not true.

Clearly, by (2.1.5) and (2.1.6), when $p = 3$

$$D(a, \|\theta\|) = 2 \int_0^R ((R^2 - y^2)/d_{R,\|\theta\|}(y))[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2(R^2 - y^2)]\, dy - 2a \int_0^R (1/d_{R,\|\theta\|}(y))[R^4 - \|\theta\|^2R^2 + 2\|\theta\|^2(R^2 - y^2)]\, dy \,. \tag{2.2.1}$$

However, applying (6.1.3) and

$\int_0^R (1/d_{R,\|\theta\|}(y))\, dy = (4\|\theta\|(R^2 + \|\theta\|^2))^{-1} \log ((R + \|\theta\|)/(R - \|\theta\|))^2$ to (2.2.1)

we get

$$D(a, \|\theta\|) = (R/3)[R^2 + (\tfrac{3}{2})(R^2 - \|\theta\|^2 - 2a) \times [1 + ((R^2 - \|\theta\|^2)/4\|\theta\|R) \log ((R + \|\theta\|)/(R - \|\theta\|))^2]] \,. \tag{2.2.2}$$

Therefore, when $\|\theta\| \geqq R$

$$\begin{aligned} D(R^2/3, \|\theta\|) &= (R/3)\left[R^2 + (\tfrac{1}{2})(R^2 - 3\|\theta\|^2)\left[1 + ((R^2 - \|\theta\|^2)/\|\theta\|R) \sum_{n=1}^{\infty} \frac{(R/\|\theta\|)^{2n-1}}{2n-1}\right]\right] \\ &= ((\|\theta\|^2 - R^2)/6\|\theta\|)\left[-3R\|\theta\| + (3\|\theta\|^2 - R^2) \sum_{n=1}^{\infty} \frac{(R/\|\theta\|)^{2n-1}}{2n-1}\right] \\ &= (4R^2(\|\theta\|^2 - R^2)/6\|\theta\|) \sum_{n=1}^{\infty} \left(\frac{n-1}{(2n+1)(2n-1)}\right)(R/\|\theta\|)^{2n-1} \geqq 0 \,. \end{aligned}$$

When $\|\theta\| \leqq R$,

$$\begin{aligned} D(R^2/3, \|\theta\|) &= ((R^2 - \|\theta\|^2)/6\|\theta\|)\left[3R\|\theta\| + (R^2 - 3\|\theta\|^2) \sum_{n=1}^{\infty} \frac{(\|\theta\|/R)^{2n-1}}{2n-1}\right] \\ &= (4(R^2 - \|\theta\|^2)R/6)\left[1 - \sum_{n=1}^{\infty} \left(\frac{n+1}{(2n+1)(2n-1)}\right)(\|\theta\|/R)^{2n}\right] \\ &= (2(R^2 - \|\theta\|^2)R/3)D_3(R^2/3, \|\theta\|) \,. \end{aligned}$$

When $k \geqq 4$, $\sum_{n=1}^{k} (n + 1)/(2n + 1)(2n - 1) \geqq 1$, indicating that $D_3(R^2/3, \|\theta\|) \leqq 0$, when $R^2 - \varepsilon \leqq \|\theta\|^2 \leqq R^2$, for some $\varepsilon > 0$. Clearly, this implies $D(R^2/3, \|\theta\|) \leqq 0$ for the same $\|\theta\|$'s. Hence, since there exists a $\|\theta\|$ such that $R(X, \theta) - R(\delta_a(X), \theta) = aMD(a, \|\theta\|) \leqq 0$ when $a = R^2/3$, $\delta_a(X)$ is not minimax for all a satisfying $0 \leqq a \leqq R^2/3$.

By a simple inequality,

$$\begin{aligned} D_3(R^2/3, \|\theta\|) &= 1 - \sum_{n=1}^{\infty} \left(\frac{n+1}{(2n+1)(2n-1)}\right)(\|\theta\|/R)^{2n} \\ &\geqq 1 - (2\|\theta\|^2/3R^2) - (\tfrac{1}{5})[-1 - (\|\theta\|^2/R^2) + (R^2/(R^2 - \|\theta\|^2))] \\ &\geqq 0 \quad \text{when} \quad 0 \leqq \|\theta\| \leqq ((25 - (205)^{\frac{1}{2}})/14)^{\frac{1}{2}} R \,. \end{aligned}$$

Since $((25 - (205)^{\frac{1}{2}})/14)^{\frac{1}{2}} > .85$, we conclude that

$$D(R^2/3, \|\theta\|) \geqq 0 \quad \text{when} \quad 0 \leqq \|\theta\| \leqq .85R \quad \text{and} \quad \|\theta\| \geqq R \,. \tag{2.2.3}$$

For several values of $\|\theta\|$ between $.85R$ and R, we calculate $b(\|\theta\|)$ given by (2.1.8) when $R = 1$. Our findings are summarized in Table 2.1.1.

TABLE 2.2.1
Evaluation of $b(\|\theta\|)$
(*given by* (2.1.8) *when* $R = 1$)

$\|\theta\|$	$b(\|\theta\|)$
.85	.3751
.90	.3493
.95	.3293
.96	.3268
.97	.3252
.98	.3248
.99	.3264
1.00	.3333

From Table 2.2.1 we see that $b(\|\theta\|)$ is in fact $< \frac{1}{3}$ when the $\|\theta\|$ is close to 1. In Theorem 2.2.1, we will prove $\delta_a(X)$ is minimax for $0 \leqq a \leqq (.75)/E_0(\|X\|^{-2}) = R^2/4$. However, since the smallest value of $b(\|\theta\|)$ we compute is .3248, it seems almost certain that $\delta_a(X)$ is minimax for a larger class of a's, namely, $\delta_a(X)$ is minimax for $0 \leqq a \leqq b_0/3R^2 = (b_0)/E_0(\|X\|^{-2})$ where $b_0 \approx .96$. This is also stated in Theorem 2.2.1.

THEOREM 2.2.1. *If X has a 3-dimensional spherical uniform distribution then $\delta_a(X)$ given by* (2.2.1) *is better than X provided* $0 < a \leqq R^2/4 = (.75)/E_0(\|X\|^{-2})$ *and the loss is given by* (1.1). (*It seems almost certain that $\delta_a(X)$ is minimax for* $0 \leqq a \leqq .32R^2 = (.96)/E_0(\|X\|^{-2})$.)

PROOF. When $0 \leqq \|\theta\| \leqq .85R$ and $\|\theta\| \geqq R$, we have already shown, by (2.2.3), $R(X, \theta) - R(\delta_a(X), \theta) = aMD(a, \|\theta\|) \geqq aMD(R^2/4, \|\theta\|) \geqq aMD(R^2/3, \|\theta\|) \geqq 0$.

We now show $D(R^2/4, \|\theta\|) \geqq 0$ when $.85R \leqq \|\theta\| \leqq R$. From (2.2.2),

$$\begin{aligned} &D(R^2/4, \|\theta\|) \\ &\quad = (R/3)[R^2 + (\tfrac{3}{4})(R^2 - 2\|\theta\|^2) \\ &\qquad \times [1 + ((R^2 - \|\theta\|^2)/4\|\theta\|R) \log((R + \|\theta\|)/(R - \|\theta\|))^2]] \\ &\quad = (R/3)\left[(R^2 + (\tfrac{3}{4})(R^2 - 2\|\theta\|^2)/\|\theta\|^2)\left[\|\theta\|^2 + (R^2 - \|\theta\|^2)\textstyle\sum_{n=1}^{\infty}\dfrac{(\|\theta\|/R)^{2n}}{2n-1}\right]\right]. \end{aligned}$$

Note,

$$\begin{aligned} &\|\theta\|^2 + (R^2 - \|\theta\|^2)\textstyle\sum_{n=1}^{\infty}\dfrac{(\|\theta\|/R)^{2n}}{2n-1} \\ &\quad \leqq (2R^2 - \|\theta\|^2)(\|\theta\|^2/R^2) + ((R^2 - \|\theta\|^2)/3)[-1 - (\|\theta\|^2/R^2) + (R^2/(R^2 - \|\theta\|^2))] \\ &\quad = (2\|\theta\|^2/3R^2)(3R^2 - \|\theta\|^2). \end{aligned}$$

Therefore,

$$\begin{aligned} D(R^2/4, \|\theta\|) &\geqq (R/3)[R^2 + ((\tfrac{1}{2})R^{-2})(R^2 - 2\|\theta\|^2)(3R^2 - \|\theta\|^2)] \\ &= (1/6R)[5R^4 - 7\|\theta\|^2R^2 + 2\|\theta\|^4] \geqq 0. \end{aligned}$$

Hence, we have proven that $R(\delta_a(X), \theta) \leqq R(X, \theta)$ for all θ when $0 \leqq a \leqq (R^2/4)$.

To show $\delta_a(X)$ is actually better than X when $0 < a \leqq (R^2/4)$, we prove the risk of X at $\theta = 0$ is strictly greter than the risk of $\delta_a(X)$ at $\theta = 0$,

$$R(X, 0) - R(\delta_a(X), 0) = a[2 - (3a/R^2)] \geqq a[2 - (\tfrac{3}{4})] = (\tfrac{5}{4})a > 0 .$$

This completes the proof. □

2.3. *A larger class of minimax estimators when $p \geqq 3$.* We now consider a new class of estimators for the location parameter θ when X has a p-dimensional spherical uniform distribution. The estimators are of the form considered by Baranchik [2] and are given by

$$\delta_{a,r}(X) = (1 - a(r(||X||^2)/||X||^2))X . \tag{2.3.1}$$

If $r(||X||^2)$ is nondecreasing, $0 \leqq r(\cdot) \leqq 1$, and $0 \leqq a \leqq (2b_0)/E_0(||X||^{-2})$ where

$$\begin{aligned} b_0 &= p/(p+2) \quad &\text{when} \quad p \geqq 4 \\ &= .375 \quad &\text{when} \quad p = 3 \end{aligned} \tag{2.3.2}$$

then we will prove $\delta_{a,r}(X)$ given by (2.3.1) is minimax.

When $r(\cdot) \equiv 1$, this result coincides with those given in Theorems 2.1.1 and 2.2.1. Hence, we have a larger class of minimax estimators.

THEOREM 2.3.1. *If $X = [X_1, X_2, \cdots, X_p]'$ has a p-dimensional spherical uniform distribution about θ, then the risk of $\delta_{a,r}(X)$, where $\delta_{a,r}(X)$ is defined by* (2.3.1), *dominates* (*is less than or equal to*) *the risk of X with respect to quadratic loss* (1.1), *provided $r(||X||^2)$ is nondecreasing, and $0 \leqq a \leqq (2b_0)/E_0(||X||^{-2})$, where b_0 is defined by* (2.3.2).

PROOF. Note that $(2b_0)/E_0(||X||^{-2}) = 2c_0R^2$ when $p \geqq 3$ where $c_0 = (p-2)/(p+2)$ when $p \geqq 4$ and $c_0 = .125$ when $p = 3$ as given by (2.1.2). Since $0 \leqq r(\cdot) \leqq 1$ and $0 \leqq a \leqq (2b_0)/E_0(||X||^{-2}) = 2c_0R^2$,

$$\begin{aligned} R(X, \theta) - R(\delta_{a,r}(X), \theta) &= aE_\theta[r(||X||^2)(2X'(X - \theta))||X||^{-2} - ar^2(||X||^2)||X||^{-2}] \\ &\geqq 2aE_\theta[r(||X||^2)[X'(X-\theta)||X||^{-2} - c_0R^2||X||^{-2}]] \\ &= 2aE_{||\theta||}[r(||X||^2)[1 - ||\theta||X_1||X||^{-2} - c_0R^2||X||^{-2}]] . \end{aligned}$$

Note that the difference in risks depends only on $\theta = [||\theta||, 0, 0, \cdots, 0]'$. We will show

$$E_{||\theta||}[r(||X||^2)[1 - ||\theta||X_1||X||^{-2} - c_0R^2||X||^{-2}]] \geqq 0 .$$

We first consider $||\theta||^2 \geqq (1 - 2c_0)R^2$.

CASE 1. $||\theta||^2 \geqq (1 - 2c_0)R^2$.

Lemma 6.2.2 states for each fixed $||\theta||$ satisfying $||\theta||^2 \geqq (1 - 2c_0)R^2$, $E_{||\theta||}[(||\theta||X_1 + c_0R^2)||X||^{-2} \mid ||X||^2]$ is nonincreasing in $||X||^2$. Since $r(||X||^2)$ is a nondecreasing function,

$$\begin{aligned} &E_{||\theta||}[r(||X||^2)[1 - (||\theta||X_1 + c_0R^2)||X||^{-2}]] \\ &\quad = E_{||\theta||}[r(||X||^2)[1 - E_{||\theta||}((||\theta||X_1 + c_0R^2)||X||^{-2} \mid ||X||^2)]] \\ &\quad \geqq (E_{||\theta||}r(||X||^2))[(R(X, \theta) - R(\delta_{2c_0R^2}(X), \theta))/4c_0R^2] , \end{aligned}$$

where, of course, $\delta_{2c_0R^2}(X) = (1 - (2c_0R^2/\|X\|^2))X$. This is nonnegative for all θ since $\delta_{2c_0R^2}$ is better than X (Theorems 2.1.1 and 2.2.1).

Since $1 - 2c_0 = (6 - p)/(p + 2)$ when $p \geqq 4$ and $1 - 2c_0 = .75$ when $p = 3$, the proof of the theorem is complete for all $\|\theta\|$ when $p \geqq 6$, for $\|\theta\|^2 \geqq ((6 - p)/(p + 2))R^2$ when $p = 4, 5$ and when $\|\theta\|^2 \geqq .75R^2$ when $p = 3$.

CASE 2. $\|\theta\|^2 \leqq .75R^2$ and $p = 3$.

If $\|X\|^2 = Z$, we see by Lemma 6.1.6, when $p = 3$, the joint density of X_1 and Z, $f_{\|\theta\|}(x_1, z)$, is given by

$$f_{\|\theta\|}(x_1, z) = (3/4R^3)I_{S_1 \cup S_2}(x_1, z)$$

where

$$S_1 = ((x_1, z): ((z - R^2 + \|\theta\|^2)/2\|\theta\|) \leqq x_1 \leqq z^{\frac{1}{2}},\quad (R - \|\theta\|)^2 \leqq z \leqq (R + \|\theta\|)^2)$$

and

$$S_2 = ((x_1, z): -z^{\frac{1}{2}} \leqq x_1 \leqq z^{\frac{1}{2}}, 0 \leqq z \leqq (R - \|\theta\|)^2)\,.$$

Hence,

$$\begin{aligned} E_{\|\theta\|} r(\|X\|^2)[1 - (\|\theta\|X_1 + c_0R^2)\|X\|^{-2}] &= (3/4R^3) \textstyle\int_0^{(R-\|\theta\|)^2} (2r(z)/z^{\frac{1}{2}})(z - .125R^2)\, dz \\ &\quad + (3/4R^3) \textstyle\int_{(R-\|\theta\|)^2}^{(R+\|\theta\|)^2} (r(z)/4z)(z^{\frac{1}{2}} - ((z - R^2 + \|\theta\|^2)/2\|\theta\|)) \\ &\quad \times (3z - 2\|\theta\|z^{\frac{1}{2}} + .5R^2 - \|\theta\|^2)\, dz\,. \end{aligned}$$

If

$$\begin{aligned} h(z) &= z - .125R^2 &&\text{when}\quad 0 \leqq z^{\frac{1}{2}} \leqq R - \|\theta\| \\ &= 3z - 2\|\theta\|z^{\frac{1}{2}} + (.5R^2 - \|\theta\|^2) &&\text{when}\quad R - \|\theta\| \leqq z^{\frac{1}{2}} \leqq R + \|\theta\| \end{aligned}$$

then for $0 \leqq \|\theta\| \leqq .65R$

$$\begin{aligned} h(z) &\leqq 0 &&\text{when}\quad 0 \leqq z^{\frac{1}{2}} \leqq (.125)^{\frac{1}{2}}R \\ &\geqq 0 &&\text{when}\quad (.125)^{\frac{1}{2}}R \leqq z^{\frac{1}{2}} \leqq R + \|\theta\|\,. \end{aligned}$$

Since $r(z)$ is nondecreasing, $r(z)h(z) \geqq r(.125R^2)h(z)$. For $\|\theta\| \geqq .65R$

$$\begin{aligned} h(z) &\leqq 0 &&\text{when}\quad 0 \leqq z^{\frac{1}{2}} \leqq (\|\theta\|/3) + ((4\|\theta\|^2 - 1.5R^2)^{\frac{1}{2}}/3) \\ &\geqq 0 &&\text{when}\quad (\|\theta\|/3) + ((4\|\theta\|^2 - 1.5R^2)^{\frac{1}{2}}/3) \leqq z^{\frac{1}{2}} \leqq R + \|\theta\| \end{aligned}$$

and so

$$r(z)h(z) \geqq r(((\|\theta\|/3) + ((4\|\theta\|^2 - 1.5R^2)^{\frac{1}{2}}/3))^2)h(z)\,.$$

Hence, in either case, there exists a z_0 such that

$$\begin{aligned} E_{\|\theta\|} r(\|X\|^2)[1 - (\|\theta\|X_1 + c_0R^2)\|X\|^{-2}] &= E_{\|\theta\|} r(z)h(z) \\ &\geqq r(z_0)E_{\|\theta\|}h(z) \\ &\geqq r(z_0)E_{\|\theta\|}[1 - (\|\theta\|X_1 + c_0R^2)\|X\|^{-2}] \\ &= r(z_0)(R(X, \theta) - R(\delta_{2c_0R^2}(X), \theta))/4c_0R^2 \\ &\geqq 0 \quad \text{by Theorem 2.2.1}\,. \end{aligned}$$

The proof is complete for this case.

CASE 3. $0 \leqq \|\theta\| \leqq (2(p-1)/p(p+2))R$ and $p = 4, 5$.

Since $X_1 \leqq \|X\|$ and $r(\|X\|^2)$ is nondecreasing,

$$E_{\|\theta\|} r(\|X\|^2)[1 - \|\theta\| X_1 \|X\|^{-2} - c_0 R^2 \|X\|^{-2}] \geqq E_{\|\theta\|} r(\|X\|^2)[1 - E_{\|\theta\|}(\|\theta\| \|X\|^{-1} + c_0 R^2 \|X\|^{-2})] .$$

Since $\|X\|$, by Lemma 6.2.3, is stochastically ordered in $\|\theta\|$,

$$\begin{aligned} E_{\|\theta\|}[\|\theta\| \|X\|^{-1} + c_0 R^2 \|X\|^{-2}] &\leqq E_0[\|\theta\| \|X\|^{-1} + c_0 R^2 \|X\|^{-2}] \\ &= (p/(p-1))(\|\theta\|/R) + (p/(p+2)) \\ &\leqq 1 \quad \text{when} \quad \|\theta\| \leqq (2(p-1)/p(p+2))R . \end{aligned}$$

Hence, $E_{\|\theta\|} r(\|X\|^2)[1 - \|\theta\| X_1 \|X\|^{-2} - c_0 R^2 \|X\|^{-2}] \geqq 0$. `Hence, this implies the remaining cases are $(R^2/16) \leqq \|\theta\|^2 \leqq (R^2/3)$ when $p = 4$ and $(64R^2/((49)(25))) \leqq \|\theta\|^2 \leqq (R^2/7)$ when $p = 5$.

CASE 4. $(R^2/16) \leqq \|\theta\|^2 \leqq (R^2/3)$ and $p = 4$.

Utilizing (6.1.15), Lemma 6.1.5 and (2.1.5), for $p = 4$, simple calculations imply if

$$A(b) = E_{\|\theta\|}(2 - 2\|\theta\| X_1 \|X\|^{-2} - bR^2 \|X\|^{-2}) \qquad \text{then}$$
$$3R^4 A(b) = 6(1-b)R^4 + 3(b-2)\|\theta\|^2 R^2 + 2\|\theta\|^4 .$$

For fixed $b < 1$, there exists an a_0 such that $A(b) \geqq 0$ when $0 \leqq \|\theta\|^2 \leqq a_0 R^2$. Lemma 6.2.2 states that $E_{\|\theta\|}((\|\theta\| X_1 + (b/2)R^2)\|X\|^{-2} \mid \|X\|^2)$ is nonincreasing in $\|X\|$ for $\|\theta\|^2 \geqq (1-b)R^2$. Hence, when $b \geqq 2c_0 = \frac{2}{3}$ and $a_0 R^2 \geqq \|\theta\|^2 \geqq (1-b)R^2$,

$$\begin{aligned} E_{\|\theta\|} r(\|X\|^2)&[2 - 2(\|\theta\| X_1 + c_0 R^2)\|X\|^{-2}] \\ &\geqq E_{\|\theta\|} r(\|X\|^2)[2 - 2(\|\theta\| X_1 + (b/2)R^2)\|X\|^{-2}] \\ &\geqq E_{\|\theta\|} r(\|X\|^2) A(b) \geqq 0 . \end{aligned}$$

Calculations show for $b = \frac{15}{16}$, $a_0 > \frac{1}{8}$. Therefore, for

$$(R^2/16) \leqq \|\theta\|^2 \leqq (R^2/8) , \qquad E_{\|\theta\|} r(\|X\|^2)[2 - 2(\|\theta\| X_1 + c_0 R^2)\|X\|^{-2}] \geqq 0 .$$

Similarly, since for $b = \frac{7}{8}$, $a_0 > \frac{1}{4}$ and for $b = \frac{3}{4}$, $a_0 > \frac{1}{3}$, then

$$E_{\|\theta\|} r(\|X\|^2)[2 - 2(\|\theta\| X_1 + c_0 R^2)\|X\|^{-2}] \geqq 0$$

when $(R^2/8) \leqq \|\theta\|^2 \leqq (R^2/4)$ and $(R^2/4) \leqq \|\theta\|^2 \leqq (R^2/3)$, implying the desired conclusion for this case.

CASE 5. $(64R^2/((25)(49))) \leqq \|\theta\|^2 \leqq (R^2/7)$ and $p = 5$.

As we did in the previous case, when $b \geqq 2c_0 = \frac{6}{7}$ and $\|\theta\|^2 \geqq (1-b)R^2$,

$$\begin{aligned} E_{\|\theta\|} r(\|X\|^2)&[2 - 2(\|\theta\| X_1 + c_0 R^2)\|X\|^{-2}] \\ &\geqq E_{\|\theta\|} r(\|X\|^2)[2 - 2(\|\theta\| X_1 + (b/2)R^2)\|X\|^{-2}] \\ &\geqq E_{\|\theta\|} r(\|X\|^2) A(b) . \end{aligned}$$

With respect to the density $g_{p,\|\theta\|}(y)$ for $p = 5$ given by (2.1.9), we have using (2.1.5)

$$A(b)\Big/\left(M \int_0^R \frac{(R^2 - y^2)^2}{d_{R,\|\theta\|}(y)} dy\right) = A^*(b)$$
$$= (R^4 - 3\|\theta\|^2R^2) + 4\|\theta\|^2E_{\|\theta\|}(R^2 - y^2)$$
$$- (2bR^2/3)[(R^4 - \|\theta\|^2R^2)E_{\|\theta\|}(R^2 - y^2)^{-1} + 2\|\theta\|^2]$$

where $M = [(\frac{2}{5}) \int_0^R (R^2 - y^2)\, dy]^{-1}$ is given by (2.1.7). By Lemma 6.2.1, $g_{p,\|\theta\|}(y)$ has monotone likelihood ratio nondecreasing in y and hence, since $\|\theta\|^2 \leqq R^2/7 \leqq R^2$, when $p = 5$,

$$E_{\|\theta\|}(R^2 - Y^2) \geqq E_R(R^2 - Y^2) = (\tfrac{4}{5})R^2 \quad \text{and}$$
$$E_{\|\theta\|}(R^2 - Y^2)^{-1} \leqq E_R(R^2 - Y^2)^{-1} = (3/2R^2)\,,$$

implying

$$A^*(b) = (5R^4 + \|\theta\|^2R^2)/5 - (bR^2/3)(3R^2 + \|\theta\|^2)\,.$$

When $b = \frac{24}{25}$, $A^*(\frac{24}{25}) \geqq 0$ for $\|\theta\|^2 \leqq R^2/3$. Hence, when $R^2/25 \leqq \|\theta\|^2 \leqq R^2/3$,

$$E_{\|\theta\|}[2 - 2(\|\theta\|X_1 + c_0R^2)\|X\|^{-2}] \geqq E_{\|\theta\|}r(\|X\|^2)A(\tfrac{24}{25})$$
$$= E_{\|\theta\|}r(\|X\|^2)[M \textstyle\int_0^R (R^2 - y^2)^2\, dy]A^*(\tfrac{24}{25})$$
$$\geqq 0\,.$$

The interval, $R^2/25 \leqq \|\theta\|^2 \leqq R^2/3$ includes the interval $(64R^2/((25)(49))) \leqq \|\theta\|^2 \leqq R^2/7$; thus the proof is complete. □

3. Minimax estimators for the mean vector of a p-dimensional spherically symmetric unimodal distribution with respect to quadratic loss.

3.1. *A characterization of a spherically symmetric unimodal distribution.*

DEFINITION 3.1.1. A random vector X is said to have a p-dimensional spherically symmetric unimodal (s.s.u.) distribution about θ if the density of X with respect to Lebesgue measure is a nonincreasing function of $\|X - \theta\|$.

In this section we will give necessary and sufficient conditions for a random vector to have a s.s.u. distribution about θ in accordance with Definition 3.1.1.

LEMMA 3.1.1. *If X is a $p \times 1$ random vector ($p \geqq 1$) with a density $g(\|x - \theta\|)$ with respect to Lebesgue measure, then $g(\cdot)$ is a nonincreasing function of $\|x - \theta\|$ if and only if*

$$g(\|x - \theta\|) = \int c(R)I_S(x, R)\, dF(R)$$

where $I_S(x, R)$ is given in (1.4), *$c(R)$ given in* (1.4) *equals c/R^p, c is a positive constant and $F(\cdot)$ is a* cdf *on* $(0, \infty)$.

PROOF. Simply by (1.4), if

$$g(\|x - \theta\|) = \int c(R)I_S(x, R)\, dF(R) = \int_{\|x-\theta\|}^{\infty} c(R)\, dF(R)$$

then $g(\|x - \theta\|)$ is a nonincreasing function of $\|x - \theta\|$.

Conversely, suppose $g(\cdot)$ is a nonincreasing function of $\|x-\theta\|$. From Lemma 6.1.1,

$$\begin{aligned} H(R) &= P(\|X-\theta\| \leqq R) \\ &= \int_{(\|x-\theta\|\leqq R)} g(\|x-\theta\|)\,dx \\ &= (M_0/c(R)) \int_0^R r^{p-1} g(r)\,dr\,, \end{aligned}$$

where $M_0 = P/R^p$. Hence, since $c(R) = c/R^p$, $M_0/c(R) = P/c$, and therefore,

$$H(R) = (P/c)\int_0^R r^{p-1} g(r)\,dr\,. \tag{3.1.1}$$

Consider the function, $F(R)$, given by

$$F(R) = H(R) - (R^p/c)g(R)\,. \tag{3.1.2}$$

It follows from (3.1.1) that

$$\begin{aligned} F(R) &= (P/c)\int_0^R r^{p-1} g(r)\,dr - (R^p/c)g(R) \\ &= (P/c)\int_0^R y^{p-1}[g(y)-g(R)]\,dy\,. \end{aligned} \tag{3.1.3}$$

We now state, without further proof, that $F(R)$, given by (3.1.2), is the cdf that characterizes the density $g(\|x-\theta\|)$, i.e., $g(\|x-\theta\|) = \int c(R) I_S(x, R)\,dF(R)$. □

3.2. *Estimators with smaller risks than the risk of one observation on a spherically symmetric unimodal distribution.* Let X be a $p \times 1$ random vector with a density with respect to Lebesgue measure given by

(3.2.1) $g(\|x-\theta\|) = \int c(R) I_S(x, R)\,dF(R)$ where $F(\cdot)$ is a known cdf on $(0, \infty)$ and $c(R)$ and $I_S(x, R)$ are defined in (1.4).

According to Lemma 3.1.1, X has a spherically symmetric unimodal distribution about θ.

Hence, since the density of X is a mixture of spherical uniforms, we may consider the random vector $X \mid R$ to have a spherical uniform distribution

$$(X \mid R \sim U\{\|X-\theta\|^2 \leqq R^2\})\,.$$

Therefore, directly by Lemma 6.1.1,

$$E_0(\|X\|^{-2}) = E[E_0(\|X\|^{-2} \mid R)] = (p/(p-2))E(R^{-2})\,.$$

Consider $\delta_a(X)$ given by (2.1.1), $\delta_a(X) = (1 - (a/\|X\|^2))X$, for $0 \leqq a \leqq (2b_0)/E_0(\|X\|^{-2}) = (2c_0)/E(R^{-2})$ where

$$\begin{aligned} b_0 &= p/(p+2) & p &\geqq 4 \\ &= .375 & p &= 3 \end{aligned}$$

and

$$\begin{aligned} c_0 &= (p-2)/(p+2) & p &\geqq 4 \\ &= .125 & p &= 3\,. \end{aligned}$$

With respect to quadratic loss (1.1), when $0 \leqq a \leqq (2c_0)/E(R^{-2})$

$$\begin{aligned} [R(X,\theta) - R(\delta_a(X),\theta)]/a &= E_\theta[2X'(X-\theta)\|X\|^{-2} - a\|X\|^{-2}] \\ &\geqq E_\theta[2X'(X-\theta)\|X\|^{-2} - (2c_0/ER^{-2})\|X\|^{-2}]\,. \end{aligned} \tag{3.2.2}$$

The minimaxity of $\delta_{2c_0R^2}(X)$ for the spherical uniform distribution (see Theorems 2.1.1 and 2.2.1) implies

$$\begin{aligned} & E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_0 R^2\|X\|^{-2}] \\ (3.2.3) \qquad & \quad = E[E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_0 R^2\|X\|^{-2} \mid R]] \\ & \quad \geqq 0 \,. \end{aligned}$$

Thus, if

$$\begin{aligned} E_\theta\|X\|^{-2} - ER^{-2}E_\theta(R^2\|X\|^{-2}) &= \mathrm{Cov}\,(E_\theta(R^2\|X\|^{-2} \mid R), R^{-2}) \\ &\leqq 0 \,, \end{aligned}$$

then

$$\begin{aligned} & E_\theta[2X'(X-\theta)\|X\|^{-2} - (2c_0/E(R^{-2}))\|X\|^{-2}] \\ & \qquad \geqq E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_0 R^2\|X\|^{-2}] \,, \end{aligned}$$

clearly implying, by (3.2.2) and (3.2.3), that $\delta_a(X)$ is minimax for $0 \leqq a \leqq (2c_0/E(R^{-2}))$.

However, $E_\theta(R^2\|X\|^{-2} \mid R) = E_{\theta(R)}(\|Z\|^{-2} \mid R)$ where $Z = X/R$ and thus $Z \mid R \sim U\{\|Z - \theta(R)\|^2 \leqq 1\}$ and $\theta(R) = [\|\theta\|/R, 0, 0, \cdots, 0]'$. Lemma 6.2.3 implies $\|Z\|^2$ is stochastically ordered in $\|\theta(R)\|$ and hence, $E_{\theta(R)}[\|Z\|^{-2} \mid R]$ is a nonincreasing function of $\|\theta(R)\|$ (see Lehmann [14], pages 73–74).

Therefore, for fixed $\|\theta\|$, $E_{\theta(R)}[\|Z\|^{-2} \mid R]$ is a nondecreasing function of R and since R^{-2} is a nonincreasing function of R, $\mathrm{Cov}\,(E_{\theta(R)}(\|Z\|^{-2} \mid R), R^{-2}) \leqq 0$. We summarize this result in the following theorem.

THEOREM 3.2.1. *If X is one observation on a* s.s.u. *distribution about θ with a density given by* (3.2.1), *$\delta_a(X)$, defined by* (2.1.1) *is minimax provided $p \geqq 3$, $E_0\|X\|^{-2}$ is finite, the loss is sum of squared errors,* (1.1), *and $0 \leqq a \leqq 2b_0/E\|X\|^{-2}$, where $b_0 = p/(p+2)$ or .375 according as $p \geqq 4$ or $p = 3$.*

James and Stein [11] proved, for X one observation on a p-variate normal distribution with mean vector θ and covariance matrix the identity, that $\delta_a(X) = (1 - (a/\|X\|^2))X$ given by (2.1.1), is minimax for $0 \leqq a \leqq 2(p-2) = 2/(E_0\|X\|^{-2})$. In the normal case, the James–Stein class of estimators includes our class. However, for $p \geqq 4$, the only estimators in the James–Stein class which are not in our class are for values of a such that

$$(2p/(p+2))/E_0(\|X\|^{-2}) \leqq a \leqq 2/E_0(\|X\|^{-2}) \,.$$

Similar statements hold when comparing our results to those of Strawderman [16] given on "variance" mixtures of normal distributions.

Since $p/(p+2) \to 1$, as $p \to \infty$, our class of estimators is, in a sense, approaching the James–Stein class for large p. Additionally, the best estimator in the normal case occurs when $a = (1/E_0(\|X\|^{-2}))$ which is always in our class. Furthermore, our bounds on a are the best possible which can be obtained for the whole class of s.s.u. distributions when $p \geqq 4$, since we have already seen in Section 2.1 that our bounds are the best possible for the spherical uniform distribution.

3.3. *A larger class of minimax estimators.* In this section we will consider estimators of the mean vector of a s.s.u. distribution, given by

$$\delta_{a,r}(X) = (1 - a(r(\|X\|^2)/\|X\|^2))X .$$

In the following theorem we will present sufficient conditions for $\delta_{a,r}(X)$ to be minimax.

THEOREM 3.3.1. *If X is a single observation on a p-dimensional distribution of the form* (3.2.1) *and $\delta_{a,r}(X)$ is defined by* (2.3.1), *then provided*:

(1) $0 \leqq a \leqq (2b_0)/E_0(\|X\|^{-2})$ *(b_0 is given by* (2.3.2)),
(2) $r(\|X\|^2)$ *is nondecreasing*,
(3) $r(\|X\|^2)/\|X\|^2$ *is nonincreasing, and*
(4) $E_0(\|X\|^{-2})$ *is finite*,

the risk of $\delta_{a,r}(X)$ dominates (is less than or equal to) the risk of X for $p \geqq 3$ and quadratic loss (1.1).

PROOF. For $0 \leqq r(\cdot) \leqq 1$ and $0 \leqq a \leqq (2b_0)/E_0(\|X\|^{-2}) = (2c_0)/E(R^{-2})$,

$$\begin{aligned} &[R(X, \theta) - R(\delta_{a,r}(X), \theta)]/a \\ (3.3.1) \qquad &\geqq E_\theta[r(\|X\|^2)[(2X'(X' - \theta) - a)\|X\|^{-2}]] \\ &\geqq E_\theta[r(\|X\|^2)[2X'(X - \theta)\|X\|^{-2} - (2c_0/E(R^{-2}))\|X\|^{-2}]] \end{aligned}$$

where c_0 is defined by (2.1.2). As we noted in the previous section, $X \mid R$ may be considered as a spherical uniform random vector. Hence, since Theorem 2.3.1 implies the minimaxity of $\delta_{2c_0R^2,r}(X)$ for the spherical uniform distribution,

$$\begin{aligned} &E_\theta[r(\|X\|^2)[2X'(X - \theta)\|X\|^{-2} - 2c_0R^2\|X\|^{-2}]] \\ (3.3.2) \qquad &= E_\theta[E[r(\|X\|^2)(2X'(X - \theta)\|X\|^{-2} - 2c_0R^2\|X\|^{-2}) \mid R]] \\ &\geqq 0 . \end{aligned}$$

Hence, as in the proof of Theorem 3.2.1, (3.3.1) and (3.3.2) imply

$$R(X, \theta) - R(\delta_{a,r}(X), \theta) \geqq 0$$

for $0 \leqq a \leqq (2c_0)/E(R^{-2})$, if

$$\begin{aligned} E_\theta(r(\|X\|^2)\|X\|^{-2}) - E(R^{-2})E_\theta(r(\|X\|^2)R^2\|X\|^{-2}) \\ = \operatorname{Cov}(E_\theta(r(\|X\|^2)R^2\|X\|^{-2} \mid R), R^{-2}) \\ \leqq 0 . \end{aligned}$$

Since R^{-2} is nonincreasing in R, the proof will be complete if

$$E_\theta(r(\|X\|^2)R^2\|X\|^{-2} \mid R) = E_{\theta(R)}(r(R^2\|Z\|^2)\|Z\|^{-2} \mid R)$$

is nondecreasing in R, where $Z = X/R$ and $\theta(R) = [\|\theta\|/R, 0, 0, \cdots, 0]'$. Hence, $Z \mid R \sim U\{\|Z - \theta(R)\|^2 \leqq 1\}$. By the properties of $r(\cdot)$ in the statement of this theorem (properties (2) and (3)) and the stochastic ordering of $\|Z\|^2$ in

$\|\theta(R)\| = \|\theta\|/R$ (see Lemma 6.2.3), if $R_1 \leqq R_2$ then

$$\begin{aligned} E_{\theta(R_1)}(r(R_1^2\|Z\|^2)\|Z\|^{-2} \mid R_1) &\leqq E_{\theta(R_2)}(r(R_1^2\|Z\|^2)\|Z\|^{-2} \mid R_2) \\ &\leqq E_{\theta(R_2)}(r(R_2^2\|Z\|^2)\|Z\|^{-2} \mid R_2) . \end{aligned}$$

Thus, $E_{\theta(R)}(r(R^2\|Z\|^2)\|Z\|^{-2} \mid R)$ is nondecreasing in R, completing this proof.

4. Minimax estimators of the location parameter of a p-dimensional ($p \geqq 3$) spherically symmetric unimodal distribution with respect to general quadratic loss. Throughout Sections 2 and 3, the only loss function considered was quadratic loss given by (1.1). In this section, we will explicitly extend the results of Sections 2 and 3 to the case of general quadratic loss given by (1.2).

4.1. *Minimax estimators for the mean vector of a spherical uniform distribution with known radius.* Analogous results to those given in Section 2 for estimating the mean vector of a p-dimensional ($p \geqq 3$), spherical uniform distribution are presented in this section when the loss is general quadratic loss, given, as in (1.2), by

$$L(\delta, \theta) = (\delta - \theta)'D(\delta - \theta)$$

where D is a known $p \times p$ positive definite symmetric matrix.

Consider one observation X having a p-dimensional spherical uniform distribution with a density given by (1.4). Let $\delta_a(X)$ be defined by (2.1.1), i.e., $\delta_a(X) = (1 - (a/\|X\|^2))X$. The loss throughout will be general quadratic loss.

We will prove that $\delta_a(X)$ is minimax when $0 \leqq a \leqq 2a_0((\text{trace } D/d_L) - 2)R^2$ where $d_L =$ maximum eigenvalue of D and

$$\begin{aligned} a_0 &= 1/(p+2) && \text{for } p \geqq 4 \\ &= .125 && \text{for } p = 3 . \end{aligned} \tag{4.1.1}$$

Note that when D is the identity, $L(\delta, \theta)$ is just quadratic loss and the result we will prove coincides with those proven in Sections 2.1 and 2.2.

If RD denotes the risk with respect to general quadratic loss, since X is minimax, $\delta_a(X)$ is minimax if the difference in risks, $RD(X, \theta) - RD(\delta_a(X), \theta)$, is nonnegative for all θ.

With respect to general quadratic loss (1.2),

$$\begin{aligned} &RD(X, \theta) - RD(\delta_a(X), \theta) \\ &\quad = E_\theta(X - \theta)'D(X - \theta) \\ &\qquad - E_\theta(X - \theta - (a/\|X\|^2)X)'D(X - \theta - (a/\|X\|^2)X) \\ &\quad = aE_\theta[2X'D(X - \theta)\|X\|^{-2} - a(X'DX)\|X\|^{-4}] \\ &\quad = aE_0[(2X'DX + 2\theta'DX)\|X + \theta\|^{-2} \\ &\qquad - a(X'DX + 2\theta'DX + \theta'D\theta)\|X + \theta\|^{-4}] . \end{aligned} \tag{4.1.2}$$

Since D is a positive definite symmetric matrix, there exists an orthogonal matrix Q such that $Q'DQ = D_1 =$ the diagonal matrix whose entries along the diagonal are the eigenvalues $d_1, d_2, \cdots, d_p$ of D (see Anderson [1], pages 338–341).

If we transform X, letting $Z = Q'X$ and $\theta^* = Q'\theta$

$$\begin{aligned} RD(X, \theta) - RD(\delta_a(X), \theta) &= aE_0[(2Z'D_1Z + 2(\theta^*)'D_1Z)\|Z + \theta^*\|^{-2}] \\ &\quad - a^2E_0[(Z'D_1Z + 2(\theta^*)'D_1Z + (\theta^*)'D_1(\theta^*))\|Z + \theta^*\|^{-4}] . \end{aligned}$$

So, we may assume without loss of generality that D is this diagonal matrix.

Hence, $X = [X_1, X_2, \cdots, X_p]'$ and $\theta = [\theta_1, \theta_2, \cdots, \theta_p]'$ then $RD(X, \theta) - RD(\delta_a(X), \theta) = aE_0[\sum_{i=1}^p d_i[(2X_i^2 + 2\theta_i X_i)\|X + \theta\|^{-2} - a(X_i + \theta_i)^2\|X + \theta\|^{-4}]]$.

Immediately from Lemma 6.1.7 when $r(\cdot) = 1$,

$$\begin{aligned} &RD(X, \theta) - RD(\delta_a(X), \theta) \\ &\quad = (\theta'D\theta a/((p-1)\|\theta\|^2))[E_0[(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}] \\ &\qquad - aE_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &\qquad + (\operatorname{tr} Da/(p-1))[E_0[2\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\qquad - aE_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \end{aligned}$$

where $\|Y\|^2 = \sum_{i=2}^p X_i^2$ and $\operatorname{tr} D = \operatorname{trace} D = \sum_{i=1}^p d_i$.

From (6.1.6) and (6.1.8)

$$\begin{aligned} &E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\quad = 2(p-1)[1 - E_0(\|\theta\|(X_1 + \|\theta\|))((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\qquad - 2pE_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\quad = 4M \int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))(R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2(R^2 - y^2))\, dy \\ &\qquad - 4M \int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))(R^4 - \|\theta\|^2R^2 + 2\|\theta\|^2(R^2 - y^2))\, dy \end{aligned}$$

where, as in (2.1.6), $d_{R,\|\theta\|}(y) = (R^2 - \|\theta\|^2)^2 + 4\|\theta\|^2(R^2 - y^2)$ and $M = [(2R^2/p) \int_0^R (R^2 - y^2)^{(p-3)/2}\, dy]^{-1}$. Therefore,

$$\begin{aligned} \text{(4.1.3)} \quad &E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\quad = -8M\|\theta\|^2 \int_0^R ((y^2(R^2 - y^2)^{(p-1)/2})/d_{R,\|\theta\|}(y))\, dy . \end{aligned}$$

Lemma 6.1.8 states that $E_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}] \geqq 0$ and clearly, (4.1.3) implies $E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \leqq 0$. Hence, if d_L = maximum eigenvalue of D, then

$$\begin{aligned} &RD(X, \theta) - RD(\delta_a(X), \theta) \\ &\quad \geqq (d_L a/(p-1)) \\ &\qquad \times [E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ \text{(4.1.4)} \quad &\qquad - aE_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &\qquad + (\operatorname{tr} Da/(p-1))[E_0[2\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\qquad - aE_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &\quad = [RD(X, \theta) - RD(\delta_a(X), \theta)]^* . \end{aligned}$$

If

$$b^*(\|\theta\|) = \frac{E_0[(d_L(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2) + 2 \operatorname{tr} D\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}]}{E_0[(d_L((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2) + \operatorname{tr} D\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]} ,$$

then (4.1.4) implies

$$RD(X, \theta) - RD(\delta_a(X, \theta)) \geqq [RD(X, \theta) - RD(\delta_a(X), \theta)]^* \geqq 0 \quad \text{if} \quad 0 \leqq a \leqq b^*(\|\theta\|) .$$

Writing $b^*(\|\theta\|)$ in integral form using (6.1.6)—(6.1.9) we obtain the following:

$$b^*(\|\theta\|) = A(\|\theta\|)/B(\|\theta\|) , \qquad \text{where}$$

$$\begin{aligned} A(\|\theta\|) = &-8\|\theta\|^2 d_L \textstyle\int_0^R (y^2(R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy \\ &+ (4\,\mathrm{tr}\, D/p) \textstyle\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y)) \\ &\times (R^4 - \|\theta\|^2 R^2 + 2\|\theta\|^2(R^2 - y^2))\, dy , \\ B(\|\theta\|) = &\; d_L \textstyle\int_0^R ((R^2 - y^2)^{(p-3)/2}/d_{R,\|\theta\|}(y)) \\ &\times ((1-p)R^4 + (p-1)R^2\|\theta\|^2 + (pR^2 - (p-2)\|\theta\|^2)(R^2 - y^2))\, dy \\ &+ (\mathrm{tr}\, D/(p-2)) \textstyle\int_0^R ((R^2 - y^2)^{(p-3)/2}/d_{R,\|\theta\|}(y)) \\ &\times ((p-1)R^4 + (1-p)R^2\|\theta\|^2 + (p\|\theta\|^2 - (p-2)R^2)(R^2 - y^2))\, dy \end{aligned}$$

and $d_{R,\|\theta\|}(y)$ is defined by (2.1.6).

Simple calculations using (6.1.1) lead to

$$\lim_{\|\theta\|\to\infty} b^*(\|\theta\|) = (2/(p+2))((\mathrm{tr}\, D/d_L) - 2)R^2 .$$

Hence, if trace $D < 2d_L$, there does not exist a minimax estimator of the form (2.1.1) for $a \geqq 0$.

We will now prove the following theorem:

THEOREM 4.1.1. *If X is a single observation on a p-dimensional spherical uniform distribution and $\delta_a(X)$ is given by* (2.1.1), *then with respect to general quadratic loss* (1.2), *the risk of $\delta_a(X)$ dominates* (*is less than or equal to*) *the risk of X when $p \geqq 3$ provided*

$$0 \leqq a \leqq 2a_0^*((\mathrm{tr}\, D/d_L) - 2)/E_0(\|X\|^{-2}) = 2a_0((\mathrm{tr}\, D/d_L) - 2)R^2$$

where

$$\begin{aligned} a_0^* &= p/((p+2)(p-2)) \qquad &\textit{for} \quad p \geqq 4 \\ &= .375 &\textit{for} \quad p = 3 \end{aligned} \tag{4.1.5}$$

and a_0 is given by (4.1.1), tr D = trace $D \geqq 2d_L$ *and d_L = maximum eigenvalue of D.*

PROOF. Suppose tr $D/d_L = q$, therefore, $2 \leqq q \leqq p$. If

$$[RD(X, \theta) - RD(\delta_{2a_0(q-2)R^2}(X), \theta)]^*/(2a_0(q-2)d_L R^2) = \Delta_q ,$$

then for $0 \leqq a \leqq 2a_0(q-2)R^2$, by (4.1.4), $[RD(X, \theta) - RD(\delta_a(X), \theta)]^*/(ad_L) \geqq \Delta_q$. In addition, (4.1.4) clearly implies

$$\begin{aligned} \Delta_q = &\; 1/(p-1)[E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &- 2a_0(q-2)R^2 E_0[((p-1)(X_1+\|\theta\|)^2 - \|Y\|^2)((X_1+\|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &+ q/(p-1)[E_0[2\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &- 2a_0(q-2)R^2 E_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] . \end{aligned} \tag{4.1.6}$$

Since $RD(X, \theta) - RD(\delta_a(X), \theta) \geqq [RD(X, \theta) - RD(\delta_a(X), \theta)]^* \geqq ad_L \Delta_q$, it is clear that the proof will be complete if $\Delta_q \geqq 0$ for all $\|\theta\|$.

However, for fixed $\|\theta\|$,

$$(d^2/dq^2)(\Delta_q) = (-2R^2 a_0/(p-1))E_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}] \leqq 0\,,$$

implying that for fixed $\|\theta\|$, Δ_q is a concave function of q. Since $2 \leqq q \leqq p$, the concavity of Δ_q implies $\Delta_q \geqq$ minimum (Δ_2, Δ_p).

When $q = p$,

$$\begin{aligned}\Delta_p = 2[1 &- E_0(\|\theta\|(X_1 + \|\theta\|)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}) \\ &- a_0(p-2)R^2 E_0((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}]\,.\end{aligned}$$

Note first that by the definition of a_0, given by (4.1.1), $(p-2)a_0 = c_0$, where c_0, as in (2.1.2), is given by

$$\begin{aligned} c_0 &= (p-2)/(p+2) \quad &\text{when} \quad p \geqq 4 \\ &= .125 \quad &\text{when} \quad p = 3\,. \end{aligned}$$

Hence, by (2.1.4), Δ_p simply equals (difference in risks for $a = 2c_0 R^2)/(2c_0 R^2)$ when the risk is quadratic loss (1.1). Clearly, Theorems 2.1.1 and 2.2.1 imply this is nonnegative.

We will now show that Δ_2 is also nonnegative.

As (4.1.6) implies,

$$\Delta_2 = (2/(p-1))E_0[((p-1)(X_1^2 + \|\theta\|X_1) + \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1})]\,.$$

Substituting in the expressions for the expected values given in (6.1.6) and (6.1.8)

$$\begin{aligned}\Delta_2 &\propto \textstyle\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))[R^4 - (p+1)\|\theta\|^2 R^2 + (p+2)\|\theta\|^2(R^2 - y^2)]\,dy \\ &= \Delta_2^*\,.\end{aligned}$$

We will show Δ_2^* is nonnegative for two cases:

CASE 1. $\|\theta\|^2 \leqq (p/2)R^2$.

With respect to the density

$$\begin{aligned} g_{p,\|\theta\|}(y) &= ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))/\textstyle\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\,dy \\ &\qquad \text{for} \quad 0 \leqq y \leqq R \\ &= 0 \quad \text{elsewhere}, \end{aligned}$$

which, according to Lemma 6.2.1, has MLR nondecreasing in y for $\|\theta\| \leqq R$ and MLR nonincreasing in y for $\|\theta\| \geqq R$,

$$\begin{aligned} \Delta_2^*/\textstyle\int_0^R & ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\,dy \\ &= R^4 - (p+1)R^2\|\theta\|^2 + (p+2)\|\theta\|^2 E_{\|\theta\|}(R^2 - Y^2) \\ &\geqq R^4 - (p+1)R^2\|\theta\|^2 + (p+2)\|\theta\|^2 E_R(R^2 - Y^2) \\ &= R^4 - (p+1)R^2\|\theta\|^2 + ((p+2)(p-1)/p)R^2\|\theta\|^2 \\ &= (R^2/p)[pR^2 - 2\|\theta\|^2] \geqq 0\,. \end{aligned}$$

CASE 2. $\|\theta\|^2 \geqq (p/2)R^2$.

Using (6.1.3) and Lemma 6.1.5, we may rewrite $\Delta_2{}^*$ as follows:

$$\begin{aligned}\Delta_2{}^* &= (\tfrac{1}{4})[(p+2)\textstyle\int_0^R (R^2-y^2)^{(p-1)/2}\,dy \\ &\quad - [(p-2)R^4 + 2pR^2\|\theta\|^2 + (p+2)\|\theta\|^4]\textstyle\int_0^R ((R^2-y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\,dy] \\ &= \textstyle\int_0^R ((R^2-y^2)^{(p-1)/2}/4\|\theta\|^2)\,dy \\ &\quad \times [(p+2)\|\theta\|^2 - [(p-2)R^2 + (p+2)\|\theta\|^2]\textstyle\sum_{i=0}^\infty (-1)^i b_i (R^2/\|\theta\|)^i]\end{aligned}$$

where $b_0 = 1$ and $b_i = [(p-2i)/(p+2i)]b_{i-1}$, $i = 0, 1, 2, \cdots$.

Applying simple calculations we obtain

$$\Delta_2{}^* \propto \textstyle\sum_{i=0}^\infty (-1)^i c_i (R^2/\|\theta\|^2)^i\,, \qquad \text{where} \quad c_i = (p-2)b_{i+1} - (p+2)b_{i+2}$$

for $i = 0, 1, 2, \cdots$.

For p an even number, c_i is zero for $i \geqq (p-2)/2$ and c_i is nonnegative for $i \leqq (p-4)/2$. If $p = 4i+3$ for $i = 1, 2, \cdots$ then $(-1)^i c_i$ is positive for $i \geqq (p-3)/2$ and c_i is positive for $i \leqq (p-5)/2$. Similarly, if $p = 4j+1$ for $j = 1, 2, \cdots$ then $(-1)^i c_i$ is negative for $i \geqq (p-3)/2$ and c_i is positive for $i \leqq (p-5)/2$. So,

$$|c_{i+1}| = [((i+2)(p-2(i+2)))/((i+1)(p+2(i+3)))]|c_i|$$

when $i \leqq (p-4)/2$;

$$|c_{i+1}| = [((i+2)(2(i+2)-p))/((i+1)(p+2(i+3)))]|c_i|$$

when $i \geqq (p-3)/2$;

$$|c_{i+1}| \leqq |c_i| \qquad \text{for} \quad i \geqq (p-3)/2$$

and

$$|c_{i+1}| \leqq (p/2)|c_i| \qquad \text{for} \quad i \leqq (p-4)/2\,.$$

By an analogous argument to the one used in Case 3 of the proof of Theorem 2.1.1 we have

$$\textstyle\sum_{i=0}^\infty (-1)^i c_i (R^2/\|\theta\|^2)^i \geqq 0\,.$$

We have shown that $\Delta_2 \geqq 0$ and $\Delta_p \geqq 0$, therefore, $\Delta_q \geqq$ minimum $(\Delta_2, \Delta_p) \geqq 0$. The proof of the theorem is now complete. □

We now expand this class of estimators by considering the estimators Baranchik [2] considered, namely estimators of the form

$$\delta_{a,r}(X) = (1 - a(r(\|X\|^2)/\|X\|^2))X\,,$$

as given by (2.3.1).

THEOREM 4.1.2. *If $X = [X_1, X_2, \cdots, X_p]'$ is a $p \times 1$ random vector with a spherical uniform distribution and $\delta_{a,r}(X)$ is given by* (2.3.1), *then provided $r(\|X\|^2)$ is nondecreasing and $p \geqq 3$, $\delta_{a,r}(X)$ is minimax with respect to general quadratic loss* (1.2) *when $0 \leqq a \leqq 2R^2 a_0((\mathrm{tr}\, D/d_L) - 2) = 2R^2 a_0{}^*((\mathrm{tr}\, D/d_L) - 2)/E_0(\|X\|^{-2})$, where*

a_0 and a_0 are defined by* (4.1.1) *and* (4.1.5) *respectively, and* trace $D = \operatorname{tr} D \geqq 2d_L = 2$ *(maximum eigenvalue of D). (For $p = 3$, also assume $r(\|X\|^2)/\|X\|^2$ is nonincreasing.)*

PROOF. Since $r(\|X\|^2)$ is nondecreasing,

$$\begin{aligned} &RD(X, \theta) - RD(\delta_{a,r}(X), \theta) \\ &\quad = aE_\theta[r(\|X\|^2)[2X'D(X-\theta)\|X\|^{-2} - ar(\|X\|^2)(X'DX)\|X\|^{-4}]] \\ &\quad \geqq aE_\theta[r(\|X\|^2)[2X'D(X-\theta)\|X\|^{-2} - a(X'DX)\|X\|^{-4}]] \\ &\quad = aE_0 r(\|X+\theta\|^2)[2X'D(X+\theta)\|X+\theta\|^{-2} \\ &\qquad - a(X'DX + 2\theta'DX + \theta'D\theta)\|X+\theta\|^{-4}]\,. \end{aligned} \tag{4.1.7}$$

Clearly, we may assume without loss of generality that D is a diagonal matrix. Immediately by (4.1.7), Lemma 6.1.7 and Lemma 6.1.2, if $\|Y\|^2 = \sum_{i=2}^p X_i^2$, then

$$\begin{aligned} &[RD(X, \theta) - RD(\delta_{a,r}(X), \theta)]/a \\ &\quad \geqq (\theta'D\theta/((p-1)\|\theta\|^2)) \\ &\qquad \times [E_{\|\theta\|}[r(\|X\|^2)(2(p-1)(X_1^2 - \|\theta\|X_1) - 2\|Y\|^2)\|X\|^{-2}] \\ &\qquad - aE_{\|\theta\|}[r(\|X\|^2)((p-1)X_1^2 - \|Y\|^2)\|X\|^{-4}]] \\ &\qquad + (\operatorname{tr} D/(p-1))[E_{\|\theta\|}[2r(\|X\|^2)\|Y\|^2\|X\|^{-2}] \\ &\qquad - aE_{\|\theta\|}[r(\|X\|^2)\|Y\|^2\|X\|^{-4}]] \end{aligned} \tag{4.1.8}$$

where $E_{\|\theta\|}$ denotes the expected value when $\theta = [\|\theta\|, 0, 0, \cdots, 0]'$.

Suppose we define $E_{\|\theta\|} r(X)h(X)$ as follows:

$$\begin{aligned} E_{\|\theta\|}(r(X)h(X)) &= E_{\|\theta\|}[r(\|X\|^2)(2(p-1)(X_1^2 - \|\theta\|X_1) - 2\|Y\|^2)\|X\|^{-2}] \\ &\quad - aE_{\|\theta\|}[r(\|X\|^2)((p-1)X_1^2 - \|Y\|^2)\|X\|^{-4}]\,. \end{aligned} \tag{4.1.9}$$

Clearly, then, (4.1.8) is equivalent to

$$\begin{aligned} &[RD(X, \theta) - RD(\delta_{a,r}(X), \theta)]/a \\ &\quad \geqq (\theta'D\theta/((p-1)\|\theta\|^2))E_{\|\theta\|}(r(X)h(X)) \\ &\qquad + (\operatorname{tr} D/(p-1))[E_{\|\theta\|}[2r(\|X\|^2)\|Y\|^2(\|X\|^2 - (a/2))\|X\|^{-4}]]\,. \end{aligned} \tag{4.1.10}$$

CASE 1. If $E_{\|\theta\|} r(X)h(X) \geqq r(a/2)E_{\|\theta\|} h(X)$.

Since $r(\cdot)$ is nondecreasing,

$$E_{\|\theta\|}[2r(\|X\|^2)\|Y\|^2(\|X\|^2 - (a/2))\|X\|^{-4}] \geqq r(a/2)E_{\|\theta\|}[2\|Y\|^2(\|X\|^2 - (a/2))\|X\|^{-4}]\,.$$

Hence, clearly, by (4.1.10),

$$[RD(X, \theta) - RD(\delta_{a,r}(X), \theta)] \geqq r(a/2)[RD(X, \theta) - RD(\delta_a(X), \theta)] \geqq 0$$

when $0 \leqq a \leqq 2a_0^*((\operatorname{tr} D/d_L) - 2)/E_0(\|X\|^{-2})$ (see Theorem 4.1.1).

CASE 2. If $E_{\|\theta\|}(r(X)h(X)) \leqq r(a/2)E_{\|\theta\|} h(X)$.

By (4.1.3),

$$\begin{aligned} &E_{\|\theta\|}[(2(p-1)(X_1^2 - \|\theta\|X_1) - 2\|Y\|^2)\|X\|^{-2}] \\ &\quad = E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \leqq 0\,. \end{aligned}$$

Similarly, Lemma 6.1.8 states

$$E_{\|\theta\|}[((p-1)X_1^2 - \|Y\|^2)\|X\|^{-4}]$$
$$= E_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]$$

is nonnegative.

Hence, clearly by the definition of $E_{\|\theta\|}(r(X)h(X))$ given by (4.1.9), $E_{\|\theta\|}(r(X)h(X)) \leqq 0$. Hence, if d_L = maximum eigenvalue of D, then

$$\begin{aligned}[RD(X, \theta) - RD(\delta_{a,r}(X), \theta)]/a \\ &\geqq (d_L/(p-1))E_{\|\theta\|}(r(X)h(X)) \\ &\quad + (\operatorname{tr} D/(p-1))[E_{\|\theta\|}[2r(\|X\|^2)\|Y\|^2(\|X\|^2 - (a/2))\|X\|^{-4}]] \\ &= [(RD(X, \theta) - RD(\delta_{a,r}(X), \theta))/a]^* .\end{aligned}$$

For $0 \leqq a \leqq 2R^2a_0((\operatorname{tr} D/d_L) - 2)$, $\delta_{a,r}(X)$ will be minimax if $[(RD(X, \theta) - RD(\delta_{a,r}(X), \theta))/a]^* \geqq 0$ when $a = 2R^2a_0((\operatorname{tr} D/d_L) - 2)$ or equivalently, if $\Delta_q(r(\|X\|^2)) \geqq 0$, where $q = \operatorname{tr} D/d_L$, and $\Delta_q(r(\|X\|^2)) = (d_L)^{-1}[(RD(X, \theta) - RD(\delta_{a,r}(X), \theta))/a]^*$ when $a = 2R^2a_0(q-2)$. Hence,

$$\begin{aligned}\Delta_q(r(\|X\|^2)) &= 1/(p-1)[E_{\|\theta\|}[r(\|X\|^2)(2(p-1)(X_1^2 - \|\theta\|X_1) - 2\|Y\|^2)\|X\|^{-2}] \\ &\quad - 2a_0R^2(q-2)E_{\|\theta\|}[r(\|X\|^2)((p-1)X_1^2 - \|Y\|^2)\|X\|^{-4}]] \\ &\quad + (2q/(p-1))[E_{\|\theta\|}[r(\|X\|^2)\|Y\|^2\|X\|^{-2}] \\ &\quad - a_0R^2(q-2)E_{\|\theta\|}[r(\|X\|^2)\|Y\|^2\|X\|^{-4}]] .\end{aligned} \tag{4.1.11}$$

For fixed $\|\theta\|$, $\Delta_q(r(\|X\|^2))$ is a concave function of q clearly implying that $\Delta_q(r(\|X\|^2)) \geqq$ minimum $(\Delta_2(r(\|X\|^2)), \Delta_p(r(\|X\|^2)))$. If we show that $\Delta_2(r(\|X\|^2)) \geqq 0$ and $\Delta_p(r(\|X\|^2)) \geqq 0$ then $\Delta_q(r(\|X\|^2)) \geqq 0$ for $2 \leqq q \leqq p$, and the proof will be complete.

Allowing $q = p$ in (4.1.11) and since $2a_0R^2(p-2) = 2c_0R^2$, where c_0 is given by (2.1.2), we have $\Delta_p(r(\|X\|^2)) = 2E_{\|\theta\|}[r(\|X\|^2)(1 - (\|\theta\|X_1 + c_0R^2)\|X\|^{-2})] \geqq 0$, as proven in Theorem 2.3.1.

Before proceeding, note that when $r(\cdot) \equiv 1$, a shift of X to the origin, clearly shows that $\Delta_2(r(\|X\|^2)) = \Delta_2(1) = \Delta_2$, where Δ_2 is given by (4.1.6) when $q = 2$. In the proof of Theorem 4.1.1, Δ_2 was shown to be nonnegative.

SUBCASE 2.1. $\|\theta\|^2 \geqq R^2$.

From (4.1.11), $\Delta_2(r(\|X\|^2)) = (2/(p-1))E_{\|\theta\|}[r(\|X\|^2)((p-1)(X_1^2 - \|\theta\|X_1) + \|Y\|^2)\|X\|^{-2}]$. According to Lemma 6.1.9,

$$\begin{aligned}E_{\|\theta\|}[r(\|X\|^2)(X_1^2\|X\|^{-2})] &= ((p-1)/p)E_{\|\theta\|}[r(\|X\|^2)((\|X\|^2 - R^2 + \|\theta\|^2)/2\|\theta\|)(X_1\|X\|^{-2}) \\ &\quad + (1/p)E_{\|\theta\|}r(\|X\|^2) .\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta_2(r(\|X\|^2)) \\ &\quad = (2/p)[2E_{\|\theta\|}r(\|X\|^2) + ((p-2)/2\|\theta\|)E_{\|\theta\|}(r(\|X\|^2)X_1) \\ &\qquad - (((p-2)R^2 + (p+2)\|\theta\|^2)/2\|\theta\|)E_{\|\theta\|}(r(\|X\|^2)(X_1\|X\|^{-2}))] .\end{aligned} \tag{4.1.12}$$

By Lemma 6.2.4, $E_{||\theta||}(X_1 \,|\, ||X||^2)$ is nondecreasing in $||X||^2$ when $||\theta|| \geqq R$. Moreover, Lemma 6.2.2 implies $E_{||\theta||}(X_1||X||^{-2} \,|\, ||X||^2)$ is nonincreasing in $||X||^2$ when $||\theta|| \geqq R$. Combining these two facts with the assumption that $r(||X||^2)$ is a nondecreasing function of $||X||^2$, (4.1.12) implies

$$\Delta_2(r(||X||^2)) \geqq [E_{||\theta||} r(||X||^2)]\Delta_2 .$$

As we noted earlier, Δ_2 is nonnegative, hence the proof is complete for this subcase.

SUBCASE 2.2. $0 \leqq ||\theta||^2 \leqq 4a_0 R^2$.

From (4.1.11), we may rewrite $\Delta_2(r(||X||^2))$ as follows:

$$\begin{aligned}\Delta_2(r(||X||^2)) = (2/(p-1))[&E_{||\theta||} r(||X||^2) - E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \\ &+ (p-2)E_{||\theta||}[r(||X||^2)(X_1^2 - ||\theta||X_1)||X||^{-2}]] .\end{aligned} \tag{4.1.13}$$

For $||\theta||^2 \leqq 4a_0 R^2$,

$$\begin{aligned}E_{||\theta||}[r(||X||^2)(X_1^2 - ||\theta||X_1)||X||^{-2}] &\geqq -(||\theta||^2/4)E_{||\theta||}[r(||X||^2)||X||^{-2}] \\ &\geqq -a_0 R^2 E_{||\theta||}[r(||X||^2)||X||^{-2}] .\end{aligned}$$

Thus, clearly, by (4.1.13),

$$\Delta_2(r(||X||^2)) \geqq (1/(p-1))\Delta_p(r(||X||^2)) \geqq 0 .$$

SUBCASE 2.3. $4a_0 R^2 \leqq ||\theta||^2 \leqq R^2$.

Suppose $E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \geqq E_{||\theta||}[r(||X||^2)(||\theta||^2 - a_0 R^2)||X||^{-2}]$; then

$$\begin{aligned}E_{||\theta||}[r(||X||^2)(X_1^2 - ||\theta||X_1)||X||^{-2}] &\geqq E_{||\theta||}[r(||X||^2)(||\theta||X_1 - ||\theta||^2)||X||^{-2}] \\ &\geqq -a_0 R^2 E_{||\theta||}[r(||X||^2)||X||^{-2}] ,\end{aligned}$$

and so,

$$\Delta_2(r(||X||^2)) \geqq (1/(p-1))\Delta_p(r(||X||^2)) \geqq 0 .$$

To complete the proof, we will now show that $\Delta_2(r(||X||^2)) \geqq 0$ if $E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \leqq E_{||\theta||}[r(||X||^2)(||\theta||^2 - a_0 R^2)||X||^{-2}]$. Lemma 6.2.5 states that

$$E_{||\theta||}[r(||X||^2)((||X||^2 - R^2 - ||\theta||^2)/2||\theta||)(X_1||X||^{-2})] \geqq -cE_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||_{-}^{2}]$$

when $||\theta||^2 \geqq ((2/cp) - 1)R^2$. Substituting in $c = (2a_0/(1 - a_0))$, for $((1 - (p+1)a_0)/pa_0)R^2 \leqq ||\theta||^2 \leqq R^2$, we have that

$$\begin{aligned}&E_{||\theta||}[r(||X||^2)((||X||^2 - R^2 - ||\theta||^2)/2||\theta||)(X_1||X||^{-2})] \\ &\quad \geqq -(2a_0/(1 - a_0))E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \\ &\quad \geqq -(2a_0/(1 - a_0))E_{||\theta||}[r(||X||^2)(||\theta||^2 - a_0 R^2)||X||^{-2}] \\ &\quad \geqq -2a_0 R^2 E_{||\theta||}[r(||X||^2)||X||^{-2}] .\end{aligned}$$

By rewriting (4.1.12) we obtain

$$\begin{aligned}\Delta_2(r(||X||^2)) = (2/p)[&2E_{||\theta||} r(||X||^2) - 2||\theta||E_{||\theta||}[r(||X||^2)X_1||X||^{-2}] \\ &+ (p-2)E_{||\theta||}[r(||X||^2)((||X||^2 - ||\theta||^2 - R^2)/2||\theta||) \\ &\times (X_1||X||^{-2})]] .\end{aligned} \tag{4.1.14}$$

Using (4.1.14), we see that

$$\begin{aligned}\Delta_2(r(\|X\|^2)) &\geqq (2/p)[2E_{\|\theta\|}r(\|X\|^2) - 2\|\theta\|E_{\|\theta\|}[r(\|X\|^2)X_1\|X\|^{-2}] \\ &\quad - 2a_0(p-2)R^2E_{\|\theta\|}[r(\|X\|^2)\|X\|^{-2}]] \\ &= (2/p)(\Delta_p(r(\|X\|^2))) \geqq 0.\end{aligned}$$

When $p \geqq 4$, $((1-(p+1)a_0)/pa_0) = (1/p) \leqq 4a_0$. Hence, for $p \geqq 4$, $\Delta_2(r(\|X\|^2)) \geqq 0$ when $R^2/p \leqq \|\theta\|^2 \leqq R^2$ implying $\Delta_2(r(\|X\|^2)) \geqq 0$ when $4a_0R^2 \leqq \|\theta\|^2 \leqq R^2$.

For $p = 3$ and $.5R^2 \leqq \|\theta\|^2 \leqq R^2$, by (4.1.12), $\Delta_2(r(\|X\|^2)) = 2/3[2E_{\|\theta\|}r(\|X\|^2) + E_{\|\theta\|}r(\|X\|^2)((\|X\|^2 - R^2 - 5\|\theta\|^2)/2\|\theta\|)(X_1\|X\|^{-2})]$. By Lemmas (6.24) and (6.25) and the fact that $\Delta_p(r(\|X\|^2)) \geqq 0$ and for $p = 3$, $E_{\|\theta\|}(X_1 \mid \|X\|^2)$ is nondecreasing in $\|X\|^2$,

$$\begin{aligned}\Delta_2(r(\|X\|^2)) &\geqq 2/3[.25R^2E_{\|\theta\|}r(\|X\|^2)\|X\|^{-2} - (\tfrac{4}{9})E_{\|\theta\|}r(\|X\|^2)\|X\|^{-2}E_{\|\theta\|}X_1] \\ &= 2/3E_{\|\theta\|}(r(\|X\|^2)\|X\|^{-2})[.25R^2 - (\tfrac{4}{9})(\tfrac{3}{2})\|\theta\|^2(R^2 - \|\theta\|^2)R^{-2}] \\ &\propto (.75R^4 - 2\|\theta\|^2R^2 + 2\|\theta\|^4) \\ &\geqq .25R^4 \qquad \text{for} \quad .5R^2 \leqq \|\theta\|^2 \leqq R^2 .\end{aligned}$$

Hence, for $p \geqq 3$ and for all $\|\theta\|$, $\Delta_2(r(\|X\|^2)) \geqq 0$ and $\Delta_p(r(\|X\|^2)) \geqq 0$ thus implying $\Delta_q(r(\|X\|^2)) \geqq \text{minimum}\,(\Delta_2(r(\|X\|^2)), \Delta_p(r(\|X\|^2))) \geqq 0$.

The proof of the theorem is complete. □

4.2. *Minimax estimators of the mean vector of a spherically symmetric unimodal distribution.* In this section we consider X one observation on a p-dimensional spherically symmetric unimodal (s.s.u.) distribution. Hence, as in (3.2.1), the density of X is given by

$$g(\|x - \theta\|) = \int c(R)I_S(x, R)\, dF(R)$$

where $F(\cdot)$ is a known cdf on $(0, \infty)$ and $c(R)$ and $I_S(X, R)$ are defined in (1.4).

We will show that with respect to general quadratic loss (1.2), $\delta_a(X) = (1 - (a/\|X\|^2))X$ is minimax when $0 \leqq a \leqq (2a_0/E(R^{-2}))((\text{tr}\, D/d_L) - 2)$ where a_0 is defined by (4.1.1), d_L = maximum eigenvalue of D and tr D = trace $D \geqq 2d_L$.

Since the random vector $X \mid R$ has a spherical uniform distribution ($X \mid R \sim U\{\|X - \theta\|^2 \leqq R^2\}$), proceeding exactly as in Section 4.1, clearly the same inequality (4.1.4) holds for the difference in risks $RD(X, \theta) - RD(\delta_a(X), \theta)$ as did for the uniform distribution. That is,

$$\begin{aligned}&RD(X, \theta) - RD(\delta_a(X), \theta) \\ &\quad = aE[E_\theta[2X'D(X - \theta)\|X\|^{-2} - a(X'DX)\|X\|^{-4} \mid R]] \\ &\quad \geqq (d_La/(p-1))[E_0[(2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\qquad - aE_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &\qquad + (\text{tr}\, Da/(p-1))[E_0[2\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &\qquad - aE_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &\quad = [RD(X, \theta) - RD(\delta_a(X), \theta)]^* .\end{aligned}$$

If $q = \text{tr } D/d_L$, $2 \leqq q \leqq p$, and $\Delta_q{}^* = [RD(X, \theta) - RD(\delta_a(X), \theta)]^*/ad_L$ when $a = 2a_0(q-2)/E(R^{-2})$, then

$$\begin{aligned}\Delta_q{}^* = &(1/(p-1))[E_0[2(p-1)(X_1^2 + \|\theta\|X_1) - 2\|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &- (2a_0(q-2)/E(R^{-2}))E_0[((p-1)(X_1 + \|\theta\|)^2 - \|Y\|^2) \\ &\times ((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] \\ &+ (q/(p-1))[E_0[2\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &- (2a_0(q-2)/E(R^{-2}))E_0[\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2}]] ,\end{aligned}$$

and the risk of $\delta_a(X)$ is less than or equal to the risk of X for $0 \leqq a \leqq 2a_0(q - 2)/E(R^{-2})$ if $\Delta_q{}^* \geqq 0$. For fixed $\|\theta\|$, $\Delta_q{}^*$ is a concave function of q and since $2 \leqq q \leqq p$, it follows that $\Delta_q{}^* \geqq$ minimum $(\Delta_2{}^*, \Delta_p{}^*)$.

However, when $q = p$, $2a_0(q-2) = 2c_0$, where c_0 is defined by (2.1.2) and $\Delta_p{}^* = (E_\theta\|X - \theta\|^2 - E_\theta\|\delta_a(X) - \theta\|^2)/a$ for $a = 2c_0/E(R^{-2})$. Hence, by Theorem 2.3.1, $\Delta_p{}^* \geqq 0$.

When $q = 2$

$$\Delta_2{}^* = (2/(p-1))E_0[((p-1)(X_1^2 + \|\theta\|X_1) + \|Y\|^2)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] .$$

When $q = 2$ and X has a spherical uniform distribution, the above expectation is just Δ_2, defined by (4.1.6) which for that case was shown to be nonnegative in the proof of Theorem 4.1.1. Hence, since $X \mid R \sim U\{\|X - \theta\|^2 \leqq R^2\}$, it is clear that $\Delta_2{}^* \geqq 0$.

Thus, $\delta_a(X)$ given by (2.1.1) is minimax with respect to general quadratic loss for $0 \leqq a \leqq (2a_0/E(R^{-2}))((\text{tr } D/d_L) - 2)$. In the following theorem we will formally state this result.

THEOREM 4.2.1. *If X is a $p \times 1$ random vector ($p \geqq 3$) with a density given by* (3.2.1), *then with respect to general quadratic loss* (1.2), *$\delta_a(X)$ given by* (2.1.1) *is minimax provided*

(4.2.1) $0 \leqq a \leqq (2a_0/E(R^{-2}))((\text{tr } D/d_L) - 2) = 2a_0{}^*((\text{tr } D/d_L) - 2)/E_0(\|X\|^{-2})$, *where* a_0 *and* $a_0{}^*$ *are defined by* (4.1.1) *and* (4.1.5) *respectively, and* $\text{tr } D \geqq 2d_L =$ 2(*maximum eigenvalue of* D) .

We next expand this class of estimators.

THEOREM 4.2.2. *If X is one observation on p-dimensional* s.s.u. *distribution about θ with a density given by* (3.2.1) *and $\delta_{a,r}(X) = (1 - a(r(\|X\|^2)/\|X\|^2))X$, as defined by* (2.3.1), *then provided $p \geqq 3$, $(r(\|X\|^2)/\|X\|^2)$ is nonincreasing, $r(\|X\|^2)$ is nondecreasing and* (4.2.1) *is satisfied, the risk of $\delta_{a,r}(X)$ dominates the risk of X with respect to general quadratic loss* (1.2).

PROOF. Since $X \mid R$ may be considered to be a spherical uniform random vector, proceeding exactly as in the proof of Theorem 4.1.2, we see that this theorem will be proven if $\Delta_q^*(r(\|X\|^2))$ is nonnegative for $2 \leqq q = (\mathrm{tr}\, D/d_L) \leqq p$ and

$$\begin{aligned}\Delta_q^*(r(\|X\|^2)) = {} & (1/(p-1))[E_{\|\theta\|}[r(\|X\|^2)[(2(p-1)(X_1^2 - \|\theta\|X_1) - 2\|Y\|^2)(\|X\|)^{-2}]] \\ & - (2(q-2)a_0/E(R^{-2}))E_{\|\theta\|}[r(\|X\|^2)[((p-1)X_1^2 - \|Y\|^2)\|X\|^{-4}]]] \\ & + (q/(p-1))[E_{\|\theta\|}[r(\|X\|^2)2\|Y\|^2\|X\|^{-2}] \\ & - (2(q-2)a_0/E(R^{-2}))E_{\|\theta\|}[r(\|X\|^2)\|Y\|^2\|X\|^{-4}]]\ .\end{aligned}$$

Clearly, $\Delta_q^*(r(\|X\|^2))$ is a concave function of q for fixed $\|\theta\|$.

For $q = p$, $2(q-2)a_0 = 2c_0$, where c_0 is defined by (2.1.2) and hence, by Theorem 3.3.1,

$$\Delta_p^* = E_{\|\theta\|}[r(\|X\|^2)[2 - (2\|\theta\|X_1 + (2c_0/E(R^{-2})))\|X\|^{-2}]] \geqq 0\ .$$

Moreover, since $X \mid R \sim U\{\|X-\theta\|^2 \leqq R^2\}$, the proof of Theorem 4.1.2 immediately shows that $\Delta_2^*(r(\|X\|^2)) \geqq 0$.

Hence, by the concavity of $\Delta_q^*(r(\|X\|^2))$, $\Delta_q^*(r(\|X\|^2)) \geqq$ minimum $(\Delta_2^*(r(\|X\|^2)), \Delta_p^*(r(\|X\|^2))) \geqq 0$, thus completing the proof of this theorem. □

5. Remarks and conclusions. We conclude with some observations on the multiple observation case and a discussion of the merits of using these improved estimators.

Consider n observations, $X_1, X_2, \cdots, X_n$, on a p-dimensional s.s.u. distribution about θ. For the normal distribution the problem is easily reduced by sufficiency to one observation, $\bar{X}$. However, this is not the case here. Pitman's estimator given by $\delta(X_1, X_2, \cdots, X_n) = X_1 - E_0[X_1 \mid Y_2, Y_3, \cdots, Y_n]$ where $Y_i = X_i - X_i$, $i = 2, 3, \cdots, n$, is the best invariant estimator and hence, there exist estimators which are better than it. For the normal distribution, Pitman's estimator is $\bar{X}$ and in fact, in one dimension when $n \geqq 3$, Pitman's estimator being $\bar{X}$ characterizes the normal distribution (see Kagan, Linnik and Rao [12], Chapter 7). In general, for $n \geqq 3$, Pitman's estimator is not $\bar{X}$. It is clear from the definition of Pitman's estimator that it is $\bar{X}$ for $n = 1, 2$.

If the distribution of Pitman's estimator is spherically symmetric unimodal about θ, the problem is reduced to the case of one observation. We investigated this question and, as yet, have only proven that Pitman's estimator has a s.s.u. distribution about θ when sampling from a spherical uniform distribution about θ. The more general problem is still under investigation. Note that $\bar{X}$, which is a convolution of random vectors having s.s.u. distributions, has a s.s.u. distribution. Hence, we may use the estimators of Sections 2—4 to improve on $\bar{X}$ with respect to quadratic and general quadratic loss.

We now return to the case when X is one observation on a p-dimensional s.s.u. distribution about θ. Consider the estimator $\delta_a^+(X) = \max(0, (1 - (a/\|X\|^2)))X$. It is clear by the work of Baranchik [2] that $\delta_a^+(X)$ is better than $\delta_a(X) = (1 - (a/\|X\|^2))X$ with respect to quadratic loss (1.1). For $0 \leqq a \leqq (2b_0)/E_0(\|X\|^{-2})$, where $b_0 = (p/(p+2))$ when $p \geqq 4$ and $b_0 = .375$ when $p = 3$, $\delta_a(X)$ and hence $\delta_a^+(X)$ is better than X (see Theorem 3.2.1). These new estimators are certainly not very difficult to calculate and the improvements over X, in some instances, can be very large. Consider, for example, the case of estimating θ, when X has a p-dimensional spherical uniform distribution about θ, with respect to quadratic loss (1.1). According to Lemma 6.1.1, the risk of X equals $E_\theta(\|X-\theta\|^2) = (p/R^p)\int_0^R r^{p+1}\,dr = (p/(p+2))R^2$, for all θ. Again, by Lemma 6.1.1, when $\theta = [0, 0, \cdots, 0]'$, $R(\delta_a(X), 0) = E_0(\|X\|^2) - a[2 - aE_0(\|X\|^{-2})] = (p/(p+2))R^2 - a[2 - a(p/(p-2))R^{-2}]$. Since $R(\delta_a(X), 0)$ is a convex function of a and for $p \geqq 4$, $0 \leqq a \leqq (2b_0)/E_0(\|X\|^{-2}) = 2((p-2)/(p+2))R^2$, $R(\delta_a(X), 0) \geqq R(\delta_{((p-2)/p)R^2}(X), 0) = (4/(p(p+2)))R^2$. Clearly, this risk becomes very small as p becomes large. Moreover, $R(\delta_{((p-2)/p)R^2}(X), 0)/R(X, 0) = (4/p^2)$. Therefore, for $p \geqq 4$, the risk of $\delta_a(X)$ when $a = ((p-2)/p)R^2$ is at most $(\frac{1}{4})$ of the risk of X at the origin. Note too, that $R(\delta_a(X), \theta)$ is a convex function of a and so is minimized at $a = \frac{1}{2}b(\|\theta\|) \geqq \frac{1}{2}b(\|\infty\|)$. So, δ_a is dominated by $\delta_{(p-2)/(p+2)}R^2$, for $0 \leqq a \leqq \frac{1}{2}b(\|\infty\|) = (p-2)/(p+2)R^2$. But, this is not true for $\delta_a{}^+$, which would normally be preferred to δ_a.

If we now wish further improvement, we may consider $\delta_a{}^+(X)$. When $\|X\|^2 \geqq a$, $\delta_a{}^+(X) = \delta_a(X)$ and when $\|X\|^2 \leqq a$, $\delta_a{}^+(X) = 0$. Since the risks only depend on $\theta = [\|\theta\|, 0, 0, \cdots, 0]'$, Lemma 6.1.6 implies that when $\|\theta\| \geqq R + a^{\frac{1}{2}}$, $\|X\|^2$ is always greater than or equal to a, and thus, $\delta_a{}^+(X)$ coincides with $\delta_a(X)$. Similarly, if $\|\theta\| \leqq a^{\frac{1}{2}} - R$ then $\|X\|^2$ is always less than or equal to a and thus $\delta_a{}^+(X) = 0$. Therefore, for $p \geqq 6$ and $R^2 \leqq a \leqq 2((p-2)/(p+2))R^2$, $R(\delta_a{}^+(X), 0) = 0$.

We thus see that when X is one observation on a p-dimensional, $(p \geqq 4)$, spherical uniform distribution, there exists an a, $0 \leqq a \leqq (2(p-2)/(p+2))R^2$, for which $\delta_a{}^+(X)$ improves over X for all $\|\theta\|$ with a large improvement at the origin. Moreover, when $p \geqq 6$, there exists an a for which $\delta_a{}^+(X)$ is minimax and $R(\delta_a{}^+(X), 0) = 0$ with respect to quadratic loss (1.1). Since these new improved estimators do not present any difficulties in calculation, they may easily be used in place of X for estimating the mean of a spherical uniform and more generally, the mean of a s.s.u. distribution.

We complete this section with a note on the robustness of these estimators. Our improved estimators are robust in the sense that for any p-dimensional, $(p \geqq 3)$, distribution about θ satisfying $E_0(\|X\|^{-2}) \leqq c$, when c is a given constant, $\delta_a(X)$ is minimax with respect to general quadratic loss (1.2) for any a such that $0 \leqq a \leqq 2a_0{}^*((\text{tr } D/d_L) - 2)/c$ where, as in (4.1.5), $a_0{}^* = (p/((p+2)(p-2)))$ for $p \geqq 4$ and $a_0{}^* = .48$ for $p = 3$ and $\text{tr } D > 2d_L$. Hence, we need not always know exactly what $E_0(\|X\|^{-2})$ is in order to use these improved estimators.

It is hoped that these results add some insight into the behavior of the James–Stein type estimators and with this, perhaps those using these estimators will do so more confidently.

6. Appendix.

6.1. *Integral expressions, expectations and density derivations.* In this section we present useful integral evaluations and various densities as well as many expected values which aid in the calculation of important expressions.

$$\int_0^R (R^2 - y^2)^q \, dy = (2q/2q + 1)R^2 \int_0^R (R^2 - y^2)^{q-1} \, dy \tag{6.1.1}$$

$$\int_0^R (R^2 - y^2)^{\frac{1}{2}} \, dy = (\pi R^2/4) \tag{6.1.2}$$

$$\begin{aligned} &4\|\theta\|^2 \int_0^R ((R^2 - y^2)^q / d_{R,\|\theta\|}(y)) \, dy \\ &\quad = \int_0^R (R^2 - y^2)^{q-1} \, dy - (R^2 - \|\theta\|^2)^2 \int_0^R ((R^2 - y^2)^{q-1}/d_{R,\|\theta\|}(y)) \, dy \end{aligned} \tag{6.1.3}$$

where

$$d_{R,\|\theta\|}(y) = (R^2 - \|\theta\|)^2 + 4\|\theta\|^2(R^2 - y^2) \,. \tag{6.1.4}$$

LEMMA 6.1.1. *If X is a $p \times 1$ random vector with a spherical uniform distibution ($X \sim U\{\|X - \theta\|^2 \leqq R^2\}$) then for any integrable function $g(\|X - \theta\|)$, $E_\theta g(\|X - \theta\|) = M_0 \int_0^R r^{p-1} g(r) \, dr$, where $M_0 = p/R^p$. In particular, $E_0(\|X\|^{-2}) = (p/(p-2))R^{-2}$ and $E_0(\|X\|^{-1}) = (p/(p-1))R^{-1}$.*

PROOF. Results follow by a conversion to spherical coordinates. □

LEMMA 6.1.2. *If $X = [X_1, X_2, \cdots, X_p]' \sim U\{\|X - \theta\|^2 \leqq R^2\}$ then $E_\theta g(X'\theta, \|X\|^2) = E_{\|\theta\|} g(X_1\|\theta\|, \|X\|^2)$ where $E_{\|\theta\|}$ denotes the expected value when $\theta = [\|\theta\|, 0, \cdots, 0]'$. Moreover, if $\theta = [\|\theta\|, 0, \cdots, 0]'$,*

$$\begin{aligned} &E_{\|\theta\|} g(X_1\|\theta\|, \|X\|^2, \|Y\|^2) \\ &\quad = E_0 g(\|\theta\|(X_1 + \|\theta\|), (X_1 + \|\theta\|)^2 + \|Y\|^2, \|Y\|^2) \\ &\quad = M \int_0^R \int_{-(R^2-r^2)^{\frac{1}{2}}}^{(R^2-r^2)^{\frac{1}{2}}} r^{p-2} g(\|\theta\|(x_1 + \|\theta\|), (x_1 + \|\theta\|)^2 + r^2, r^2) \, dx_1 \, dr \end{aligned}$$

where $\|Y\|^2 = \sum_{i=2}^p X_i^2$ and

$$M = [(2/p)R^2 \int_0^R (R^2 - y^2)^{(p-3)/2} \, dy]^{-1} \,. \tag{6.1.5}$$

PROOF. The first part is true, by a simple transformation of variables by applying the $p \times p$ orthogonal matrix P which is such that $P\theta = [\|\theta\|, 0, \cdots, 0]$.

The second part is easily obtained by translating to the origin and then transforming $(x_2, x_3, \cdots, x_p)$ into spherical coordinates. □

LEMMA 6.1.3. *If $X = [X_1, X_2, \cdots, X_p]'$ has a spherical uniform distribution then*

$$\begin{aligned} E_\theta(2X'(X - \theta)\|X\|^{-2}) &= 2 - 2E_0[\|\theta\|(X_1 + \|\theta\|)((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1}] \\ &= (4M/(p-1)) \int_0^R ((R^2 - y^2)^{(p-1)/2} / d_{R,\|\theta\|}(y)) \\ &\quad \times [R^4 - 3\|\theta\|^2 R^2 + 4\|\theta\|^2(R^2 - y^2)] \, dy \end{aligned} \tag{6.1.6}$$

and

$$\begin{aligned} E_\theta(\|X\|^{-2}) &= E_0((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1} \\ &= (2M/(p-2)) \int_0^R ((R^2 - y^2)^{(p-3)/2} / d_{R,\|\theta\|}(y)) \\ &\quad \times [R^4 - \|\theta\|^2 R^2 + 2\|\theta\|^2(R^2 - y^2)] \, dy \end{aligned} \tag{6.1.7}$$

where $\|Y\|^2 = \sum_{i=2}^p X_i^2$, $d_{R,\|\theta\|}(y)$ is defined by (6.1.4) *and M is defined by* (6.1.5).

PROOF. We obtain the desired results by applying Lemma 6.1.2, integrating with respect to x_1 and then integrating by parts and transforming variables to $y = (R^2 - r^2)^{\frac{1}{2}}$. □

LEMMA 6.1.4. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{\|X - \theta\|^2 \leqq R^2\}$ *and* $\theta = [\|\theta\|, 0, \cdots, 0]'$, *then*

$$E_{\|\theta\|}(\|Y\|^2\|X\|^{-2}) = E_0(\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-1})$$

$$(6.1.8) \qquad = (2M/p) \int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y)) \times [R^4 - \|\theta\|^2R^2 + 2\|\theta\|^2(R^2 - y^2)]\, dy$$

and

$$E_{\|\theta\|}(\|Y\|^2\|X\|^{-4}) = E_0(\|Y\|^2((X_1 + \|\theta\|)^2 + \|Y\|^2)^{-2})$$

$$(6.1.9) \qquad = (M/(p-2)) \int_0^R ((R^2 - y^2)^{(p-3)/2}/d_{R,\|\theta\|}(y)) \times [(p-1)R^4 - (p-1)\|\theta\|^2R^2 + (p\|\theta\|^2 - (p-2)R^2)(R^2 - y^2)]\, dy$$

where $\|Y\|^2 = \sum_{i=2}^p X_i^2$ *and* $d_{R,\|\theta\|}(y)$ *and* M *are defined by* (6.1.4) *and* (6.1.5) *respectively.*

LEMMA 6.1.5. *When* $p \geqq 3$,

$$\int_0^R ((R^2 - y^2)^{(p-3)/2}/(d_{R,\|\theta\|}(y)))\, dy$$
$$= \left(\int_0^R (R^2 - y^2)^{(p-3)/2}\, dy\right)[h(\|\theta\|, R)]_p \qquad \textit{when} \quad \|\theta\| \geqq R$$
$$= \left(\int_0^R (R^2 - y^2)^{(p-3)/2}\, dy\right)[h(R, \|\theta\|)]_p \qquad \textit{when} \quad \|\theta\| \leqq R$$

where

$$(6.1.10) \qquad [h(\|\theta\|, R)]_p = (\|\theta\|^2(R^2 + \|\theta\|^2))^{-1} \sum_{i=0}^\infty (-1)^i a_i (R^2/\|\theta\|^2)^i$$

and

$$(6.1.11) \qquad [h(R, \|\theta\|)]_p = (R^2(R^2 + \|\theta\|^2))^{-1} \sum_{i=0}^\infty (-1)^i a_i (\|\theta\|^2/R^2)^i$$

and $a_0 = 1$, $a_i = [(p - 2(i+1))/(p + 2(i-1))]a_{i-1}$ *for* $i = 1, 2, \cdots$ *and* $d_{R,\|\theta\|}(y)$ *is defined by* (6.1.4).

PROOF. The proof is a proof by induction by first assuming the lemma is true for p and utilizing (6.1.3) to prove it is true for $p + 2$. In order to prove the lemma for $p \geqq 3$, it is straightforward to show it is true for $p = 3$ and $p = 4$.

When $p = 3$, by the definition of $d_{R,\|\theta\|}(y)$ given in (6.1.4)

$$\int_0^R (R^2 - y^2)^{(p-3)/2}/(d_{R,\|\theta\|}(y))\, dy$$
$$= \int_0^R [(R^2 + \|\theta\|^2)^2 - 4\|\theta\|^2y^2]^{-1}\, dy$$
$$= (4\|\theta\|(R^2 + \|\theta\|^2))^{-1} \log [(R + \|\theta\|)^2/(R - \|\theta\|)^2]$$
$$= (\|\theta\|(R^2 + \|\theta\|^2))^{-1} \sum_{n=1}^\infty (R/\|\theta\|)^{2n-1}/(2n-1) \qquad \text{when} \quad \|\theta\| \geqq R$$
$$= (\|\theta\|(R^2 + \|\theta\|^2))^{-1} \sum_{n=1}^\infty (\|\theta\|/R)^{2n-1}/(2n-1) \qquad \text{when} \quad \|\theta\| \leqq R\,.$$

So, when $\|\theta\| \geqq R$

$$\int_0^R [(R^2 + \|\theta\|^2)^2 - 4\|\theta\|^2y^2]^{-1}\, dy = \left[\int_0^R dy/(\|\theta\|^2(R^2 + \|\theta\|^2))\right] \sum_{n=0}^\infty (R^2/\|\theta\|^2)^n/(2n+1)$$
$$= \int_0^R dy\, [[h(\|\theta\|, R)]_3] \quad \text{as defined in (6.10)}\,.$$

Similarly, when $\|\theta\| \leqq R$

$$\int_0^R [(R^2 + \|\theta\|^2)^2 - 4\|\theta\|^2 y^2]^{-1} dy = \int_0^R dy\, [[h(R, \|\theta\|)]_3] .$$

So, the theorem is true for $p = 3$.

Again, by integrating properly, for $p = 4$,

$$\begin{aligned} \int_0^R (R^2 - y^2)^{(p-3)/2}/(d_{R,\|\theta\|}(y))\, dy &= \int_0^R (R^2 - y^2)^{\frac{1}{2}}((R^2 + \|\theta\|^2)^2 - 4\|\theta\|^2 y^2)^{-1} dy \\ &= \pi R^2(4\|\theta\|^2(R^2 + \|\theta\|^2))^{-1} \quad \text{when} \quad \|\theta\| \geqq R \\ &= \pi(R^2 + \|\theta\|^2)^{-1} \quad \text{when} \quad \|\theta\| \leqq R . \end{aligned}$$

Since, by (6.1.2), $\int_0^R (R^2 - y^2)^{\frac{1}{2}} dy = (\pi R^2/4)$

$$\begin{aligned} &\int_0^R (R^2 - y^2)^{\frac{1}{2}}/(d_{R,\|\theta\|}(y))\, dy \\ &\quad = \int_0^R (R^2 - y^2)^{\frac{1}{2}}\, dy/(\|\theta\|^2(R^2 + \|\theta\|^2)) \quad \text{when} \quad \|\theta\| \geqq R \\ &\quad = \int_0^R (R^2 - y^2)^{\frac{1}{2}}/(R^2(R^2 + \|\theta\|^2)) \quad \text{when} \quad \|\theta\| \leqq R . \end{aligned}$$

Thus, this implies $a_0 = 1$ and $a_i = 0$ for $i = 1, 2, \cdots$, which implies the theorem is true for $p = 4$.

Assume the lemma is true for p; to show it is true for $p + 2$, we must show that

$$\begin{aligned} &\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy \\ &\quad = \int_0^R (R^2 - y^2)^{(p-1)/2} dy\, [h(\|\theta\|, R)]_{p+2} \quad \text{when} \quad \|\theta\| \geqq R \\ &\quad = \int_0^R (R^2 - y^2)^{(p-1)/2} dy\, [h(R, \|\theta\|)]_{p+2} \quad \text{when} \quad \|\theta\| \leqq R \end{aligned}$$

where

$$[h(\|\theta\|, R)]_{p+2} = (\|\theta\|^2(R^2 + \|\theta\|^2))^{-1} \textstyle\sum_{i=0}^{\infty} (-1)^i b_i (R^2/\|\theta\|^2)^i ,$$

$$[h(R, \|\theta\|)]_{p+2} = (R^2(R^2 + \|\theta\|^2))^{-1} \textstyle\sum_{i=0}^{\infty} (-1)^i b_i (\|\theta\|^2/R^2)^i ,$$

and

$$b_0 = 1 , \qquad b_i = [(p - 2i)/(p + 2i)] b_{i-1} \qquad i = 1, 2, \cdots .$$

Directly,

$$\begin{aligned} &4\|\theta\|^2 \int_0^R (R^2 - y^2)^{(p-1)/2}/((R^2 - \|\theta\|^2) + 4\|\theta\|^2(R^2 - y^2))\, dy \\ &\quad = \int_0^R (R^2 - y^2)^{(p-3)/2} dy - (R^2 - \|\theta\|^2)^2 \int_0^R ((R^2 - y^2)^{(p-3)/2}/d_{R,\|\theta\|}(y))\, dy . \end{aligned}$$

Since we assume the lemma is true for p,

$$\begin{aligned} &4\|\theta\|^2 \int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy \\ &\quad = \int_0^R (R^2 - y^2)^{(p-3)/2} dy \left(1 - \frac{(R^2 - \|\theta\|^2)^2}{\|\theta\|^2(R^2 + \|\theta\|^2)} \textstyle\sum_{i=0}^{\infty} (-1)^i a_i (R^2/\|\theta\|^2)^i \right) \\ &\qquad \text{if} \quad \|\theta\| \leqq R \quad \text{and} \\ &\quad = \int_0^R (R^2 - y^2)^{(p-3)/2} dy \left(1 - \frac{(R^2 - \|\theta\|^2)^2}{R^2(R^2 + \|\theta\|^2)} \textstyle\sum_{i=0}^{\infty} (-1)^i a_i (\|\theta\|^2/R^2)^i \right) \\ &\qquad \text{if} \quad \|\theta\| \leqq R \end{aligned} \tag{6.1.12}$$

where $a_0 = 1$ and

$$a_i = [(p - 2(i + 1))/(p + 2(i - 1))] a_{i-1} \qquad \text{for} \quad i = 1, 2, \cdots .$$

CASE 1. $\|\theta\| \geqq R$.

$$\left[1 - \frac{(R^2 - \|\theta\|^2)^2}{\|\theta\|^2(R^2 + \|\theta\|^2)} \sum_{i=0}^{\infty} (-1)^i a_i (R^2/\|\theta\|^2)^i\right]$$
$$= (R^2/(R^2 + \|\theta\|^2))[1 + (\|\theta\|^2/R^2)$$
$$- ((R^2/\|\theta\|^2) - 2 + (\|\theta\|^2/R^2)) \sum_{i=0}^{\infty} (-1)^i a_i (R^2/\|\theta\|^2)^i]$$
$$= (R^2/(R^2 + \|\theta\|^2))[3 + a_1 + \sum_{i=0}^{\infty} (-1)^{i+1} a_i (R^2/\|\theta\|^2)^{i+1}$$
$$+ 2 \sum_{i=1}^{\infty} (-1)^i a_i (R^2/\|\theta\|^2)^i + \sum_{i=2}^{\infty} (-1)^{i+1} a_i (R^2/\|\theta\|^2)^{i-1}]$$
$$= (R^2/(R^2 + \|\theta\|^2))[(3 + a_1) + \sum_{i=1}^{\infty} (-1)^i [a_{i-1} + 2a_i + a_{i+1}](R^2/\|\theta\|^2)^i] .$$

This, together with (6.1.12) implies

$$\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy$$
$$(6.1.13) \qquad = R^2 \int_0^R (R^2 - y^2)^{(p-3)/2}\, dy\, (4\|\theta\|^2(R^2 + \|\theta\|^2))^{-1}$$
$$\times [(3 + a_1) + \sum_{i=1}^{\infty} (-1)^i [a_{i-1} + 2a_i + a_{i+1}](R^2/\|\theta\|^2)^i] .$$

Now, directly by (6.1.1),

$$\int_0^R (R^2 - y^2)^{(p-3)/2}\, dy = (p/(p - 1))R^{-2} \int_0^R (R^2 - y^2)^{(p-1)/2}\, dy ,$$

so, (6.1.13) becomes

$$\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,\|\theta\|}(y))\, dy$$
$$= \int_0^R (R^2 - y^2)^{(p-1)/2}\, dy\, (\|\theta\|^2(R^2 + \|\theta\|^2))^{-1} \sum_{i=0}^{\infty} (-1)^i b_i (R^2/\|\theta\|^2)^i$$

where $b_0 = (p/(4(p - 1))(3 + a_1)$ and

$$b_i = (p/4(p - 1))[a_{i-1} + 2a_i + a_{i+1}] \qquad i = 1, 2, \cdots .$$

Since $a_i = [(p - 2(i + 1))/(p + 2(i - 1))]a_{i-1}$ for $i = 1, 2, \cdots$, and $a_0 = 1$,

$$(4(p - 1)/p)b_i = \frac{4(p^2 - 3p + 2)a_{i-1}}{(p + 2i)(p + 2(i - 1))}$$
$$= \frac{4(p^2 - 3p + 2)(p - 2i)a_{i-2}}{(p + 2(i - 2))(p + 2(i - 1))(p + 2i)} \qquad \text{for} \quad i = 2, 3, \cdots ,$$

and

$$(4(p - 1)/p)b_{i-1} = \frac{4(p^2 - 3p + 2)a_{i-2}}{(p + 2(i - 1))(p + 2(i - 2))} \qquad \text{for} \quad i = 2, 3, \cdots ,$$

so clearly,

$$b_i = ((p - 2i)/(p + 2i))b_{i-1} \qquad \text{for} \quad i = 2, 3, \cdots .$$

Moreover, $b_0 = (p/4(p - 1))(3 + (p - 4)/p) = 1$ and

$$b_1 = (p/4(p - 1))[(p^2 - 3p + 2)/((p - 4)(p + 2))]((p - 4)/p) = (p - 2)/(p + 2) .$$

Thus,

$$b_i = [(p - 2i)/(p + 2i)]b_{i-1} \qquad \text{for} \quad i = 1, 2, \cdots$$

and

$$b_0 = 1 .$$

For $||\theta|| \geqq R$,

$$\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,||\theta||}(y))\, dy = \int_0^R (R^2 - y^2)^{(p-1)/2}\, dy\, [h(||\theta||, R)]_{p+2}\,,$$

which is what we want to show.

CASE 2. $||\theta|| \leqq R$.

Proceeding as we did in Case 1, when $||\theta|| \leqq R$,

$$\left[1 - \frac{(R^2 - ||\theta||^2)^2}{R^2(R^2 + ||\theta||^2)} \sum_{i=0}^{\infty} (-1)^i a_i(||\theta||^2/R^2)^i\right]$$
$$= (||\theta||^2/(R^2 + ||\theta||^2))[(3 + a_1) + \sum_{i=0}^{\infty} (-1)^i [a_{i-1} + 2a_i + a_{i+1}](||\theta||^2/R^2)^i]\,.$$

So, (6.1.12) becomes, for $||\theta|| \leqq R$,

$$\int_0^R (R^2 - y^2)^{(p-1)/2}\, dy\, (R^2(R^2 + ||\theta||^2))^{-1} \sum_{i=0}^{\infty} (-1)^i b_i(R^2/||\theta||^2)^i$$

where $b_0 = 1$ and

$$b_i = [(p - 2i)/(p + 2i)]b_{i-1} \qquad \text{for} \quad i = 1, 2, \cdots.$$

Thus, for $||\theta|| \leqq R$,

$$\int_0^R ((R^2 - y^2)^{(p-1)/2}/d_{R,||\theta||}(y))\, dy = \int_0^R (R^2 - y^2)^{(p-1)/2}\, dy\, [h(R, ||\theta||)]_{p+2}\,.$$

Hence, we have the desired result for both cases.

Since the lemma is true for $p = 3$ and $p = 4$, the induction implies it is true for $p \geqq 3$. □

LEMMA 6.1.6. *Suppose* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X - \theta||^2 \leqq R^2\}$ *and* $\theta = [||\theta||, 0, 0, \cdots, 0]'$. *If* $Z = ||X||^2$,

$$S_1 = ((x_1, z): ((z - R^2 + ||\theta||^2)/2||\theta||) \leqq x_1 \leqq z^{\frac{1}{2}}, (R - ||\theta||)^2 \leqq z \leqq (R + ||\theta||)^2)$$

and

$$S_2 = ((x_1, z): -z^{\frac{1}{2}} \leqq x_1 \leqq z^{\frac{1}{2}}, 0 \leqq z \leqq (R - ||\theta||)^2)\,;$$

then the joint density of X_1 *and* Z *is given by*

$$\text{(6.1.14)} \qquad f_{||\theta||}(x_1, z) = (M/2)(z - x_1^2)^{(p-3)/2} I_{S_1}(x_1, z) \qquad \textit{when} \quad ||\theta|| \geqq R$$

and

$$\text{(6.1.15)} \qquad f_{||\theta||}(x_1, z) = (M/2)(z - x_1^2)^{(p-3)/2} I_{S_1 \cup S_2}(x_1, z) \qquad \textit{when} \quad ||\theta|| \leqq R$$

where M *is given by* (6.1.5) *and* $I_S(x_1, z)$ *and* $I_{S_1 \cup S_2}(x_1, z)$ *are the indicator functions over the sets* S_1 *and* $(S_1 \cup S_2)$, *respectively.*

PROOF. We obtain these densities by taking $(d/dx_1)(d/dz)P_{||\theta||}(X_1 \leqq x_1, ||X||^2 \leqq z)$. □

LEMMA 6.1.7. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X - \theta||^2 \leqq R^2\}$ *where* $\theta =$

$[\theta_1, \theta_2, \cdots, \theta_p]'$ *and* $||Y||^2 = \sum_{i=2}^p X_i^2$, *then for any integrable function* $r(\cdot)$

$$\begin{aligned}(p-1)E_0[r(||X+\theta||^2)[(2X_i^2+2\theta_i X_i)||X+\theta||^{-2} - a(X_i+\theta_i)^2||X+\theta||^{-4}]] \\ = (\theta_i^2/||\theta||^2)[E_0[r((X_1+||\theta||)^2+||Y||^2) \\ \times [(2(p-1)(X_1^2+||\theta||X_1) - 2||Y||^2)((X_1+||\theta||)^2+||Y||^2)^{-1}]] \\ - aE_0[r((X_1+||\theta||)^2+||Y||^2) \\ \times [((p-1)(X_1+||\theta||)^2 - ||Y||^2)((X_1+||\theta||)^2+||Y||^2)^{-2}]]] \\ + 2E_0[r((X_1+||\theta||)^2+||Y||^2)||Y||^2((X_1+||\theta||)^2+||Y||^2)^{-1}] \\ - aE_0[r((X_1+||\theta||)^2+||Y||^2)||Y||^2((X_1+||\theta||)^2+||Y||^2)^{-2}] .\end{aligned}$$

PROOF. We obtain the desired result by making two transformations of variables. If P is a $(p-1)\times(p-1)$ orthogonal transformation such that $P[\theta_1, \theta_2, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_p]' = [||\theta||_i, 0, \cdots, 0]'$ where $||\theta||_i = (||\theta||^2 - \theta_i^2)^{\frac{1}{2}}$, we first transform to $s = [s_2, s_3, \cdots, s_p]' = P[x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_p]'$.

There exists an orthogonal transformation Q such that $Q[\theta_i, ||\theta||_i, 0, \cdots, 0]' = [||\theta||, 0, \cdots, 0]'$, and if $z = [z_1, z_2, \cdots, z_p]' = Q[x_i, s_2, \cdots, s_p]'$ then $x_i = (\theta_i/||\theta||)z_1 - (||\theta||_i/||\theta||)z_2$, $s_2 = (||\theta||_i/||\theta||)z_1 + (\theta_i/||\theta||)z_2$ and $s_i = z_i$ for $i \geqq 3$. With this transformation and using the fact that for any constant c,

$$\int_{(||z||^2 \leqq R^2)} r((z_1+||\theta||)^2+||Y||^2)(z_2(||\theta||+cz_1))((z_1+||\theta||)^2+||Y||^2)^{-1}\,dz = 0 ,$$

we obtain the desired result. □

LEMMA 6.1.8. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X-\theta||^2 \leqq R^2\}$ *and* $||Y||^2 = \sum_{i=2}^p X_i^2$ *then* $E_0[((p-1)(X_1+||\theta||)^2 - ||Y||^2)((X_1+||\theta||)^2+||Y||^2)^{-2}] \geqq 0$ *when* $p \geqq 3$.

PROOF. Using (6.1.9) and Lemma 6.1.5 it can be shown that

$$\begin{aligned} E_0[||Y||^2((X_1+||\theta||)^2+||Y||^2)^{-2}] &\propto g_1(||\theta||, R) \quad \text{when } ||\theta|| \geqq R \\ &\propto g_2(R, ||\theta||) \quad \text{when } ||\theta|| \leqq R \end{aligned}$$

where

$$g_1(||\theta||, R) = (pR^2 - p||\theta||^2)([h(||\theta||, R)]_p) + (p||\theta||^2 - (p-2)R^2)([h(||\theta||, R)]_{p+2})$$

and

$$g_2(R, ||\theta||) = (pR^2 - p||\theta||^2)([h(R, ||\theta||)]_p) + (p||\theta||^2 - (p-2)R^2)([h(R, ||\theta||)]_{p+2}) .$$

Moreover, by (6.1.7), (6.1.9), and Lemma 6.1.5

$$\begin{aligned} E_0[((p-1)(X_1+||\theta||)^2 - ||Y||^2)((X_1+||\theta||)^2+||Y||^2)^{-2} \\ \propto g_1(R, ||\theta||) \quad \text{when } R \geqq ||\theta|| \\ \propto g_2(||\theta||, R) \quad \text{when } R \leqq ||\theta|| .\end{aligned}$$

Hence, it is nonnegative for all $||\theta||$. □

LEMMA 6.1.9. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X-\theta||^2 \leqq R^2\}$, $\theta = [||\theta||, 0, \cdots, 0]'$, *and* $r(||X||^2)$ *is any integrable function, then*

$$\begin{aligned} E_{||\theta||}[r(||X||^2)(X_1^2||X||^{-2})] = ((p-1)/p)E_{||\theta||}r(||X||^2)((||X||^2 - R^2 + ||\theta||^2)/2||\theta||)(X_1||X||^{-2}) \\ + (1/p)E_{||\theta||}r(||X||^2) .\end{aligned}$$

PROOF. Use of the expressions for the joint density of X_1 and $||X||^2$ given in Lemma 6.1.6 and a straightforward integration by parts completes the proof. □

6.2. *Auxiliary lemmas.* In this section we present lemmas which contain important properties which aid in the proof of theorems in Sections 2—4.

LEMMA 6.2.1. *If Y is a random variable with a density with respect to Lebesgue measure given by*

$$g_{2q+1}(y) = ((R^2 - y^2)^q/d_{R,||\theta||}(y))/\int_0^R ((R^2 - y^2)^q/d_{R,||\theta||}(y))\,dy \quad \text{when} \quad 0 \leqq y \leqq R$$
$$= 0 \quad \text{elsewhere}$$

where, as in (6.1.5), $d_{R,||\theta||}(y) = (R^2 - ||\theta||^2)^2 + 4||\theta||^2(R^2 - y^2)$, *then the distribution of Y has monotone likelihood ratio* (MLR) *nondecreasing in Y when $||\theta|| \leqq R$ and* MLR *nonincreasing in Y when $||\theta|| \geqq R$.*

PROOF. It is straightforward to show that for $0 \leqq ||\theta||_1 \leqq ||\theta||_2 \leqq R$,

$$(d/dy)(g_{2q+1,||\theta||_2}(y)/g_{2q+1,||\theta||_1}(y)) \geqq 0$$

and for $R \leqq ||\theta||_1 \leqq ||\theta||_2$,

$$(d/dy)(g_{2q+1,||\theta||_2}(y)/g_{2q+1,||\theta||_1}(y)) \leqq 0\,,$$

which completes the proof. □

LEMMA 6.2.2. *If $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X - \theta||^2 \leqq R^2\}$ and $Z = ||X||^2$, then for any c, and for fixed $||\theta||$ satisfying $||\theta||^2 \geqq (1 - 2c)R^2$, $E_{||\theta||}((||\theta||X_1 + cR^2)||X||^{-2} \mid ||X||^2)$ is nonincreasing in $||X||^2$.*

PROOF. For $(||\theta|| - R)^2 \leqq z \leqq (R + ||\theta||)^2$, Lemma 6.1.6 clearly implies that if $S = (||\theta||X_1 + cR^2)/Z$ then

$$f_{||\theta||}(s \mid z) = h_z(s) = \frac{(z - (sz - cR^2)^2||\theta||^{-2})^{(p-3)/2}I_A(s)}{\int_{d(z)}^{c(z)} (z - (sz - cR^2)^2||\theta||^{-2})^{(p-3)/2}\,ds}$$

where $c(z) = (||\theta||z^{\frac{1}{2}} + cR^2)/z$, $d(z) = (z - R^2 + ||\theta||^2 + 2cR^2)/2z$, $A = (s\colon d(z) \leqq s \leqq c(z))$, and

$$I_A(s) = 1 \quad \text{if} \quad s \in A$$
$$= 0 \quad \text{if} \quad s \notin A\,.$$

With respect to $h_z(s)$,

$$E_{||\theta||}[(||\theta||X_1 + cR^2)(z^{-1}) \mid z] = E_z(s) \qquad \text{for fixed} \quad ||\theta||\,.$$

If $z_1 \leqq z_2$, $c(z_1) \geqq c(z_2)$ and, provided $||\theta||^2 \geqq (1 - 2c)R^2$, $d(z_1) \geqq d(z_2)$. Therefore,

$$h_{z_2}(s)/h_{z_1}(s) \to \infty\,, \qquad \text{when} \quad d(z_2) \leqq s \leqq d(z_1)\,,$$
$$h_{z_2}(s)/h_{z_1}(s) \propto (z_2 - (sz_2 - cR^2)^2||\theta||^{-2})^{(p-3)/2}/(z_1 - (sz_1 - cR^2)^2||\theta||^{-2})^{(p-3)/2}$$
$$\text{when} \quad d(z_1) \leqq s \leqq c(z_2)\,,$$

and

$$h_{z_2}(s)/h_{z_1}(s) = 0 \qquad \text{when} \quad c(z_2) \leqq s \leqq c(z_1)\,.$$

If $h_{z_2}(s)/h_{z_1}(s)$ is nonincreasing in s, then $h_z(s)$ has MLR nonincreasing; hence $E_z(S)$ is nonincreasing in z, and the proof is complete for $(R-||\theta||)^2 \leqq z \leqq (R+||\theta||)^2$.

For $d(z_1) \leqq s \leqq c(z_2)$,

$$\begin{aligned} d/_{ds}(h_{z_2}(s)/h_{z_1}(s)) &\propto [z_1(sz_1 - cR^2)[z_2 - (sz_2 - cR^2)^2||\theta||^{-2}]] \\ &\quad - [z_2(sz_2 - cR^2)[z_1 - (sz_1 - cR^2)^2||\theta||^{-2}]] \\ &= g^*(s) . \end{aligned}$$

Clearly, $g^*(s) \leqq 0$ if $(sz_2 - cR^2) \geqq 0$ and $(sz_1 - cR^2) \leqq 0$. If $(sz_1 - cR^2)$ and $(sz_2 - cR^2)$ have the same sign,

$$g^*(s) = (z_1 - z_2)[sz_1z_2 + (sz_1 - cR^2)(sz_2 - cR^2)cR^2||\theta||^{-2}]$$

is clearly nonpositive. So, we have the MLR property satisfied.

When $0 \leqq z \leqq (R - ||\theta||)^2$,

$$E_{||\theta||}[(||\theta||X_1 + cR^2)Z^{-1} \mid z] = \frac{\int_{-z^{1/2}}^{z^{1/2}} (||\theta||x_1 + R^2)z^{-1}(z - x_1^2)^{(p-3)/2}\, dx_1}{\int_{-z^{1/2}}^{z^{1/2}} (z - x_1^2)^{(p-3)/2}\, dx_1} = \frac{cR^2}{z}$$

which is clearly nonincreasing in z. □

LEMMA 6.2.3. *If X has a p-dimensional spherical uniform distribution about θ, then $P_\theta\{||X||^2 \geqq c\} = P_{||\theta||}\{||X||^2 \geqq c\}$ is a nondecreasing function of $||\theta||$.*

PROOF. Suppose $\theta_1 = [||\theta_1||, 0, \cdots, 0]'$ and $\theta_2 = [||\theta_2||, 0, \cdots, 0]'$ and $||\theta_1|| \leqq ||\theta||_2$.

CASE 1. $||\theta||_1^2 \geqq R^2$ or $(||\theta||_2 \leqq R^2$ and $c \geqq R^2)$.

First note that, by Lemma 6.1.6, when $||\theta||^2 \geqq R^2$, $x_1 \geqq 0$ (x_1 is the first coordinate of x) and when $||\theta||^2 \leqq R^2$ but $||x||^2 \geqq c \geqq R^2$, $x_1 \geqq ||\theta||/2 \geqq 0$. This means that $||x||^2 \geqq c$ implies

$$\begin{aligned} ||x + (\theta_2 - \theta_1)||^2 &= ||x||^2 + 2(||\theta_2|| - ||\theta_1||)x_1 + (||\theta_2|| - ||\theta_1||)^2 \\ &\geqq ||x||^2 \geqq c . \end{aligned}$$

So,

$$\begin{aligned} P_{||\theta_2||}(||X||^2 \geqq c) &= P_{||\theta_1||}(||X + (\theta_2 - \theta_1)||^2 \geqq c) \\ &\geqq P_{||\theta_1||}(||X||^2 \geqq c) . \end{aligned}$$

CASE 2. $||\theta_2||^2 \leqq R^2$ and $c \leqq R^2$.

For this case we will show $P_{||\theta_2||}(||X||^2 \leqq c) \leqq P_{||\theta_1||}(||X||^2 \leqq c)$ by showing $S_3 = (x : ||x||^2 \leqq c, ||x - \theta_2||^2 \leqq R^2)$ is a subset of $S_4 = (x : ||x||^2 \leqq c, ||x - \theta_1|| \leqq R^2)$.

Suppose $x \in S_3$ and $x_1 \leqq ||\theta_1||$; then

$$\begin{aligned} ||x - \theta_1||^2 - ||x - \theta_2||^2 &\leqq -(||\theta_1|| - ||\theta||_2)^2 \leqq 0 , \\ \text{implying} \quad ||x - \theta_1||^2 &\leqq ||x - \theta_2||^2 \leqq R^2 . \end{aligned}$$

If $x_1 \geqq ||\theta_1||$ then

$$||x - \theta_1||^2 = ||x||^2 - 2||\theta_1||x_1 + ||\theta_1||^2 \leqq ||x||^2 - ||\theta_1||^2 \leqq ||x||^2 \leqq c \leqq R^2 .$$

Therefore, $S_3 \in S_4$.

CASE 3. $||\theta_1||^2 \leqq R^2 \leqq ||\theta_2||^2$.

Cases 1 and 2 imply

$$P_{||\theta_2||}(||X||^2 \geqq c) \geqq P_R(||X||^2 \geqq c) \geqq P_{||\theta_1||}(||X||^2 \geqq c)\,.$$

The proof is now complete. □

LEMMA 6.2.4. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X - \theta||^2 \leqq R^2\}$, $\theta = [||\theta||, 0, 0, \cdots, 0]'$ *and* $p \geqq 3$, *then for fixed* $||\theta||$ *satisfying* $||\theta|| \geqq R$, $E_{||\theta||}(X_1 \,|\, ||X||^2)$ *is nonincreasing in* $||X||^2$. *For* $p = 3$, $E_{||\theta||}(X_1 \,|\, ||X||^2)$ *is nondecreasing for all* $||\theta||$.

PROOF. From Lemma 6.2.2, it is clear that the density of $X_1 \,|\, ||X||^2$, $f_{||\theta||}(X_1 \,|\, ||X||^2)$ has MLR nondecreasing in X_1 for fixed $||\theta|| \geqq R$. Hence, $E_Z(X_1)$ is nondecreasing in Z. For $p = 3$, directly finding $E_{||\theta||}(X_1 \,|\, Z)$, it is clearly nondecreasing in $||X||^2$. □

LEMMA 6.2.5. *If* $X = [X_1, X_2, \cdots, X_p]' \sim U\{||X - \theta||^2 \leqq R^2\}$, $\theta = [||\theta||, 0, 0, \cdots, 0]'$, *then for* $p \geqq 3$, c *a positive constant, and* $||\theta||^2 \geqq ((2/cp) - 1)R^2$,

$$E_{||\theta||}[r(||X||^2)[((||X||^2 - R^2 - ||\theta||^2)/2||\theta||)(X_1||X||^{-2})]] \geqq -cE_{||\theta||}[r(||X||^2)(||\theta||X_1||X||^{-2})]$$

where $r(||X||^2)$ *is a nonnegative nondecreasing function.*

PROOF. If $g(z) = (z - R^2 - (1 - 2c)||\theta||^2)/2||\theta||$ where $Z = ||X||^2$, we will prove that when $||\theta||^2 \geqq ((2/cp) - 1)R^2$, $E_{||\theta||}[r(Z)g(Z)(X_1 Z^{-1})] \geqq 0$. Using the joint density for X_1 and Z given by Lemma 6.1.6, we have

$$E_{||\theta||}[r(Z)g(Z)(X_1 Z^{-1})] \geqq r((R^2 + (1 - 2c)||\theta||^2)E_{||\theta||}[g(Z)X_1 Z^{-1}]\,.$$

Expression (6.1.6) implies

$$\begin{aligned} E_{||\theta||}[g(Z)(X_1 Z^{-1})] &\propto -(R^2 - 2c||\theta||^2)(R^2 - ||\theta||^2)^2 \\ &\quad + (R^2 + (1 - 2c)||\theta||^2)(R^4 - 3||\theta||^2 R^2) + 4||\theta||^4 E_{||\theta||}(R^2 - Y^2) \\ &= E^*_{||\theta||}[g(Z)(X_1 Z^{-1})] \end{aligned}$$

where $E_{||\theta||}(R^2 - Y^2)$ is the expected value with respect to the density $g_{p,||\theta||}(y)$ given by (6.2.1). Using the MLR properties of $g_{p,||\theta||}(y)$ given in Theorem 6.2.1, we have $E_{||\theta||}(R^2 - Y^2) \geqq E_R(R^2 - Y^2) = ((p - 1)/p)R^2$ which along with the assumption $||\theta||^2 \geqq ((2/cp) - 1)R^2$ implies

$$\begin{aligned} E^*_{||\theta||}[g(Z)X_1 Z^{-1}] &\geqq (2||\theta||^4/p)[(cp - 2)R^2 + cp||\theta||^2] \\ &\geqq 0\,. \end{aligned}$$

□

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