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HYPOTHESIS TESTING FOR THE COMMON MEAN AND FOR BALANCED INCOMPLETE BLOCKS DESIGNS

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Let $X_1, X_2, \cdots, X_m$ be a random sample of size m from a normal population with mean θ and variance σ_x^2. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from a normal population with mean θ and variance σ_y^2. The X-sample and Y-sample are independent. Note that this model is appropriate in balanced incomplete blocks designs. We consider various hypotheses testing problems concerned with θ and obtain the following results:

(1) For testing $H_0: \theta = 0$ vs $H_1: \theta \neq 0$ (and the usual variants of these hypotheses), the usual t-test based on only one sample is proven to be admissible. This is somewhat surprising in light of results obtained in point and confidence estimation.

(2) For testing $H_0: \theta = 0$ vs $H_1 \neq 0$, suppose it is assumed that $\sigma_x \geqq B$, where B is any positive constant. Then a similar test is found, which is better than the t-test based only on the X-sample, if $m \geqq 3$ and $n \geqq 4$.

(3) Let $\delta = \theta/\sigma_x$ and test $H_0: \delta = 0$ vs $H_1: \delta_0 \leqq \delta \leqq \delta_1$. Here $\delta_0 > 0$ can be determined by the sample sizes and size of the t-test and δ_1 is arbitrarily large. For this separated hypothesis a test, based on improved estimators of θ, is found which is better than the usual t-test for $m \geqq 2$, $n \geqq 6$.

(4) Let σ_x^2 be known and test $H_0: \theta = 0$ vs $H_1 \theta \neq 0$. It is shown in this case that the test which rejects if $|\bar{X}| > C$, is admissible.

1. Introduction and summary. Point and confidence estimation of a common mean and the related problem of recovery of interblock information has been studied recently by Brown and Cohen [2]. They indicate conditions under which improved estimates and improved confidence intervals can be found and also offer the improved procedures. Bhattacharya [1] has made some improvements on the Brown–Cohen results.

In this paper we prove that the usual t-test based on data from only one population (or based on the intra-block estimate) for testing the hypothesis that a treatment contrast is zero, is admissible. The proof of admissibility is accomplished by applying a theorem due to Stein [6].

When the variance of the first population is close to zero, the t-test cannot be beaten uniformly. However, competitors to the t-test can be found which are better than the t-test except at extreme points of the parameter space. One competitor is a similar test which is better than the t-test as long as the variance of the first population is bounded away from zero and sample sizes are at least

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3 and 4 respectively for the first and second populations. Another competitor exists which is appropriate for testing a separated hypothesis. That is, suppose the standardized mean for the first population is δ, and the null hypothesis is $\delta = 0$. Suppose further that the alternative hypothesis is $0 < \delta_0 \leqq \delta \leqq \delta_1$. The number δ_0 can depend on the sample sizes and size of the t-test and can be determined before carrying out the test. The number δ_1 can be made arbitrarily large. Under these conditions a test can be found which is uniformly better than the t-test as long as sample sizes of at least 2 and 6 are taken. This latter test is based on the improved estimators of the mean developed in Brown and Cohen.

This study represents the first attempt at tackling the recovery of information problem from a testing point of veiw. The results are also significant in several other senses. The first admissibility result is somewhat surprising in light of previous inadmissibility results for the counterparts in estimation and confidence intervals. Intuitively one would expect data in the second sample to help in inferences pertaining to the mean. Another interesting feature regarding the first admissibility result is that the Stein method is exploited to its fullest advantage. That is, the t-test based on the first sample is admissible only because it cannot be dominated when the variance of the first population tends to zero. The second main result of this paper proves this. In previous applications of the Stein method, the tests shown to be admissible have also been shown to be admissible by other methods and not because they could not be dominated at extreme points of the parameter space.

The second main result yields, for each $B > 0$, a similar test which is better than the t-test everywhere provided σ_x, the variance of the first population, is greater than B. These similar tests are not invariant (under scale transformations), but it can be shown, using arguments of Section 2, that no invariant test can be better than the one sample t-test in such a situation. Thus the particular choice of test must depend on some knowledge of B. In practice this means that we must know what unit of measurement is being used or else we could not guarantee that $\sigma_x \geqq B$, for any $B > 0$, and so there would be no hope of improving uniformly on the one sample t-test. However, if we do know our unit of measurement (and now also B), we can do better than using the one sample t-test. At present the practical value of this result is limited as the improved procedures derived in Section 3 are probably not the best improvements possible and the amount of improvement may be slight.

The result concerning the separated hypothesis is important for the following reasons. First, it represents a demonstration of a nontrivial inadmissible test for a realistic and important problem. The only other result of this type, reflecting inadmissibility of a test, appears in Portnoy and Stein [5], where the problem is artificial. Secondly, whereas the result is in terms of the existence of a better test, the form of the test is given. This means that the result has potential for practical application.

Still another result of interest concerns the model where the variance of the first population is known. Consider the test which rejects if the sample mean, based on the first population, is large. It is quite surprising that in this instance such a test is also proven to be admissible. An adaptation of Stein's theorem can be used to obtain this result.

The model and admissibility result are given in the next section. A competitive similar test is given in Section 3, while the better test for the separated hypothesis is given in Section 4. The model where the variance of the first population is known is discussed in Section 5. All results will be given for the common mean problem. For the connection with balanced incomplete blocks designs the reader is referred to the beginning of Section 3 of Brown and Cohen [2].

2. Admissibility of t-test. Let $X_1, X_2, \cdots, X_m$ be a random sample from a normal population with unknown mean θ and unknown variance σ_x^2. Let $\bar{X} = \sum_{i=1}^m X_i/m$, $s_x^2 = \sum_{i=1}^m (X_i - \bar{X})^2/(m-1)$, $T_x = \sum_{i=1}^m X_i^2$, $s_{\bar{x}}^2 = s_x^2/m$. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a normal population with unknown mean θ and unknown variance σ_y^2. The Y sample is independent of the X sample. Define $\bar{Y}$, s_y^2, T_y, $s_{\bar{y}}^2$ in analogy with their counterparts in the X sample. We assume $m \geqq 2$ and $n \geqq 2$. The problem is to test $H_0: \theta = 0$ vs $H_1: \theta \neq 0$. If $t_x = m^{\frac{1}{2}}\bar{X}/s_x$, the usual t-test of size α based on the X sample is to reject H_0 if $|t_x| > t_{m-1}(\alpha)$, where $t_{m-1}(\alpha)$ is the two-tailed α-percent critical value determined from Student's t-distribution with $(m-1)$ degrees of freedom. We assume throughout that $\alpha \leqq \frac{1}{2}$.

Clearly the joint probability density of the sufficient statistic $Z' = (\bar{X}, T_x, \bar{Y}, T_y)$ is multivariate exponential family with respect to μ, a measure, absolutely continuous with respect to Lebesgue measure, and with natural parameters $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) = (m\theta/\sigma_x^2, -\frac{1}{2}\sigma_x^2, n\theta/\sigma_y^2, -\frac{1}{2}\sigma_y^2)$. (See for example, Lehmann [3], page 168.) Now let $\mathscr{X}$ be the sample space, $\mathscr{X}'$ be the adjoint space, $\boldsymbol{\Theta}$ be a subset of $\mathscr{X}'$ such that for each $\xi \in \Theta$, $\int e^{\xi z}\, d\mu(z) < \infty$ and $[\xi_1/m\xi_2] = [\xi_3/n\xi_4]$. Also let $\Theta_0 = \{\xi \in \Theta : \xi = (0, -\frac{1}{2}\sigma_x^2, 0, -\frac{1}{2}\sigma_y^2)\}$. We now paraphrase Stein's theorem as

LEMMA 2.1. *Let Z be distributed as multivariate exponential family. Let A be a closed convex subset of $\mathscr{X}$ such that for every $\xi \in \mathscr{X}'$ and real C for which*

$$\{z : \xi z > C\} \cap A = \emptyset \quad (\textit{the empty set}), \tag{2.1}$$

there exists $\omega \in \tilde{\Theta} = \{\xi : \int e^{\xi z}\, d\mu(z) < \infty\}$, such that there exists arbitrarily large λ for which $\omega + \lambda\xi \in \Theta - \Theta_0$. Then the test φ_0, defined by

$$\begin{aligned} \varphi_0(z) &= 0 \quad \textit{if} \quad z \in A \\ &= 1 \quad \textit{if} \quad z \notin A \end{aligned} \tag{2.2}$$

is admissible for testing $\xi \in \Theta_0$ vs $\xi \in \Theta - \Theta_0$.

PROOF. See Stein [6], page 617.

We now prove

THEOREM 2.1. *For testing H_0 vs H_1, the t-test is admissible.*

PROOF. It can be verified that the acceptance region for the t-test is equivalent to the intersection of all half spaces of the form

$$\left\{(\bar{X}, T_x, \bar{Y}, T_y) : \eta\bar{X} - \frac{\eta^2}{2}T_x < \frac{C}{2}\right\}, \tag{2.3}$$

and $T_x > 0$, where η ranges over the reals, $\eta \neq 0$. The constant C is the critical value of the equivalent t-test expressed as; accept if $(\bar{X}^2/T_x) \leqq C$. (See Stein [5], page 620.) Thus those ξ such that the half spaces $\{z: \xi z > C\} \cap A = \emptyset$, are of the form $(\xi_1, \xi_2, 0, 0)$ where ξ_2 is negative. To prove that the t-test is admissible we must find an $\omega \in \Theta$ such that $\omega + \lambda\xi$ lies in the alternative space for arbitrarily large λ. We may write $(\omega + \lambda\xi) = (\omega_1 + \lambda\xi_1, \omega_2 + \lambda\xi_2, \omega_3, \omega_4)$. Choose ω such that

$$[(\omega_1 + \lambda\xi_1)/(\omega_2 + \lambda\xi_2)] = [m\omega_3/n\omega_4]. \tag{2.4}$$

That is, let $\omega_1 = \xi_1$, $\omega_2 = \xi_2$, $\omega_3 = n\xi_1$, $\omega_4 = m\xi_2$. Since $\xi_1 \neq 0$, $\xi_2 < 0$, such a choice of ω lies in $\tilde{\Theta}$. Also (2.4) implies that $\omega + \lambda\xi \in \Theta - \Theta_0$ for all λ. This completes the proof of the theorem.

REMARK 2.1. The above proof can establish a similar result for the one-sided case and for the case of common mean vector for multivariate normal distributions. That is, Hotelling's T^2-test based on a sample from one population would be admissible even if another independent sample were available from a population with the same mean vector but with different covariance matrix.

3. Improved similar test when σ_x^2 is bounded below. The model in this section is an in Section 2. We consider $H_0: \theta = 0$ vs $H_1: \theta \neq 0$. We prove that for $m \geqq 3$ and $n \geqq 4$, there exists a test which is better than the usual t-test based on t_x provided $\sigma_x \geqq B$, for B, any given positive constant. In Remark 3.2 at the end of the section, we briefly indicate the rationale for the choice of the better test to be given. We need some preliminaries before stating what the improved test is. The method of proof makes use of a Taylor series expansion.

The joint density of $\bar{X}$ and T_x is

$$\begin{aligned} f_{\theta,\sigma_x}(\bar{x}, T_x) &= K(\theta, \sigma_x)(T_x - m\bar{x}^2)^{(m-3)/2} \exp(-T_x/2\sigma_x^2 + m\theta\bar{x}/\sigma_x^2) \\ &\qquad \text{if} \quad T_x \geqq m\bar{x}^2 \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{3.1}$$

where $K(\theta, \sigma_x) = K\exp(-m\theta^2/2\sigma_x^2)/\sigma_x^m$. The marginal density of T_x is

$$\begin{aligned} f_{\theta,\sigma_x}(T_x) = K(\theta, \sigma_x) \exp(-T_x/2\sigma_x^2) T_x^{(m-3)/2}(T_x/m)^{\frac{1}{2}} \textstyle\int_{-1}^{1} (1 - v^2)^{(m-3)/2} \\ \times \exp(m\theta v(T_x)^{\frac{1}{2}}/m^{\frac{1}{2}}\sigma_x^2)\, dv . \end{aligned} \tag{3.2}$$

Let $\delta_x = m\theta/\sigma_x^2$ and let $h_{\delta_x}(\bar{x} \mid T_x)$ denote the conditional density function of $\bar{X}$

given T_x. (This density depends on (θ, σ_x^2) only through δ_x.) Also let $H_{\delta_x}(\cdot \mid T_x)$ denote the conditional cdf of $\bar{X}$ given T_x so that

$$H_{\delta_x}(u \mid T_x) = \frac{\int_{-1}^{m^{\frac{1}{2}}u/(T_x)^{\frac{1}{2}}} (1 - v^2)^{(m-3)/2} \exp(\delta_x v(T_x)^{\frac{1}{2}}/m^{\frac{1}{2}})\, dv}{\int_{-1}^{1} (1 - v^2)^{(m-3)/2} \exp(\delta_x v(T_x)^{\frac{1}{2}}/m^{\frac{1}{2}})\, dv} . \tag{3.3}$$

Let $\varphi_{0,\alpha}(\bar{x}, T_x, \bar{y}, T_y)$ denote the t-test based on t_x of size α. That is, $\varphi_{0,}(\bar{x}, T_x, \bar{y}, T_y) = 1$ if $m^{\frac{1}{2}}|\bar{x}|/(T_x)^{\frac{1}{2}} > k_{0,\alpha}$, and is zero otherwise, where $k_{0,\alpha}$ is the two-sided α-percent critical value. (Note $k_{0,\alpha}$ is related to the two-sided α-percent critical value from Student's t, t_α, by $k_{0,\alpha}^2 = t_\alpha^2/(m - 1 + t_\alpha^2)$ and so $k_{0,\alpha}$ satisfies $0 < k_{0,\alpha} < 1$.) We may also write $\varphi_{0,\alpha}(\bar{x}, T_x, \bar{y}, T_y)$ as follows:

$$\begin{aligned} \varphi_{0,\alpha}(\bar{x}, T_x, \bar{y}, T_y) &= 0 \qquad \text{if } \; C_0^L(T_x) < \bar{x} < C_0^U(T_x) \\ &= 1 \qquad \text{otherwise,} \end{aligned} \tag{3.4}$$

where

$$C_0^L(T_x) = H_0^{-1}((\alpha/2) \mid T_x) = -k_{0,\alpha}(T_x/m)^{\frac{1}{2}}$$

and

$$C_0^U(T_x) = H_0^{-1}((1 - (\alpha/2)) \mid T_x) = k_{0,\alpha}(T_x/m)^{\frac{1}{2}} .$$

Now for any constant a such that $0 \leqq a \leqq 1$, let

$$\begin{aligned} \varphi_{a,\alpha}(\bar{x}, T_x, \bar{y}, T_y) &= 0 \qquad \text{if } \; C_a^L(T_x, \bar{y}, T_y) < \bar{x} < C_a^U(T_x, \bar{y}, T_y) \\ &= 1 \qquad \text{otherwise,} \end{aligned} \tag{3.5}$$

where

$$C_a^L(T_x, \bar{y}, T_y) = H_0^{-1}(((\alpha/2) - a(T_x)^{\frac{1}{2}}(\operatorname{sgn}\bar{y})/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)) \mid T_x)$$

and

$$C_a^U(T_x, \bar{y}, T_y) = H_0^{-1}(((1 - \alpha/2) - a(T_x)^{\frac{1}{2}}(\operatorname{sgn}\bar{y})/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)) \mid T_x) .$$

In (3.5), $\operatorname{sgn}\bar{y}$ is $-1, 0, 1$ when $\bar{y}$ is $< 0, = 0, > 0$ respectively, and C is a fixed constant such that $C > 1/\alpha$. Clearly $\varphi_{a,\alpha}$ are tests of size α. Now let us note some properties of C_a^L andC_a^U. The form of $H_0(u \mid T_x)$ given in (3.3) implies that the corresponding density $h_0(u \mid T_x)$ is symmetric, which in turn implies

$$C_a^L(T_x, \bar{y}, T_y) = -C_a^U(T_x, -\bar{y}, T_y) . \tag{3.6}$$

There exist functions $k_{a,\alpha}^-(T_x, T_y)$ and $k_{a,\alpha}^+(T_x, T_y)$ and constants k^- and k^+ such that

$$0 < k^+ \leqq k_{a,\alpha}^+(T_x, T_y) \leqq k_{0,\alpha} \leqq k_{a,\alpha}^-(T_x, T_y) \leqq k^- < 1 , \tag{3.7}$$

and

$$\begin{aligned} C_a^U(T_x, \bar{y}, T_y) &= k_{a,\alpha}^-(T_x, \bar{y}, T_y)(T_x/m)^{\frac{1}{2}} \qquad \text{if } \; \bar{y} < 0 \\ &= k_{a,\alpha}^+(T_x, \bar{y}, T_y)(T_x/m)^{\frac{1}{2}} \qquad \text{if } \; \bar{y} > 0 . \end{aligned}$$

This is easily seen, since if $\bar{y} > 0$, the conditional acceptance interval shifts to the left. Also $k_{a,\alpha}^- \leqq k^- < 1$. This follows since $C > 1/\alpha$, which in turn implies $[a(T_x)^{\frac{1}{2}}(\operatorname{sgn}\bar{y})/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)] < \alpha^* < \alpha/2$. Finally (3.7) implies that there exist constants $0 < m_1 < m_2 < 1$ such that

$$m_1 T_x \leqq T_x - m(C_a^U)^2 \leqq m_2 T_x \,, \tag{3.8}$$

and

$$m_1 T_x \leqq T_x - m(C_a^L)^2 \leqq m_2 T_x \,.$$

Denote the power function of $\varphi_{a,\alpha}$ by

$$\begin{aligned}\beta(\theta, \sigma_x, \sigma_y; a) &= E_{(\theta,\sigma_x,\sigma_y)}\{\varphi_{a,\alpha}(\bar{x}, T_x, \bar{y}, T_y)\} \\ &= 1 - E_{(\theta,\sigma_x,\sigma_y)}\{H_{\delta_x}(C_a{}^U(T_x, \bar{y}, T_y) \mid T_x) \\ &\quad - H_{\delta_x}(C_a{}^L(t_x, \bar{y}, T_y) \mid T_x)\} \,.\end{aligned} \tag{3.9}$$

We are now ready to state

THEOREM 3.1. *Let $m \geqq 3$, $n \geqq 4$ and let $\sigma_x \geqq B$. Then there exists an a sufficiently small, such that the test $\varphi_{a,\alpha}$ is better than the test $\varphi_{0,\alpha}$ for testing H_0* vs *H_1.*

PROOF. Expand the power function in a Taylor series about $a = 0$ with a second order remainder term so that

$$\begin{aligned}\beta(\theta, \sigma_x, \sigma_y; a) &= \beta(\theta, \sigma_x, \sigma_y; 0) + a\beta'(\theta, \sigma_x, \sigma_y; 0) \\ &\quad + [a^2/2]\beta''(\theta, \sigma_x, \sigma_y; a^*) \,,\end{aligned} \tag{3.10}$$

where $a^* = a^*(\theta, \sigma_x, \sigma_y; a)$ is some value between 0 and a. Since $\beta(\theta, \sigma_x, \sigma_y; 0)$ is the power of the test $\varphi_{0,\alpha}$, we can prove the theorem by showing that $\beta'(\theta, \sigma_x, \sigma_y, 0) > 0$ for all $(\theta, \sigma_x, \sigma_y)$, $\theta \neq 0$, and $|\beta'(\theta, \sigma_x, \sigma_y; 0)/\beta''(\theta, \sigma_x, \sigma_y; a^*)| \geqq M$, for all $0 \leqq a^* \leqq 1$, all $(\theta, \sigma_x, \sigma_y)$, $\theta \neq 0$, where M is a positive constant. We proceed to show these properties of β' and β''.

From (3.9), (3.5), (3.6), the symmetry of $h_0(u \mid T_x)$, (3.2) and (3.3) we find

$$\begin{aligned}\beta'(\theta, \sigma_x, \sigma_y; 0) &= E_{(\theta,\sigma_x,\sigma_y)}\{[(h_{\delta_x}(C_0{}^U(T_x) \mid T_x) \\ &\quad - h_{\delta_x}(-C_0{}^U(T_x) \mid T_x))/h_0(C_0{}^U(T_x) \mid T_x)] \\ &\quad \times [(T_x)^{\frac{1}{2}} \operatorname{sgn} \bar{y}/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)]\} \\ &= (1/2m^{\frac{1}{2}})K(\theta, \sigma_x) \textstyle\int_{-1}^{1} (1 - v^2)^{(m-3)/2}\, dv \\ &\quad \times \{[1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)]E_{\sigma_y}[1/(C + s_y)] \\ &\quad \times \textstyle\int_0^\infty [T_x{}^{(m-1)/2}/(1 + (T_x)^{\frac{1}{2}})^3] \\ &\quad \times [\exp(m^{\frac{1}{2}}\theta k_0(T_x)^{\frac{1}{2}}/\sigma_x{}^2) - \exp(-m^{\frac{1}{2}}\theta k_0(T_x)^{\frac{1}{2}}/\sigma_x{}^2] \\ &\quad \times \exp(-T_x/2\sigma_x{}^2)\, dT_x\} \,.\end{aligned} \tag{3.11}$$

From (3.11) it is easily seen that $\beta'(\theta, \sigma_x, \sigma_y; 0) = \beta'(-\theta, \sigma_x, \sigma_y; 0)$ and that $\beta'(\theta, \sigma_x, \sigma_y; 0) > 0$.

Now from (3.9), (3.5), (3.2), and (3.3) we find

$$\begin{aligned}&\beta''(\theta, \sigma_x, \sigma_y; a^*) \\ &\quad = -\delta_x E_{(\theta,\sigma_x,\sigma_y)}\{[h_{\delta_x}(C_{a^*}^U(T_x, \bar{y}, T_y) \mid T_x)/h_0^2(C_{a^*}^U(T_x, \bar{y}, T_y) \mid T_x) \\ &\qquad - h_{\delta_x}(C_{a^*}^L(T_x, \bar{y}, T_y) \mid T_x)/h_0^2(C_{a^*}^L(T_x, \bar{y}, T_y) \mid T_x)] \\ &\qquad \times [(T_x)^{\frac{1}{2}} \operatorname{sgn} \bar{y}/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)]^2\}\end{aligned} \tag{3.12}$$

$$= -(1/4m)K(\theta, \sigma_x)(\int_{-1}^{1} (1-v^2)^{(m-3)/2}\,dv)^2\delta_x$$
$$\times \int_0^\infty [T_x^{m-1}/(1+(T_x)^{\frac{1}{2}})^6]E_{\theta,\sigma_y}\{[(T_x - m(C^U_{a^*}(T_x, \bar{y}, T_y))^2)^{-(m-3)/2}$$
$$\times \exp((m\theta/\sigma_x^2)C^U_{a^*}(T_x, \bar{y}, T_y))$$
$$- (T_x - m(C^L_{a^*}(T_x, \bar{y}, T_y))^2)^{-(m-3)/2} \exp((m\theta/\sigma_x^2)C^L_{a^*}(T_x, \bar{y}, T_y))]$$
$$\times (1/(C+s_y))^2\} \exp(-T_x/2\sigma_x^2)\,dT_x\,.$$

Since the distribution of $\bar{y}$ under $-\theta$ is the same as the distribution of $-y$ under θ it can be verified, using (3.6), that $\beta''(\theta, \sigma_x, \sigma_y; a^*) = \beta''(-\theta, \sigma_x, \sigma_y; a^*)$. This, plus the fact that $\beta'(\theta, \sigma_x, \sigma_y; 0) = \beta'(-\theta, \sigma_x, \sigma_y; 0)$ enables us to consider only the case $\theta > 0$ in the remainder of the proof.

The remainder of the proof is devoted to showing that $|\beta'/\beta''| \geqq M$. The needed computations are lengthy and detailed. To ease matters we will list several lemmas, deferring their proofs to the Appendix. Also we use subscripted capital letters K and M to denote positive finite constants (not depending on θ, σ_x, σ_y, a, or a^*).

LEMMA 3.1. *If* $n \geqq 4$, $E[(C+s_y)^{-1}]/\sigma_y{}^i E[(C+s_y)^{-2}] \geqq K_1$, $i = 0, 1$, *where* $(n-1)s_y^2/\sigma_y^2$ *has a chi-square distribution with* $(n-1)$ *degrees of freedom.*

LEMMA 3.2. *Let* Φ *denote the standard normal* cdf. *If* $\theta/\sigma_y \leqq 1$, *then* $[1 - 2\Phi(-m^{\frac{1}{2}}\theta/\sigma_y)]/\theta \geqq K_2/\sigma_y$.

LEMMA 3.3. *If* $\theta/\sigma_y \geqq 1$, *then* $[1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] \geqq K_3$.

LEMMA 3.4. *If* $\theta/\sigma_x \leqq 1$, *then*

$$[1 - \exp(-2m^{\frac{1}{2}}\theta k_0 Z/\sigma_x^2)]\sigma_x/\theta \geqq K_4(1 - e^{-K_5 Z/\sigma_x})\,, \tag{3.13}$$

and

$$[\exp(m^{\frac{1}{2}}\theta(k^{\pm}_{a^*} - k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k^{\pm}_{a^*} + k_0)Z/\sigma_x^2)]\sigma_x/\theta \leqq e^{K_6 Z/\sigma_x}\,, \tag{3.14}$$

where $k^{\pm}_{a^*}$ *means taking either* $k^+_{a^*}$ *or* $k^-_{a^*}$ *in both terms of* (3.14).

LEMMA 3.5. *For* $Z \geqq 0$,

$$k^-_{a^*}(Z^2, T_y) - k_0 \leqq K_7[Z/(1+Z)^3(C+s_y)] \tag{3.15}$$

and

$$[k^-_{a^*}(Z^2, T_y) - k_0]Z \leqq K_7\,. \tag{3.16}$$

LEMMA 3.6. *Let* Z *be a normal random variable with mean* $m^{\frac{1}{2}}\theta k_0$ *and variance* σ_x^2. $I(Z) = 1$ *if* $Z \geqq 0$, *and* $I(Z) = 0$ *otherwise, then*

$$E\{I(Z)[Z^n/(1+Z)^3](1 - e^{-K_5 Z})\} \geqq K_8 E\{I(Z)Z^{n+2}/(1+Z)^6\}\,, \tag{3.17}$$

and

$$E\{I(Z)[Z^n/(1+Z)^3](1 - e^{-K_5 Z})\} \geqq K_8 E\{I(Z)Z^{n+1}/(1+Z)^4\}\,. \tag{3.18}$$

LEMMA 3.7. *Let* V *be a normal random variable with mean* $m^{\frac{1}{2}}k_0$ *and variance*

$(\sigma_x/\theta)^2$. *If* $(\theta/\sigma_x^2) \geqq 1$, $\sigma_x \geqq B$, *then*

$$E\{I(V)[V^{n+1}/(\theta^{-1} + V)^4]\} \geqq K_9, \tag{3.19}$$

and

$$E\{I(V)[V^{n+2}/(\theta^{-1} + V)^6] \exp([K_{10}/\sigma_x^2(\theta^{-1} + V)]\} \leqq K_{11}. \tag{3.20}$$

We now return to the proof of the theorm. In (3.12), use (3.7) and (3.8) and take the expectation over the variable $\bar{y}$ and find that

$$\begin{aligned} &|\beta''(\theta, \sigma_x, \sigma_y; a^*)| \\ &\quad \leqq [\theta/\sigma_x^2]K(\theta, \sigma_x)[\textstyle\int_{-1}^{1} (1 - v^2)^{(m-3)/2}\, dv]^2 K_{12} \\ &\qquad \times [\{\textstyle\int_0^\infty [T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6] E_{\sigma_y}([1 - \Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] \\ &\qquad \times |\exp(m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^+/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^+/\sigma_x^2)| \\ &\qquad \times [1/(C + s_y)^2]) \exp(-T_x/2\sigma_x^2)\, dT_x\} \\ &\qquad + \{\textstyle\int_0^\infty [T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6] E_{\sigma_y}([1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] \\ &\qquad \times \exp(-m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^+/\sigma_x^2)[1/(C + s_y)^2]) \exp(-T_x/2\sigma_x^2)\, dT_x\} \\ &\qquad + \{\textstyle\int_0^\infty [T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6] E_{\sigma_y}(\Phi(-n^{\frac{1}{2}}\theta/\sigma_y) \\ &\qquad \times |\exp(m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^-/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^-/\sigma_x^2)| \\ &\qquad \times [1/(C + s_y)^2]) \exp(-T_x/2\sigma_x^2)\, dT_x\} \\ &\qquad + \{\textstyle\int_0^\infty T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6 E_{\sigma_y}([1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] \\ &\qquad \times \exp(-m^{\frac{1}{2}}\theta(T_x)^{\frac{1}{2}}k_{a^*}^-/\sigma_x^2)[1/(C + s_y)^2]) \exp(-T_x/2\sigma_x^2)\, dT_x\}]. \end{aligned} \tag{3.21}$$

Denote the curly bracketed expression on the r.h.s. of (3.11) by A and denote the four curly bracketed expressions on the r.h.s. of (3.21) by A_i, $i = 1, 2, 3, 4$, respectively. To complete the proof of the theorem we must show that $(A/\delta_x A_i)$ is bounded away from zero uniformly in $(\theta, \sigma_x, \sigma_y; a^*)$ for $i = 1, 2, 3, 4$.

For $i = 2, 4$, from (3.21) and (3.11) we have

$$\begin{aligned} A_i \leqq [1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] E_{\sigma_y}[1/(C + s_y)^2] \textstyle\int_0^\infty [T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6] \\ \times \exp(-T_x/2\sigma_x^2)\, dT_x, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} (A/\delta_x) &\geqq [1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] E_{\sigma_y}(1/(C + s_y)) m^{\frac{1}{2}} k_0 \textstyle\int_0^\infty [T_x^{m/2}/(1 + (T_x)^{\frac{1}{2}})^3] \\ &\qquad \times \exp(-T_x/2\sigma_x^2)\, dT_x \\ &\geqq [1 - 2\Phi(-m^{\frac{1}{2}}\theta/\sigma_y)] E_{\sigma_y}(1/(C + s_y)) m^{\frac{1}{2}} k_0 \textstyle\int_0^\infty [T_x^{(m+1)/2}/(1 + (T_x)^{\frac{1}{2}})^6] \\ &\qquad \times \exp(-T_x/2\sigma_x^2)\, dT_x. \end{aligned} \tag{3.23}$$

From (3.22), (3.23) and Lemma 3.1 we do have $(A/\delta_x A_i) \geqq K_{12}$ for $i = 2, 4$. For $i = 1, 3$, start by writing A, the bracketed term in (3.11), as

$$\begin{aligned} A &= [1 - 2\Phi(-\theta n^{\frac{1}{2}}/\sigma_y)] E_{\sigma_y}(1/(C + s_y))(2\pi)^{\frac{1}{2}}\sigma_x \exp(m\theta^2 k_0^2/2\sigma_x^2) \\ &\qquad \times \textstyle\int_0^\infty [T_x^{(m-1)/2}/(1 + (T_x)^{\frac{1}{2}})^3][1 - \exp(-2m^{\frac{1}{2}}\theta k_0 (T_x)^{\frac{1}{2}}/\sigma_x^2)] \\ &\qquad \times [\exp(-((T_x)^{\frac{1}{2}} - m^{\frac{1}{2}}\theta k_0)^2/2\sigma_x^2)/(2\pi)^{\frac{1}{2}}\sigma_x]\, dT_x \\ &= 2[1 - 2\Phi(-\theta n^{\frac{1}{2}}/\sigma_y)] E_{\sigma_y}(1/(C + s_y))(2\pi)^{\frac{1}{2}}\sigma_x \\ &\qquad \times \exp(m\theta^2 k_0^2/2\sigma_x^2) E\{[I(Z)Z^m/(1 + Z)^3] \\ &\qquad \times [1 - \exp(-2m^{\frac{1}{2}}\theta k_0 Z/\sigma_x^2)]\}, \end{aligned} \tag{3.24}$$

where Z is normal with mean $m^{\frac{1}{2}}\theta k_0$ and variance σ_x^2. Similarly,

$$A_1 \leqq (2\pi)^{\frac{1}{2}}\sigma_x \exp(m\theta^2 k_0^2/2\sigma_x^2)E\{[I(Z)Z^{m+2}/(1+Z)^6]E_{\sigma_y}[1/(C+s_y)^2] \times [\exp(m^{\frac{1}{2}}\theta(k_{\alpha*}^+ - k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k_{\alpha*}^+ + k_0)Z/\sigma_x^2)]\}, \tag{3.25}$$

and

$$A_3 \leqq (2\pi)^{\frac{1}{2}}\sigma_x \exp(m\theta^2 k_0^2/2\sigma_x^2)E\{[I(Z)Z^{m+2}/(1+Z)^6]E[1/(C+s_y)^2] \times [\exp(m^{\frac{1}{2}}\theta(k_{\alpha*}^- - k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k_{\alpha*}^- + k_0)Z/\sigma_x^2)]\}. \tag{3.26}$$

Now consider

CASE 1. $[\theta/\sigma_x^2] \leqq 1$, $[\theta/\sigma_x] \leqq 1$. By Lemma 3.4, (3.13) and (3.14) we get

$$(A/\delta_x A_i) \geqq 2K_4[1 - 2\Phi(-\theta n^{\frac{1}{2}}/\sigma_y)][E_{\sigma_y}(1/(C+s_y))/E_{\sigma_y}1/(C+s_y)^2] \times \frac{E\{[I(Z)Z^m/(1+Z)^3](1-e^{-K_5 Z/\sigma_x})\}}{\delta_x E\{[I(Z)Z^{m+2}/(1+Z)^6](e^{K_6 Z/\sigma_x})\}}. \tag{3.27}$$

Note that from Lemmas 3.1, 3.2, 3.3 and the fact that $\sigma_x \geqq B$, we have

$$\begin{aligned} &\{[1 - 2\Phi(-\theta n^{\frac{1}{2}}/\sigma_y)]E_{\sigma_y}(1/(C+s_y))/[\theta/\sigma_x^2]E_{\sigma_y}(1/(C+s_y)^2)\} \\ &\qquad \geqq B^2K_1K_2 \qquad \text{if} \quad [\theta/\sigma_y] \leqq 1 \\ &\qquad \geqq K_1K_3 \qquad \text{if} \quad [\theta/\sigma_y] > 1. \end{aligned} \tag{3.28}$$

Furthermore, since $Z^{m+2}/(1+Z)^5 \leqq Z^m/(1+Z)^3$ and $1/(1+Z)^5 \geqq 1/(1+Z)^6$,

$$\begin{aligned} &E\{[I(Z)Z^m/(1+Z)^3](1-e^{-K_5Z/\sigma_x})\}/E\{[I(Z)Z^{m+2}/(1+Z)^6]e^{K_6Z/\sigma_x}\} \\ &\qquad \geqq \frac{E\{[I(U)U^{m+2}/(\sigma_x^{-1}+U)^5](1-e^{-K_5U})\}}{E\{[I(U)U^{m+2}/(\sigma_x^{-1}+U)^5]e^{K_5U}\}}, \end{aligned} \tag{3.29}$$

where U is normal with mean $m^{\frac{1}{2}}k_0\theta/\sigma_x$ and variance 1. From (3.27), (3.28), (3.29), $m \geqq 3$, $\sigma_x \leqq B$ and $[\theta/\sigma_x] \leqq 1$ it follows that $(A/\delta_x A_i) \geqq K_{13}$, for $i = 1, 3$, in Case 1.

CASE 2. $[\theta/\sigma_x^2] \leqq 1$, $[\theta/\sigma_x] > 1$. Since $[\theta/\sigma_x] > 1$,

$$(1 - \exp(-2m^{\frac{1}{2}}\theta k_0 Z/\sigma_x^2)) \geqq (1 - \exp(-2m^{\frac{1}{2}}k_0 Z/\sigma_x)). \tag{3.30}$$

Also since $[\theta/\sigma_x^2] \leqq 1$, (3.16) of Lemma (3.5) can be used to yield

$$\exp(m^{\frac{1}{2}}\theta(k_{\alpha*}^- - k_0)Z/\sigma_x^2) - \exp(m^{\frac{1}{2}}\theta(k_{\alpha*}^- + k_0)Z/\sigma_x^2) \leqq \exp(m^{\frac{1}{2}}K_7). \tag{3.31}$$

Now one may proceed as in Case 1, noting that the only difference is that e^{K_5U} in (3.29) is replaced by $\exp(m^{\frac{1}{2}}K_7)$.

CASE 3. $[\theta/\sigma_x^2] \geqq 1$, $[\theta/\sigma_y] \leqq 1$. From (3.24), (3.25), (3.26), and (3.15) of Lemma 3.5 we get

$$\begin{aligned} (A/\delta_x A_i) \geqq{} & K_{14}[1 - 2\Phi(-\theta n^{\frac{1}{2}}/\sigma_y)]\sigma_x^2 E_{\sigma_y}\{[1/(C+s_y)]E[I(Z)Z^m/(1 \\ & + Z)^3](1 - e^{-K_{15}})\}/\theta E_{\sigma_y}[1/(C+s_y)^2]E\{[I(Z)Z^{m+2}/(1+Z)^6] \\ & \times \exp(\theta Z^2 K_7 K_{15}/(1+Z)^3\sigma_x^2)\}. \end{aligned} \tag{3.32}$$

Use (3.32), Lemmas 3.1, 3.2, and (3.18) of Lemma 3.6 to find

$$
\begin{aligned}
(A/\delta_x A_i) &\geqq K_{16}K_1K_2K_8 E[I(Z)Z^{m+1}/(1+Z)^4]/E\{[I(Z)Z^{m+2}/(1+Z)^6] \\
&\qquad \times \exp(\theta K_{17}Z^2/(1+Z)^3\sigma_x^2)\} \\
&= K_{18}\theta E\{I(V)V^{(m+1)}/(\theta^{-1}+V)^4\}/E\{[I(V)V^{(m+2)}/(\theta^{-1}+V)^6] \\
&\qquad \times \exp(K_{17}/(\theta^{-1}+V)\sigma_x^2)\},
\end{aligned} \tag{3.33}
$$

where $V = Z/\theta$ is normal with mean $m^{\frac{1}{2}}k_0$, and variance (σ_x^2/θ^2). Since $\sigma_x \geqq B$ and $[\theta/\sigma_x^2] \geqq 1$, then $\theta \geqq B^2$. Therefore from (3.33) and Lemma 3.7 we have $(A/\delta_x A_i) \geqq K_{19}$.

CASE 4. $[\theta/\sigma_x^2] \geqq 1$ and $[\theta/\sigma_y] \geqq 1$. As in Case 3 we have (3.32). Use Lemmas 3.1, 3.3, and (3.18) of Lemma 3.6, make the transformation to V, and use Lemma 3.7. This leads to $(A/\delta_x A_i) \geqq K_{20}$. This completes the proof of the theorem.

We conclude this section with some remarks.

REMARK 3.1. For the one-sided alternative hypothesis, results analogous to those obtained in Sections 2, 3 and 5 hold. The correct form of the better test corresponding to (3.5) when $\sigma_x \geqq B$, is C_α^U or C_α^L of (3.5). In the one-sided case, however, the conditional tests given $(T_x, \bar{y}, T_y)$ are no longer of size α as in the two-sided case. Nevertheless the overall size must be α.

REMARK 3.2. The motivation and intuition which led to the test given in (3.5) is best illustrated when examining the one-sided case. Using a result of Matthes and Truax [4] it follows that all (measurable) functions of the form

$$
\begin{aligned}
\phi(\bar{x}, T_x, \bar{y}, T_y) &= 0 \quad \text{if} \quad \bar{x} < C^U(T_x, \bar{y}, T_y) \\
&= 1 \quad \text{otherwise}
\end{aligned} \tag{3.34}
$$

form a complete class. Let $g(\bar{y}, T_y; \sigma_x) = E_{(0,\sigma_x)}\{\phi(\bar{x}, T_x, \bar{y}, T_y) \mid \bar{y}, T_y\}$ be the conditional size function. We now note that (for fixed $\bar{y}$, T_y) there is, essentially, a one-to-one correspondence between conditional size functions g and test functions, ϕ, of the above form. Consider the set of functions $\Omega = \{\omega(\cdot)$: there exists a test function $\phi^*(\bar{x}, T_x)$ for the one sample problem for which $E_{(0,\sigma_x)}\{\phi^*(\bar{x}, T_x)\} = \omega(\sigma_x)\}$. Then an interpretation of a test of the form (3.34) is to use $(\bar{y}, T_y)$ to choose a function, $g(\bar{y}, T_y: \cdot)$ from Ω and then use the one sample test based on $\bar{x}$, T_x which has $g(\bar{y}, T_y; \sigma_x)$ as its size function. A test of the form (3.34) will be α-similar if and only if $E_{(0,\sigma_y)}\{g(\bar{y}, T_y; \sigma_x)\} = \alpha$ for all σ_x, σ_y. It is interesting that the above characterization can be used to prove the admissibility results of Section 2 and Section 5.

We now exlain how the specific test given in (3.5) was found. This entails explaining how the expression

$$
[(T_x)^{\frac{1}{2}}(\operatorname{sgn} \bar{y})/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)] \tag{3.35}
$$

was arrived at. Clearly a logical first choice would be $\bar{y}/s_y$. But since it is necessary to bound (3.35) by $\alpha/2$, the next logical expression is $\bar{y}/2(C + s_y)$, where $C > 1/\alpha$. The quantity (sgn $\bar{y}$) enabled computations whereas $\bar{y}$ would not. The computation using (sgn $\bar{y}$)$/2(C + s_y)$ indicated a need for some function of T_x that was bounded by one and which went to zero as T_x went to 0 or ∞. Hence the expression $(T_x)^{\frac{1}{2}}/(1 + (T_x)^{\frac{1}{2}})^3$ was eventually arrived at as the proper multiplier. The power of 3 in the denominator of that expression was needed for the finiteness of certain expectations used in the lemmas.

REMARK 3.3. It should be noted that the test given in (3.5) can probably be improved upon as it depends on $\bar{y}$ in one place through (sgn $\bar{y}$). It is possible that this could be done through use of the characterization mentioned in Remark 3.2. However it is also likely that the computations would be even more involved than those of this section.

REMARK 3.4. We note that using the argument in Section 2, it can be shown that no invariant test (invariant under a scale transformation) can be better than the one-sample t-test even when $\sigma_x \geqq B$.

4. Improved test for separated hypothesis. The model is the same as in Section 2. We assume $m \geqq 2$, $n \geqq 6$. The problem is to test the hypothesis H_0: $\theta = 0$ against the alternative, H_1: $\delta_0 \leqq \delta = (\theta/\sigma_x) \leqq \delta_1$ where $\delta_0 > 0$, δ_0 will be specified later, and δ_1 is arbitrarily large. Let $t_{m-1}(\alpha)$ now denote the one-tailed α-percent critical value determined from Student's t-distribution with $(m - 1)$ degrees of freedom. We seek a test which is better than the usual t-test.

Let $z = s_{\bar{y}}^2/s_{\bar{x}}^2$, $v = s_y^2\sigma_x^2/s_x^2\sigma_y^2$, $\tau = \sigma_{\bar{y}}^2/\sigma_{\bar{x}}^2$, $u = (m - 1)^{\frac{1}{2}}s_x/\sigma_x$, $w = u^2$, $C = t_{m-1}(\alpha)/(m - 1)^{\frac{1}{2}}$, $\delta^* = \delta m^{\frac{1}{2}}$. Note that w is distributed as a chi-square variate with $(m - 1)$ degrees of freedom, v is distributed as an F-variate with $(n - 1)$ and $(m - 1)$ degrees of freedom and the joint density of w and v is

$$h(w, v) = Ke^{-wv/2}w^{[(m+n-2)/2]-1}v^{[(n-1)/2]-1}, \tag{4.1}$$

where

$$K = (1/2^{(m+n-2)/2})/\Gamma((m - 1)/2)\Gamma((n - 1)/2).$$

Now consider the test procedure which rejects if

$$\{[\bar{X} + a(\bar{Y} - \bar{X})/(1 + z)]/s_{\bar{x}}\} > t_{m-1}(\alpha), \tag{4.2}$$

where a is a constant, to be determined, such that $0 < a < 1 - \varepsilon$, for $\varepsilon > 0$. We prove the following.

THEOREM 4.1. *For δ_0 sufficiently large, there exists a positive constant a sufficiently small, such that the test given in* (4.2) *is better than the usual t-test for testing H_0* vs *H_1.*

PROOF. For the test in (4.2) it is easy to see that one minus the probability of rejection is

$$E_{\delta^*,\tau}\Phi((Cu - \delta^*)(1 + v\tau)/[(1 + v\tau - a)^2 + a^2\tau]^{\frac{1}{2}}). \tag{4.3}$$

Since the usual t-test is (4.2) with $a = 0$, we must show that there exists a δ_0^* and an a such that

(i) For $\delta^* = 0$, and the selected a value, (4.3) $\geqq$ (4.3) with $a = 0$, and,
(ii) For all δ^*, such that $\delta_0^* \leqq \delta^* \leqq \delta_1^*$, and the selected a value, (4.3) $\leqq$ (4.3) with $a = 0$.

That (i) is true follows from the proof of Theorem 5.1 in Brown and Cohen [2]. (Expression (5.9) of that paper is expression (4.3) of this paper, with $\delta^* = 0$.)

To prove (ii), expand (4.3) in a Taylor series about the point $a = 0$, to get

$$\begin{aligned}
&E_{\delta^*,\tau}\Phi(Cu - \delta^*) + (a/(2\pi)^{\frac{1}{2}})E_{\delta^*,\tau}[(Cu - \delta^*)/(1 + v\tau)]e^{-(Cu-\delta^*)^2/2} \\
&\quad + (a^2/2(2\pi)^{\frac{1}{2}})\{E_{\delta^*,\tau}\, e^{-(Cu-\delta^*)^2(1+v\tau)^2/2[(1+v\tau-a^*)^2+a^{*2}\tau]} \\
&\quad \times (1 + v\tau)/[(1 + v\tau - a^*)^2 + a^{*2}\tau]^{\frac{3}{2}} \\
&\quad \times [((\delta^* - Cu)^3(1 + v\tau)^2(a^*(\tau + 1) - (1 + v\tau))^2/[(1 + v\tau - a^*)^2 \\
&\quad + a^{*2}\tau]^2) + (\delta^* - Cu)(\tau + 1) - 3(\delta^* - CU)(a^*(\tau + 1) \\
&\quad - (1 + v\tau))^2/[(1 + v\tau - a^*)^2 + a^{*2}\tau]]\}\,,
\end{aligned} \tag{4.4}$$

for $0 < a^* < 1 - \varepsilon$. Note that the first of the three expectation terms in (4.4) is (4.3) with $a = 0$. We proceed to show that there exists a δ_0^*, sufficiently large, so that the second term is negative uniformly in τ, for all $\delta^* \geqq \delta_0^*$. Furthermore, the bracketed part of the third term will be shown to be uniformly bounded in a^*, τ, and δ^* for δ^* such that $\delta_0^* \leqq \delta^* \leqq \delta_1^*$. This will enable us to choose a sufficiently small, so that when $\delta^* = \delta_0^*$, (4.4) will be $\leqq$ (4.3) with $a = 0$. Also for δ^* any value such that $\delta_0^* \leqq \delta^* \leqq \delta_1^*$, a suitable a value can be obtained. By taking the infimum over all such a values, we have the desired result.

Now use (4.1) and change variables and find that the second term of (4.4) is a constant times

$$\begin{aligned}
&e^{-\delta^{*2}/2(1+C^2)} \int_0^\infty (Cu - \delta^*)u^m e^{-[(1+C^2)/2][u-C\delta^*/(1+C^2)]^2} \\
&\qquad \times \int_0^\infty e^{-t/2}(t^{((n-1)/2)-1}/[u^2 + t\tau])\, dt\, du\,.
\end{aligned} \tag{4.5}$$

Let $z = (1 + C^2)^{\frac{1}{2}}(u - C\delta^*/(1 + C^2))$, expand, and simplify to find that (-1) times (4.5) becomes

$$\begin{aligned}
&[1/(1 + C^2)^{\frac{1}{2}}]e^{-\delta^{*2}/2(1+C^2)} \\
&\quad \times \{\textstyle\int_{[-C\delta^*/(1+C^2)^{\frac{1}{2}}]}^\infty e^{-z^2/2} \int_0^\infty e^{-t/2}t^{((n-1)/2)-1}/[t\tau + (z/(1 + C^2)^{\frac{1}{2}} \\
&\quad + C\delta^*/(1 + C^2))^2][\delta^{*m+1}[C^m/(1 + C^2)^{m+1}] + \textstyle\sum_{i=1}^m \delta^{*i}C^{i-1}\binom{m}{i-1} \\
&\quad \times [1 - (m + 1)C^2/i(1 + C^2)]z^{m+1-i}/(1 + C^2)^{i-1+(m+1-i)/2} \\
&\quad - Cz^{m+1}/(1 + C^2)^{(m+1)/2}]\, dt\, dz\}\,.
\end{aligned} \tag{4.6}$$

Note that the term in brackets in (4.6) with δ^* raised to the highest power is a positive term. Now multiply all terms in the bracketed expression by $\delta^{*2}\tau$, and assume only for now that $\tau \geqq 1$. First note that for the term involving δ^{*m+1},

$$\begin{aligned} \tau \int_{-C\delta^*/(1+C^2)^{\frac{1}{2}}}^{\infty} e^{-z^2/2} \int_0^\infty e^{-t/2}[t^{((n-1)/2)-1}/((t\tau/\delta^{*2}) \\ + ((z/\delta^*(1+C^2)^{\frac{1}{2}}) + C/(1+C^2))^2)]\, dt\, dz \\ \geqq \int_0^\infty e^{-z^2/2} \int_0^\infty e^{-t/2}[t^{((n-1)/2)-1}/(t + ((z/(1+C^2)^{\frac{1}{2}}) \\ + C/(1+C^2))^2)]\, dt\, dz > 0\,. \end{aligned} \tag{4.7}$$

(For $0 < \tau < 1$, one can argue in a similar fashion.) Similar arguments can be used to bound from below, with positive constants, the positive parts of all other terms in (4.6). Also the negative terms can be uniformly bounded above by positive constants. This implies that there exists a δ_0^* such that for all $\delta^* \geqq \delta_0^*$, and all τ, the second term of (4.4) is negative.

We complete the proof of the theorem by giving a crude finite upper bound for the product of $(\delta^{*2}\tau)$ times the third expectation term in (4.4). Eliminating all negative terms, and arguing for $\tau \geqq 1$ we find a bound to be

$$\int_0^\infty \int_0^\infty h(w, v)\{(\delta^{*3} + 3\delta^* C^2 w)(1/\varepsilon^3 v^2) + (2\delta^*/\varepsilon v^2) + (3Cw^{\frac{1}{2}}/\varepsilon^3 v^2)\}\, dv\, dw\,. \tag{4.8}$$

The term (4.8) is finite for $n \geqq 6$. (See equation (5.15) of Brown and Cohen [2].) One can argue similarly for $\tau < 1$. This completes the proof of the theorem.

We conclude the section with the following remarks.

REMARK 4.1. In the case where σ_x is known, say $\sigma_x = 1$, one seeks a better test than the one which rejects if $m^{\frac{1}{2}}\bar{X} > Z_\alpha$, where Z_α is the critical value obtained from the standard normal. The hypothesis is $H_0: \theta = 0$ vs $H_1: 0 < \theta_0 \leqq \theta < \theta_1$, where θ_1 is arbitrarily large but fixed. The test procedure which rejects if $m^{\frac{1}{2}}[\bar{X} + a(\bar{Y} - \bar{X})/(1 + ms_{\bar{y}}^2) > Z_\alpha]$, can be shown to be better provided $\theta_0 \geqq Z_\alpha/m^{\frac{1}{2}}$, for the proper choice of a.

5. Admissibility when σ_x^2 known. In this section the model is the same as in Section 2, save that now σ_x^2 is assumed known and set equal to 1. The null hypothesis is $H_0: \theta = 0$ vs $H_1: \theta \neq 0$. We consider the test which rejects H_0 if

$$m^{\frac{1}{2}}|\bar{X}| > C_\alpha\,, \tag{5.1}$$

where C_α is the two-sided α-percent critical value obtained from a standard normal distribution. The sufficient statistics are $z = (\bar{X}, \bar{Y}, T_y)$. The joint density is

$$\begin{aligned} dP(z) = K \exp(-m(\bar{x} - \theta)^2/2) \exp(-n\theta^2/2\sigma_y^2)[1/\sigma_y^2]^{n/2}(T_y - n\bar{y}^2)^{(n-3)/2} \\ \times \exp(-T_y/2\sigma_y^2) \exp(n\theta\bar{y}/\sigma_y^2)\, d\nu(z) \\ \text{for } -\infty < \bar{x} < \infty\,, \quad 0 < T_y < \infty\,, \\ -(T_y/n)^{\frac{1}{2}} < \bar{y} < (T_y/n)^{\frac{1}{2}}\,, \end{aligned} \tag{5.2}$$

where ν is Lebesgue measure. Clearly, from (5.2) it follows that the density is multivariate exponential family with respect to μ, a measure, absolutely continuous with respect to ν and with natural parameters $\xi = (\xi_1, \xi_2, \xi_3) = (m\theta, n\theta/\sigma_y^2, -1/2\sigma_y^2)$. Thus (5.2) is a special case of the family

$$dP_\xi(z) = K \exp(-\xi_1^2/2m) \exp(\xi_2^2/4n\xi_3)(-\xi_3)^{n/2} e^{\xi z}\, d\mu(z)\,. \tag{5.3}$$

Let Θ be the set of parameter points for which $[\xi_1 \cdot \xi_3] = [-m\xi_2/2n]$. Note that (5.3) is appropriate even for those ξ not lying in Θ, as long as $\psi(\xi) = \int e^{\xi x}\, d\mu(x) < \infty$. We now prove

THEOREM 5.1. *The test given in* (5.1) *is admissible.*

PROOF. The first steps of the proof are essentially the same as those given to prove the theorem of Section 3 in Stein [6], page 618. The explanation for this is as follows: The set $A = \{z: |\bar{X}| > C_\alpha\}$. If φ is a strictly better test than that given in (5.1), then the set $B = \{z: \varphi(z) < 1\}$, is such that the set $A' \cap B$ has positive Lebesgue measure. Let us assume B contains a set of points of positive measure for which $\bar{X} > C_\alpha$. (Otherwise B would have to contain a set of positive measure for which $\bar{X} < -C_\alpha$ and we would argue from there.) Now consider the following half space, $S = \{z: \xi z > C_\alpha\}$, where $\xi = (1, 0, 0)$. We must have $\mu(A' \cap B \cap S) > \varepsilon$. Also consider a sequence of alternatives, $\underset{\sim}{\theta}_\lambda = \underset{\sim}{\theta}_{1\lambda} + \lambda\xi$, where $\underset{\sim}{\theta}_{1\lambda} = (0, [2n/\lambda m], -1/\lambda^2)$. Note $\underset{\sim}{\theta}_{1\lambda} \in \tilde{\Theta} - \Theta$. With the above choices we may compute, exactly as in Stein's equation (22), page 619, that the difference in powers of the test in (5.1) and $\varphi(z)$ at $\underset{\sim}{\theta}_\lambda$, is

$$\begin{aligned}[\psi(\underset{\sim}{\theta}_{1\lambda})/\psi(\underset{\sim}{\theta}_\lambda)]e^{\lambda C_\alpha}\{&\textstyle\int_{\{z:\xi z > C_\alpha\}} [1 - \chi_A(z) - \varphi(z)] e^{\lambda(\xi z - C_\alpha)}\, dP_{\underset{\sim}{\theta}_{1\lambda}}(z) \\ &+ \textstyle\int_{\{z:\xi z \leqq C_\alpha\}} [1 - \chi_A(z) - \varphi(z)] e^{\lambda(\xi z - C_\alpha)}\, dP_{\underset{\sim}{\theta}_{1\lambda}}(z)\}\,.\end{aligned} \tag{5.4}$$

Clearly, for every λ, the second term in the bracketed expression of (5.4) is bounded. Note that from (5.3) and the definition of $\underset{\sim}{\theta}_{1\lambda}$, we may write the first term in the bracketed expression of (5.4) as

$$K \textstyle\int_{\{z:\xi z > C_\alpha\}} [1 - \chi_A(z) - \varphi(z)] e^{\lambda(\xi z - C_\alpha)} (1/\lambda)^{n/2} e^{(nz_2/m) - z_3/\lambda}\, d\mu(z)\,. \tag{5.5}$$

Now observe that, despite the term $[1/\lambda]^{n/2}$, the term $e^{\lambda(\xi z - C_\alpha)}$ tends to infinity exponentially fast as $\lambda \to \infty$ for z's in the set $\{z: \xi z > C_\alpha\}$. This fact, the nature of μ, plus the fact that $\mu(A' \cap B \cap S) > \varepsilon$, implies that expression (5.5) approaches ∞ as $\lambda \to \infty$. The argument is now completed as in Stein's proof. This completes the proof of the theorem.

6. Appendix.

In this appendix we prove the seven lemmas stated in Section 3.

LEMMA 3.1. *If* $n \geqq 4$, $\{E[(C + s_y)^{-1}]/\sigma_y^i E[(C + s_y)^{-2}]\} \geqq K_1$, $i = 0, 1$, *where* $[(n-1)s_y^2/\sigma_y^2]$ *has a chi-square distribution with* $(n-1)$ *degrees of freedom.*

PROOF. Since $(C + s_y)^{-1} \geqq (C + s_y)^{-2}$, for $i = 0$, take $K_1 = 1$. Similarly, when $i = 1$, if $\sigma_y \leqq 1$, take $K_1 = 1$. If $\sigma_y > 1$ multiply numerator and denominator by σ_y and note

$$\begin{aligned}E\{[(C/\sigma_y) + (s_y/\sigma_y)]^{-1}\}&/E\{[(C/\sigma_y) + (s_y/\sigma_y)]^{-2}\} \\ &\geqq E\{[C + (s_y/\sigma_y)]^{-1}\}/E\{(s_y/\sigma_y)^{-2}\} \geqq K_1\,,\end{aligned}$$

if $n \geqq 4$.

LEMMA 3.2. *Let* Φ *denote the standard normal* cdf. *If* $\theta/\sigma_y \leqq 1$, *then* $[1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)]/\theta \geqq K_2/\sigma_y$.

PROOF. Note that

$$\begin{aligned}[1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)]/\theta &= 2[\Phi(0) - \Phi(-n^{\frac{1}{2}}\theta/\sigma_y)]/\theta \\ &\geqq 2n^{\frac{1}{2}}\theta\Phi'(-n^{\frac{1}{2}}\theta/\sigma_y)/\theta\sigma_y \\ &\geqq 2n^{\frac{1}{2}}\Phi'(-n^{\frac{1}{2}})/\sigma_y = K_2/\sigma_y ,\end{aligned}$$

where Φ' is the standard normal density.

LEMMA 3.3. *If* $\theta/\sigma_y \geqq 1$, $[1 - 2\Phi(-n^{\frac{1}{2}}\theta/\sigma_y)] \geqq K_3$.

PROOF. Take $K_3 = 1 - 2\Phi(-n^{\frac{1}{2}})$.

LEMMA 3.4. *If* $\theta/\sigma_x \leqq 1$, *then*

$$[1 - \exp(-2m^{\frac{1}{2}}\theta k_0 Z/\sigma_x^2)]\sigma_x/\theta \geqq K_4(1 - e^{-K_5 Z/\sigma_x}) \tag{3.13}$$

and

$$[\exp(m^{\frac{1}{2}}\theta(k_{a*}^{\pm} - k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k_{a*}^{\pm} + k_0)Z/\sigma_x^2)]\sigma_x/\theta \leqq e^{K_6 Z/\sigma_x} . \tag{3.14}$$

PPOOF. For (3.13) note that at $Z = 0$, both sides are equal. Take the derivative with respect to (Z/σ_x) on both sides and find, for $K_5 \geqq 2m^{\frac{1}{2}}$ and $K_4 K_5 < 2m^{\frac{1}{2}}k_0$, that the derivatives satisfy $2m^{\frac{1}{2}}k_0 \exp(-m^{\frac{1}{2}}\theta k_0 Z/\sigma_x^2) \geqq K_4 K_5 e^{-K_5 Z/\sigma_x}$. Thus the l.h.s. increases more rapidly than the r.h.s.

For (3.14) note that

$$\begin{aligned}&[\exp(m^{\frac{1}{2}}\theta(k_{a*}^{\pm} - k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k_{a*}^{\pm} + k_0)Z/\sigma_x^2)] \\ &\quad \leqq [\exp(m^{\frac{1}{2}}\theta(k_{a*}^{-} + k_0)Z/\sigma_x^2) - \exp(-m^{\frac{1}{2}}\theta(k_{a*}^{-} + k_0)Z/\sigma_x^2)] \\ &\quad \leqq [\exp(2m^{\frac{1}{2}}\theta Z/\sigma_x^2) - \exp(-2m^{\frac{1}{2}}\theta Z/\sigma_x^2)] .\end{aligned} \tag{6.1}$$

Use the method used to prove (3.13) above on the last term of (6.1) and the r.h.s. of (3.14) to complete the proof of the lemma.

LEMMA 3.5. *For* $Z \geqq 0$,

$$k_{a*}^{-}(Z^2, T_y) - k_0 \leqq K_7[Z/(1 + Z)^3(C + s_y)] \tag{3.15}$$

and

$$[k_{a*}^{-}(Z^2, T_y) - k_0]Z \leqq K_7 . \tag{3.16}$$

PROOF. Since (3.16) follows from (3.15) and the fact that $C \geqq 1/\alpha$, we need only prove (3.15). Toward this end, use (3.3), the definition of C_a^U in (3.5), and the definition of k_{a*}^{-} in (3.7) to conclude

$$\begin{aligned}&[\textstyle\int_{-1}^{k_{a*}^{-}} (1 - v^2)^{(m-3)/2}\, dv - \int_{-1}^{k_0} (1 - v^2)^{(m-3)/2}\, dv]/\int_{-1}^{1} (1 - v^2)^{(m-3)/2}\, dv \\ &\quad = [a(T_x)^{\frac{1}{2}}/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y)] .\end{aligned} \tag{6.2}$$

Equation (6.2) may be rewritten as

$$\begin{aligned}&\textstyle\int_{k_0}^{k_{a*}^{-}} (1 - v^2)^{(m-3)/2}\, dv \\ &\quad = a(T_x)^{\frac{1}{2}} \int_{-1}^{1} (1 - v^2)^{(m-3)/2}\, dv/2(1 + (T_x)^{\frac{1}{2}})^3(C + s_y) .\end{aligned} \tag{6.3}$$

The r.h.s. of (6.3) is less than or equal to $K_{21}[(T_x)^{\frac{1}{2}}/(1+(T_x)^{\frac{1}{2}})^3(C+s_y)]$. Also for the l.h.s. of (6.3) we have

$$\int_{k_0}^{k_{a^*}^-}(1-v^2)^{(m-3)/2}\,dv \geqq (k_{a^*}^- - k_0)K_{22}\,, \tag{6.4}$$

as over this range of integration $v \leqq k^- < 1$. (See (3.7) for the definition of k^-.) Thus combining (6.3) and (6.4) we have

$$(k_{a^*}^-(T_x, T_y) - k_0) \leqq [K_{21}/K_{22}](T_x)^{\frac{1}{2}}/(1+(T_x)^{\frac{1}{2}})^3(C+s_y)\,.$$

Since $Z = (T_x)^{\frac{1}{2}}$ the result follows.

LEMMA 3.6. *Let Z be a normal random variable with mean $m^{\frac{1}{2}}\theta k_0$ and variance σ_x^2. If $I(Z) = 1$ if $Z \geqq 0$ and $I(Z) = 0$ otherwise, then*

$$E\{I(Z)[Z^n/(1+Z)^3](1-e^{-K_5 Z})\} \geqq K_8 E\{I(Z)Z^{n+2}/(1+Z)^6\}\,, \tag{3.17}$$

and

$$E\{I(Z)[Z^n(1+Z)^3](1-e^{-K_5 Z}) \geqq K_8 E\{I(Z)Z^{n+1}/(1+Z)^4\}\,. \tag{3.18}$$

PROOF. Note that

$$\begin{aligned} Z^n(1-e^{-K_5 Z})/(1+Z)^3 &\geqq Z^{n+1}(1-e^{-K_5})/(1+Z)^3 \qquad \text{if } 0 \leqq Z \leqq 1 \\ &\geqq Z^n(1-e^{-K_5})/(1+Z)^3 \qquad \text{if } Z \geqq 1\,. \end{aligned} \tag{6.5}$$

Since $[Z^{n+1}/(1+Z)^3] \geqq [Z^{n+2}/(1+Z)^6]$ and $[Z^n/(1+Z)^3] \geqq [Z^{n+2}/(1+Z)^6]$, (3.17) follows with $K_8 = 1 - e^{-K_5}$. Essentially the same proof yields (3.18).

LEMMA 3.7. *Let V be a normal random variable with mean $m^{\frac{1}{2}}k_0$ and variance $(\sigma_x/\theta)^2$ If $(\theta/\sigma_x^2) \geqq 1$, $\sigma_x \geqq B$, then*

$$E\{I(V)[V^{n+1}/(\theta^{-1}+V)^4]\} \geqq K_9\,, \tag{3.19}$$

and

$$E\{I(V)[V^{n+2}/(\theta^{-1}+V)^6]\exp[K_{11}/\sigma_x^2(\theta^{-1}+V)]\} \leqq K_{10}\,. \tag{3.20}$$

PROOF. First note that the conditions $\theta/\sigma_x^2 \geqq 1$, $\sigma_x \geqq B$ imply $\theta \geqq B^2$ and $(\sigma_x/\theta)^2 \leqq 1/B^2$. Now (3.19) is immediate since $E\{I(V)V^{n+1}/(\theta^{-1}+V)^4\} \geqq E\{I(V)V^{n+1}/(B^{-2}+V)^4\} = K_9$. To prove (3.20) let $f(v)$ denote the density of V and break up the region of integration for the expectation into $[0, k_0 m^{\frac{1}{2}}/2]$ and $[k_0 m^{\frac{1}{2}}/2, \infty]$. First consider

$$\begin{aligned} &\int_{k_0 m^{\frac{1}{2}}/2}^{\infty}[v^{n+2}\exp[K_{11}/\sigma_x^2(\theta^{-1}+v)]/(\theta^{-1}+v)^6]f(v)\,dv \\ &\qquad \leqq \int_{k_0 m^{\frac{1}{2}}/2}^{\infty}[v^{n+2}\exp(2K_{11}/k_0 m^{\frac{1}{2}}B^2)/(k_0 m^{\frac{1}{2}}/2)^6]f(v)\,dv \\ &\qquad \leqq (2/k_0 m^{\frac{1}{2}})^6\exp(2K_{11}/B^2 k_0 m^{\frac{1}{2}})E|V|^{n+2} = K_{23}\,, \end{aligned}$$

since V is normal with mean $k_0 m^{\frac{1}{2}}$ and variance $(\sigma_x/\theta)^2 \leqq B^{-2}$. On the other hand

$$\begin{aligned} &\int_0^{k_0 m^{\frac{1}{2}}/2}[v^{n+2}\exp[K_{11}/(\theta^{-1}+v)\sigma_x^2]/(\theta^{-1}+v)^6]f(v)\,dv \\ &\qquad \leqq (k_0 m^{\frac{1}{2}}/2)^{n+2}\theta^6\exp(K_{11}\theta/\sigma_x^2)f(k_0 m^{\frac{1}{2}}/2) \\ &\qquad = \theta^6(k_0 m^{\frac{1}{2}}/2)^{n+2}\exp(K_{11}\theta/\sigma_x^2)[\theta/\sigma_x(2\pi)^{\frac{1}{2}}] \\ &\qquad\qquad \times \exp(-\theta^2([k_0 m^{\frac{1}{2}}/2] - k_0 m^{\frac{1}{2}})^2/2\sigma_x^2) \\ &\qquad \leqq [\theta^7(k_0 m^{\frac{1}{2}}/2)^{n+2}/(2\pi)^{\frac{1}{2}}B]\exp[\theta/\sigma_x^2][K_{11} - \theta k_0^2 m/8] \leqq K_{25}\,, \end{aligned} \tag{6.6}$$

since

$$[\theta/\sigma_x^2][K_{11} - \theta k_0^2 m/8] \leqq 0 \qquad \text{if} \quad [8K_{11}/k_0^2 m] \leqq \theta$$
$$\leqq 8K_{11}^2/k_0^2 m B^2 , \qquad \text{if} \quad [8K_{11}/k_0^2 m] > \theta .$$

This completes the proof of Lemma 3.7.

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