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SOME INCOMPLETE AND BOUNDEDLY COMPLETE FAMILIES OF DISTRIBUTIONS[1]

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Let $\mathscr{P}$ be a family of distributions on a measurable space such that (†) $\int u_i\,dP = c_i$, $i = 1, \cdots, k$, for all $P \in \mathscr{P}$, and which is sufficiently rich; for example, $\mathscr{P}$ consists of all distributions dominated by a σ-finite measure and satisfying (†). It is known that when conditions (†) are not present, no nontrivial symmetric unbiased estimator of zero (s.u.e.z.) based on a random sample of any size n exists. Here it is shown that (I) if $g(x_1, \cdots, x_n)$ is a s.u.e.z. then there exist symmetric functions $h_i(x_1, \cdots, x_{n-1})$, $i = 1, \cdots, k$, such that

$$g(x_1, \cdots, x_n) = \sum_{i=1}^k \sum_{j=1}^n \{u_i(x_j) - c_i\} h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n);$$

and (II) if every nontrivial linear combination of $u_1, \cdots, u_k$ is unbounded then no bounded nontrivial s.u.e.z. exists. Applications to unbiased estimation and similar tests are discussed.

1. Introduction and statement of results. Let $\mathscr{A}$ be a σ-field of subsets of a set $\mathscr{X}$, and let $\mathscr{P}$ be a family of distributions (probability measures) P on $(\mathscr{X}, \mathscr{A})$ which satisfy the conditions

$$\int u_i\,dP = c_i, \quad i = 1, \cdots, k, \tag{1.1}$$

where k is a positive integer, $u_1, \cdots, u_k$ are given $\mathscr{A}$-measurable functions, and $c_1, \cdots, c_k$ are given real numbers. Let $\mathscr{A}^{(n)}$ be the σ-field of subsets of $\mathscr{X}^n$ generated by the (cylinder) sets in $\mathscr{A}^n$, and let $\mathscr{P}^{(n)} = \{P^n : P \in \mathscr{P}\}$ denote the family of the n-fold product measures P^n on $(\mathscr{X}^n, \mathscr{A}^{(n)})$.

A family $\mathscr{Q}$ of distributions on $(\mathscr{X}^n, \mathscr{A}^{(n)})$ will be said to be complete relative to the permutation group if the condition that the $\mathscr{A}^{(n)}$-measurable symmetric real-valued function g satisfies $\int g\,dQ = 0$ for all $Q \in \mathscr{Q}$ implies $g(x_1, \cdots, x_n) = 0$ a.e. $(\mathscr{Q})$. Here g is called symmetric if it is invariant under all permutations of its arguments. The family $\mathscr{Q}$ will be said to be boundedly complete relative to the permutation group if the same conclusion holds under the additional condition that g is bounded. Informally, $\mathscr{Q}$ is [boundedly] complete relative to the permutation group if there is no nontrivial [bounded] symmetric unbiased estimator of zero. (This definition relates to the well-known notion of a [boundedly] complete family [8] as follows. Let T be a maximal invariant under the permutation group and let $\mathscr{Q}^T = \{Q^T : Q \in \mathscr{Q}\}$ be the family

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of distributions of T induced by the distributions in $\mathscr{Q}$. Then $\mathscr{Q}$ is [boundedly] complete relative to the permutation group iff the family $\mathscr{Q}^T$ is [boundedly] complete.)

It is well known (Halmos (1946), Fraser (1954a), Bell, Blackwell and Breiman (1960)) that if the conditions (1.1) are absent and $\mathscr{P}$ is sufficiently rich then $\mathscr{P}^{(n)}$ is complete relative to the permutation group. This is not true in the presence of conditions (1.1) (unless the u_i and c_i are such that the conditions impose no restriction). Indeed, if $h_1, \cdots, h_k$ are any $\mathscr{A}^{(n-1)}$-measurable symmetric functions such that $\int |h_i| \, dP^{n-1} < \infty$, $i = 1, \cdots, k$, for all $P \in \mathscr{P}$, then the function g defined by

$$(1.2) \qquad g(x_1, \cdots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} \{u_i(x_j) - c_i\} h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$$

is a symmetric unbiased estimator of zero.

In this paper two theorems (each in two versions) are proved. The first theorem shows that if $\mathscr{P}$ is sufficiently rich then a symmetric unbiased estimator of zero is necessarily of the form (1.2). The second theorem shows that although $\mathscr{P}^{(n)}$ is not complete relative to the permutation group it is boundedly complete if all nontrivial linear combinations of $u_1, \cdots, u_k$ are unbounded.

To state the theorems, we introduce the following notation. If $\mathscr{A}$ contains the one-point sets, let $\mathscr{P}_0$ be the family of all distributions P concentrated on finite subsets of $\mathscr{X}$ which satisfy conditions (1.1). If μ is a σ-finite measure on $(\mathscr{X}, \mathscr{A})$, let $\mathscr{P}_0(\mu)$ be the family of all distributions absolutely continuous with respect to μ whose densities $dP/d\mu$ are simple functions (finite linear combinations of indicator functions of sets in $\mathscr{A}$) and which satisfy conditions (1.1).

THEOREM 1A. *Let $\mathscr{A}$ contain the one-point sets and let $\mathscr{P}$ be a convex family of distributions on $(\mathscr{X}, \mathscr{A})$ which satisfy conditions* (1.1), *and such that $\mathscr{P}_0 \subset \mathscr{P}$. If g is a symmetric $\mathscr{A}^{(n)}$-measurable function such that $\int g \, dP^n = 0$ for all $P \in \mathscr{P}$ then there exist k symmetric $\mathscr{A}^{(n-1)}$-measurable functions $h_1, \cdots, h_k$ which are P^{n-1}-integrable for all $P \in \mathscr{P}$, such that* (1.2) *is satisfied for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$.*

THEOREM 2A. *If the conditions of Theorem* 1A *are satisfied and if g is bounded while every nontrivial linear combination of $u_1, \cdots, u_k$ is unbounded then $g(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$.*

The following analogs of the two theorems hold for dominated families of distributions.

We shall say that an $\mathscr{A}$-measurable function u is $\mathscr{P}$-unbounded if for every real number c there is a P in $\mathscr{P}$ such that $P(|u(x)| > c) \neq 0$.

THEOREM 1B. *Let $\mathscr{P}$ be a convex family of distributions absolutely continuous with respect to a σ-finite measure μ on $(\mathscr{X}, \mathscr{A})$, which satisfy conditions* (1.1), *and such that $\mathscr{P}_0(\mu) \subset \mathscr{P}$. If g is a symmetric $\mathscr{A}^{(n)}$-measurable function such that $\int g \, dP^n = 0$ for all $P \in \mathscr{P}$ then there exist k symmetric $\mathscr{A}^{(n-1)}$-measurable functions $h_1, \cdots, h_k$ which are P^{n-1}-integrable for all $P \in \mathscr{P}$, such that* (1.2) *holds* a.e. $(\mathscr{P}^{(n)})$.

THEOREM 2B. *If the conditions of Theorem* 1B *are satisfied and if g is bounded while every nontrivial linear combination of $u_1, \cdots, u_k$ is $\mathscr{P}$-unbounded then* $g(x_1, \cdots, x_n) = 0$ a.e. ($\mathscr{P}^{(n)}$).

REMARK 1. The assumption that the family $\mathscr{P}$ is convex is used only to prove that there are versions of the functions h_i that are integrable. Note that the families $\mathscr{P}_0$, $\mathscr{P}_0(\mu)$, and the family of all P whicha re absolutely continuous with respect to μ and satisfy conditions (1.1), are convex.

REMARK 2. Theorems 1B and 2B remain true if $\mathscr{P}_0(\mu)$ is defined as the family of all distributions absolutely continuous with respect to μ which satisfy conditions (1.1) and whose densities are finite linear combinations of indicator functions of sets in a ring which generates the σ-field $\mathscr{A}$; compare Fraser (1954a).

REMARK 3. The analogs of Theorems 1 and 2 with $\mathscr{P}$ the class of all non-atomic probability measures on $(\mathscr{X}, \mathscr{A})$ satisfying (1.1) are also true; compare Bell et al. (1960).

REMARK 4. If the assumptions of Theorems 1A or 1B are satisfied but conditions (1.1) are absent then the family $\mathscr{P}^{(n)}$ is complete relative to the permutation group. Here the assumption that $\mathscr{P}$ is convex is not needed. This is essentially known (as noted above) and is easily seen from the proofs.

REMARK 5. A special case of Theorem 1B (with $\mathscr{X} = R^1$, μ Lebesgue measure, $k = 1$, $u_1(x) = 1$ if $x < 0$, $= 0$ otherwise) is due to Fraser (1954b). I am grateful to a referee for drawing my attention to this fact.

The theorems are proved in Sections 3–6. Section 2 contains lemmas that are used in the proofs.

This section is concluded with three examples of applications of Theorems 1 and 2.

EXAMPLE 1. Let $X_1, \cdots, X_n$ be independent real-valued random variables with common probability density $p(x)$ and suppose that the first k moments, $\int x^i p(x)\,dx = c_i$, $i = 1, \cdots, k$, are known ($k \geqq 1$). Nothing else is assumed. Consider estimating $\psi(P) = P(A)$, the probability of a given set $A \subset R^1$. Theorem 2B implies that $\hat{\psi} = n^{-1} \sum_{j=1}^{n} I_A(X_j)$, where I_A is the indicator function of A, is the unique symmetric unbiased estimator of $\psi(P)$. (It is reasonable to require that the range of an estimator of $\psi(P)$ be contained in the range of $\psi(P)$. In the present example, due to Chebyshev-type inequalities, $\hat{\psi}$ may not satisfy this requirement. In such a case the use of an unbiased estimator cannot be recommended.)

EXAMPLE 2. Let $X_1, \cdots, X_n$ be independent real-valued random variables with common distribution P whose variance is known. Consider testing the hypothesis $\int x\,dP = 0$ against the alternatives $\int x\,dP > 0$. For every $n \geqq 1$, every $\alpha \in (0, 1)$, and every $\varepsilon > 0$ there exists a strictly unbiased test of size α against the alternatives $\int x\,dP \geqq \varepsilon$. (In Hoeffding (1956), page 112, a test is exhibited which, after a suitable change in notation, is strictly unbiased against $\int x\,dP = \varepsilon$. This test can be shown to be strictly unbiased against $\int x\,dP \geqq \varepsilon$.) Theorem 2 implies that against the alternatives $\int x\,dP > 0$ no nontrivial unbiased test exists. (One first shows that every unbiased test is similar; see [8]. We may assume that the test is symmetric. By Theorem 2 the only symmetric similar test of size α is trivial.)

EXAMPLE 3. Let the assumptions of Theorem 1 (A or B) be satisfied. If $\psi(P)$ admits an unbiased estimator, then the difference of any two symmetric unbiased estimators is given by (1.2). We discuss only the simplest case, $n = 1$. Let $\psi(P) = \int w\,dP$. Then any unbiased estimator $t(x)$ is given by

$$t(x) = w(x) + \sum_{i=1}^{k} h_i\{u_i(x) - c_i\}\ ,$$

where $h_1, \cdots, h_k$ are arbitrary constants. Suppose that $w, u_1, \cdots, u_k$ have finite second moments. Then

$$\mathrm{Var}_P\,(t) = \mathrm{Var}_P\,(w) + 2\sum_{i=1}^{k} h_i C_i(P) + \sum_{i=1}^{k}\sum_{j=1}^{k} h_i h_j D_{ij}(P)$$

where $C_i(P) = \mathrm{Cov}_P\,(w, u_i)$ and $D_{ij}(P) = \mathrm{Cov}_P\,(u_i, u_j)$. It is straightforward to minimize $\mathrm{Var}_P\,(t)$ with respect to $h_1, \cdots, h_k$. Let Q be a distribution in $\mathscr{P}$ such that the matrix $(D_{ij}(Q))$ is nonsingular, and let $(D^{ij}(Q))$ be its inverse. Then the unbiased estimator which has minimum variance when the distribution is Q is $t(x)$ with $h_i = \sum_j D^{ij}(Q)C_j(Q)$, and its variance at $P = Q$ is $\mathrm{Var}_Q\,(w) - \sum\sum D^{ij}(Q)C_i(Q)C_j(Q)$.

2. Lemmas. The following lemmas will be used in the proofs of the theorems. We write $\mathbf{u}(x)$ for the column vector with components $u_1(x), \cdots, u_k(x)$.

LEMMA 1A. *If, for* $(x_1, \cdots, x_n) \in \mathscr{X}^n$,

$$g(x_1, \cdots, x_n) = \sum_{i=1}^{k}\sum_{j=1}^{n} u_i(x_j)h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)\ , \tag{2.1}$$

where each h_i *is symmetric in its* $n-1$ *arguments, and if* $z_1, \cdots, z_k$ *are* k *points in* $\mathscr{X}$ *such that the* $k \times k$ *matrix*

$$\mathbf{U} = (\mathbf{u}(z_1), \cdots, \mathbf{u}(z_k)) \tag{2.2}$$

is nonsingular, then, for $(x_1, \cdots, x_n) \in \mathscr{X}^n$,

$$g(x_1, \cdots, x_n) = \sum_{m=0}^{n-1}(-1)^{n-m-1}T_{n,m}(x_1, \cdots, x_n)\ , \tag{2.3}$$

where

$$T_{n,m}(x_1, \cdots, x_n) = \sum_{m,n-m}\sum_{i_1=1}^{k}\cdots\sum_{i_{n-m}=1}^{k} g(x_{j_1}, \cdots, x_{j_m}, z_{i_1}, \cdots, z_{i_{n-m}})\, v_{i_1}(x_{j_{m+1}})\cdots v_{i_{n-m}}(x_{j_n})\ , \tag{2.4}$$

$$\mathbf{v}(x) = \mathbf{U}^{-1}\mathbf{u}(x)\ , \tag{2.5}$$

and $\sum_{m,n-m}$ *denotes summation over those permutations* $j_1, \cdots, j_n$ *of the integers* $1, \cdots, n$ *for which* $j_1 < \cdots < j_m$ *and* $j_{m+1} < \cdots < j_n$.

REMARK. Note that representation (2.3) of $g(x_1, \cdots, x_n)$ does not involve the functions $h_1, \cdots, h_k$ which appear in (2.1).

PROOF. From (2.1) and (2.5) we have

$$g(x_1, \cdots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} v_i(x_j) f_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \tag{2.6}$$

where each f_i is symmetric in its $n-1$ arguments. By (2.5)

$$v_i(z_i) = 1, \qquad v_i(z_j) = 0, \qquad i \neq j. \tag{2.7}$$

Hence, for $1 \leqq i_r \leqq k$, $r = 1, \cdots, n-m$; $n-m = 1, \cdots, n$,

$$\begin{aligned} g(x_1, \cdots, x_m, z_{i_1}, \cdots, z_{i_{n-m}}) &= \sum_{i=1}^{k} \sum_{j=1}^{m} v_i(x_j) f_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m, z_{i_1}, \cdots, z_{i_{n-m}}) \\ &\quad + f_{i_1}(x_1, \cdots, x_m, z_{i_2}, \cdots, z_{i_{n-m}}) + \cdots \\ &\quad + f_{i_{n-m}}(x_1, \cdots, x_m, z_{i_1}, \cdots, z_{i_{n-m-1}}). \end{aligned} \tag{2.8}$$

From (2.6) and (2.8), by induction on m (beginning with $m = n-1$),

$$g(x_1, \cdots, x_n) = T_{n,n-1} - T_{n,n-2} + \cdots + (-1)^{m-1} T_{n,n-m} + (-1)^m R_m, \qquad m = 0, 1, \cdots, n-1, \tag{2.9}$$

where $T_{n,r} = T_{n,r}(x_1, \cdots, x_n)$, and R_m differs from $T_{n,n-m-1}$ only in that $g(\cdots, z_{i_1}, \cdots, z_{i_{m+1}})$ is replaced by $f_{i_1}(\cdots, z_{i_2}, \cdots, z_{i_{m+1}}) + \cdots + f_{i_{m+1}}(\cdots, z_{i_1}, \cdots z_{i_m})$. In particular, by (2.8) with $m = 0$, we have $R_{n-1} = T_{n,0}$, and (2.3) follows from (2.9).

LEMMA 1B. *Let ν be a finite measure on the measurable space $(\mathscr{X}, \mathscr{A})$, let g be an $\mathscr{A}^{(n)}$-measurable function such that $\int |g|\, d\nu^n < \infty$, and let $u_1, \cdots, u_k$ be $\mathscr{A}$-measurable functions such that $\int |u_i|\, d\nu < \infty$, $i = 1, \cdots, k$. If there exist symmetric $\mathscr{A}^{(n-1)}$-measurable functions $h_1, \cdots, h_k$ such that $g(x_1, \cdots, x_n)$ can be represented in the form (2.1) for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$, and if $B_1, \cdots, B_k$ are k sets in $\mathscr{A}$ such that the $k \times k$ matrix*

$$\mathbf{U}_\nu = \left(\int_{B_1} \mathbf{u}\, d\nu, \cdots, \int_{B_k} \mathbf{u}\, d\nu\right) \tag{2.10}$$

is nonsingular, then, for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$,

$$g(x_1, \cdots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} T_{n,m}^{(\nu)}(x_1, \cdots, x_n), \tag{2.11}$$

where $T_{n,m}^{(\nu)}(x_1, \cdots, x_n)$ is defined like $T_{n,m}(x_1, \cdots, x_n)$ in (2.4) but with $g(x_{j_1}, \cdots, x_{j_m}, z_{i_1}, \cdots, z_{i_{n-m}})$ replaced by

$$\int_{B_{i_1}} d\nu(y_1) \cdots \int_{B_{i_{n-m}}} d\nu(y_{n-m})\, g(x_{j_1}, \cdots, x_{j_m}, y_1, \cdots, y_{n-m}) \tag{2.12}$$

and $\mathbf{v}(x) = (v_1(x), \cdots, v_k(x))^T$ replaced by

$$\mathbf{v}^{(\nu)}(x) = \mathbf{U}_\nu^{-1} \mathbf{u}(x). \tag{2.13}$$

The same is true with the phrase "for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$" replaced by "a.e. $\nu^{(n)}$" *in the two places where it occurs.*

The proof of Lemma 1B closely parallels that of Lemma 1A. The only difference is that any substitution, in a function $f(\cdots, x_i, \cdots)$, of z_j for x_i in the proof of Lemma 1A is replaced by integration over B_j with respect to $d\nu(x_i)$.

Incidentally, Lemma 1B contains Lemma 1A.

The functions $h_1, \cdots, h_k$ in representation (2.1) of g are not, in general, uniquely determined by the functions $g, u_1, \cdots, u_k$. For example, if $h_1(x, y)$, $h_2(x, y)$ satisfy (2.1) with $k = 2$, $n = 3$, so do

$$H_1(x, y) = h_1(x, y) + w(y)u_2(x) + w(x)u_2(y) ,$$
$$H_2(x, y) = h_2(x, y) - w(y)u_1(x) - w(x)u_1(y) ,$$

where $w(x)$ is arbitrary. The following lemma records, for future reference, a certain version of the functions $h_1, \cdots, h_k$.

LEMMA 2. *Suppose there exist symmetric functions $h_1, \cdots, h_k$ such that g has the representation* (2.1). *Under the conditions of Lemma* 1A, *$h_1, \cdots, h_k$ can be so chosen that each $h_i(x_1, \cdots, x_{n-1})$ is a finite linear combination of terms of the form*

$$g(x_{j_1}, \cdots, x_{j_m}, z_{i_1}, \cdots z_{i_{n-m}})\, u_{r_1}(x_{j_{m+1}}) \cdots u_{r_{n-m-1}}(x_{j_{n-1}}) , \tag{2.14}$$

where $(j_1, \cdots, j_{n-1})$ is a permutation of $(1, \cdots, n-1)$. Under the conditions of Lemma 1B, *each $h_i(x_1, \cdots, x_{n-1})$ can be chosen as a finite linear combination of terms of the form*

$$\int_{B_{i_1}} d\nu(y_1) \cdots \int_{B_{i_{n-m}}} d\nu(y_{n-m}) g(x_{j_1}, \cdots, x_{j_m}, y_1, \cdots, y_{n-m}) \\ u_{r_1}(x_{j_{m+1}}) \cdots u_{r_{n-m-1}}(x_{j_{n-1}}) , \tag{2.15}$$

where $(j_1, \cdots, j_{n-1})$ is a permutation of $(1, \cdots, n-1)$.

PROOF. Under the conditions of Lemma 1A, $g(x_1, \cdots, x_n)$ has the representation (2.3), where $T_{n,m}(x_1, \cdots, x_n)$ is defined in (2.4) and each $v_i(x)$ is a linear combination of $u_1(x), \cdots, u_k(x)$. Hence $g(x_1, \cdots, x_n)$ can be written as a linear combination of terms, each of which, for some i and some j, is of the form $u_i(x_j)$ multiplied with a product of the form (2.14) which does not involve x_j. This fact and the symmetry of $g(x_1, \cdots, x_n)$ imply the assertion of the lemma. Under the conditions of Lemma 1B the proof is analogous.

3. Proof of Theorem 1A. We may and shall assume that conditions (1.1) are satisfied with $c_1 = \cdots = c_k = 0$. Thus $\mathscr{P}_0$ is the family of all distributions P concentrated on finite subsets of $\mathscr{X}$ which satisfy the conditions

$$\int u_i \, dP = 0 , \qquad i = 1, \cdots, k , \tag{3.1}$$

and $\mathscr{P}$ is a convex family of distributions P on $(\mathscr{X}, \mathscr{A})$ which satisfy (3.1), and such that $\mathscr{P} \supset \mathscr{P}_0$. Let g be a symmetric $\mathscr{A}^{(n)}$-measurable function such that $\int g \, dP^n = 0$ for all $P \in \mathscr{P}$. We must show that there exist symmetric $\mathscr{A}^{(n-1)}$-measurable functions $h_1, \cdots, h_k$ such that

$$\int |h_i| \, dP^{n-1} < \infty , \qquad i = 1, \cdots, k, \text{ if } P \in \mathscr{P} \tag{3.2}$$

and, for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$,

$$g(x_1, \cdots, x_n) = \sum_{i=1}^k \sum_{j=1}^n u_i(x_j) h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) . \tag{3.3}$$

Since $\mathscr{P}_0 \subset \mathscr{P}$, we have that if N is a positive integer, $x_1, \cdots, x_N$ are points in $\mathscr{X}$, and $p_1, \cdots, p_N$ are nonnegative numbers such that

$$(3.4) \qquad p_1 + \cdots + p_N = 1 ,$$

$$(3.5) \qquad u_i(x_1)p_1 + \cdots + u_i(x_N)p_N = 0 , \qquad i = 1, \cdots, k ,$$

then

$$(3.6) \qquad \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} g(x_{i_1}, \cdots, x_{i_n}) p_{i_1} \cdots p_{i_n} = 0 .$$

It will be convenient to identify points in R^k with the corresponding column vectors. Conditions (3.4) and (3.5) show that the origin $\mathbf{0}$ of R^k is in the convex hull of the set $\mathscr{U} = \{\mathbf{u}(x) : x \in \mathscr{X}\}$.

First assume that $\mathbf{0}$ is in the interior of the convex hull of $\mathscr{U}$. Then there exist $k+1$ points $y_1, \cdots, y_{k+1}$ in $\mathscr{X}$ such that $\mathbf{0}$ is in the interior of the polytype whose vertices are $\mathbf{u}(y_1), \cdots, \mathbf{u}(y_{k+1})$. Thus there are strictly positive numbers $q_1, \cdots, q_{k+1}$ such that

$$(3.7) \qquad \begin{aligned} q_1 + \cdots + q_{k+1} &= 1 , \\ \mathbf{u}(y_1)q_1 + \cdots + \mathbf{u}(y_{k+1})q_{k+1} &= \mathbf{0} . \end{aligned}$$

Now let $x_1, \cdots, x_n$ be points in $\mathscr{X}$. Let $x_{n+i} = y_i$, $i = 1, \cdots, k+1$. There exists a positive number ε such that if

$$(3.8) \qquad 0 \leqq p_j \leqq \varepsilon , \qquad j = 1, \cdots, n ,$$

the equations (3.4), (3.5) with $N = n + k + 1$, regarded as equations for $p_{n+1}, \cdots, p_{n+k+1}$, have a positive solution. This follows, by continuity, from the fact that if $p_1 = \cdots = p_n = 0$, the solution is $p_{n+i} = q_i > 0$, $i = 1, \cdots, k+1$. The solution in the general case is of the form

$$(3.9) \qquad p_{n+i} = q_i(1 - \sum_{j=1}^{n} p_j) - \sum_{r=1}^{k} \sum_{j=1}^{n} a_{ir} u_r(x_j) p_j , \qquad i = 1, \cdots, k+1 ,$$

where the coefficients a_{ir} (and q_i) do not depend on x_j and p_j ($j = 1, \cdots, n$).

If we now insert the expressions (3.9) for $p_{n+1}, \cdots, p_{n+k}$ in the left side of (3.6) with $N = n + k + 1$, we obtain a polynomial in $p_1, \cdots, p_n$ which is zero in the range (3.8), and hence identically zero. The resulting equation may be written

$$(3.10) \qquad \sum_{m=0}^{n} \binom{n}{m} S_{n,m} = 0 ,$$

where

$$(3.11) \qquad \begin{aligned} S_{n,m} = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{i_1=1}^{k+1} \cdots \sum_{i_{n-m}=1}^{k+1} g(x_{j_1}, \cdots, x_{j_m}, \\ y_{i_1}, \cdots, y_{i_{n-m}}) p_{j_1} \cdots p_{j_m} p_{n+i_1} \cdots p_{n+i_{n-m}} \end{aligned}$$

and the p_{n+i} are given by (3.9).

The identity (3.10) will be used to show that $g(x_1, \cdots, x_n)$ can be represented in the form (3.3). If A_M denotes the sum of the coefficients of $p_1 \cdots p_M$ in (3.10), and A_0 denotes the constant term, then

$$(3.12) \qquad A_M = 0 , \quad M = 0, 1, \cdots, n .$$

It is easy to see that $A_M = A_M(x_1, \cdots, x_M)$ depends on $x_1, \cdots, x_n$ only through $x_1, \cdots, x_M$, and that the sum of the coefficients of $p_{j_1} \cdots p_{j_M}$, $1 \leqq j_1 < \cdots < j_M \leqq n$, is $A_M(x_{j_1}, \cdots, x_{j_M})$.

It is readily seen from (3.9)—(3.11) that the condition $A_0 = 0$ is equivalent to

$$\sum_{i_1=1}^{k+1} \cdots \sum_{i_n=1}^{k+1} g(y_{i_1}, \cdots, y_{i_n})\, q_{i_1} \cdots q_{i_n} = 0\,. \tag{3.13}$$

It will now be shown by induction on m that

$$\begin{aligned}\sum_{i_1=1}^{k+1} \cdots \sum_{i_{n-m}=1}^{k+1} g(x_1, \cdots, x_m, y_{i_1}, \cdots, y_{i_{n-m}})\, q_{i_1} \cdots q_{i_{n-m}} \\ = \sum_{i=1}^{k} \sum_{j=1}^{m} u_i(x_j) h_{m,i}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m)\end{aligned} \tag{3.14}$$

for $m = 1, \cdots, n$, where each $h_{m,i}(\cdot)$ is symmetric in its $m - 1$ arguments; and that (3.14) also holds with $x_1, \cdots, x_m$ replaced by $x_{j_1}, \cdots, x_{j_m}$, $1 \leqq j_1 < \cdots < j_m \leqq n$. In particular, (3.14) with $m = n$ implies the representation (3.3) of $g(x_1, \cdots, x_n)$.

That (3.14) holds for $m = 1$ (where the $h_{1,i}$ are constants) can be seen from $A_1 = 0$ and (3.13). Suppose (3.14) is true for $m \leqq M - 1$ $(2 \leqq M \leqq n)$. The products $p_1 \cdots p_M$ occur in the sums $S_{n,m}$ with $m = 0, \cdots, M$. The coefficient of $p_1 \cdots p_M$ in $S_{n,M}$ is, apart from a nonzero numerical factor, equal to the left-hand side of (3.14) with $m = M$. Hence, to prove that (3.14) holds for $m = M$, it is enough to show that for $m = 0, 1, \cdots, M - 1$, the sum of the coefficients, call it $A_{M,m}$, of $p_1 \cdots p_M$ in $S_{n,m}$ is of the form

$$\sum_{i=1}^{k} \{u_i(x_1) f_i(x_2, \cdots, x_M) + \cdots + u_i(x_M) f_i(x_1, \cdots, x_{M-1})\} \tag{3.15}$$

for some symmetric functions f_i.

It is seen from (3.11) and (3.9) that $A_{M,m}$ may be written as $A + B$, where A is the sum of those coefficients of $p_1 \cdots p_M$ in $S_{n,m}$ that contain at least one factor $u_r(x_j)$ (for some r, j), and B is the sum of the remaining coefficients. Each term containing the factor $u_r(x_j)$ is the product of $u_r(x_j)$ and a factor not depending on x_j. Also, $A_{M,m}$ is symmetric in $x_1, \cdots, x_M$. These facts imply that A is of the form (3.15).

The term B is the sum of the coefficients of $p_1 \cdots p_M$ in the sum

$$\begin{aligned}\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{i_1=1}^{k+1} \cdots \sum_{i_{n-m}=1}^{k+1} g(x_{j_1}, \cdots, x_{j_m}, y_{i_1}, \cdots, y_{i_{n-m}}) \\ (q_{i_1} \cdots q_{i_{n-m}})(p_{j_1} \cdots p_{j_m})(1 - \textstyle\sum_{j=1}^{n} p_j)^{n-m}\,.\end{aligned}$$

It follows from the induction hypothesis that B is also of the form (3.15). This completes the proof that $g(x_1, \cdots, x_n)$ is of the form (3.3) with symmetric functions $h_1, \cdots, h_k$.

We now show that the functions $h_1, \cdots, h_k$ can be so chosen that they satisfy the integrability condition (3.2). Let P_0 be the distribution which assigns probabilities $q_1, \cdots, q_{k+1}$ to the respective points $y_1, \cdots, y_{k+1}$, as defined in (3.7). Let B_i denote the set which consists of the single point y_i, for $i = 1, \cdots, k$. Since the q_j are strictly positive, the matrix $(\int_{B_1} \mathbf{u}\, dP_0, \cdots, \int_{B_k} \mathbf{u}\, dP_0)$ is nonsingular. The conditions of Lemma 1B with $\nu = P_0$ are satisfied. By Lemma 2, the functions $h_1, \cdots, h_k$ in (3.3) can be so chosen that each $h_i(x_1, \cdots, x_{n-1})$ is a linear combination of terms of the form

$$\int_{B_{i_1}} dP_0(t_1) \cdots \int_{B_{i_{n-m}}} dP_0(t_{n-m}) g(x_{j_1}, \cdots, x_{j_m}, t_1, \cdots, t_{n-m}) \\ u_{r_1}(x_{j_{m+1}}) \cdots u_{r_{n-m-1}}(x_{j_{n-1}}) .$$

Let P be a distribution in $\mathscr{P}$. The u_i are P-integrable by assumption. Hence to show that the h_i are P^{n-1}-integrable it is sufficient to show that

$$\int_{\mathscr{X}^n} |g| \, d(P_0^{n-m} P^m) < \infty \tag{3.16}$$

for $m = 0, 1, \cdots, n-1$ and all $P \in \mathscr{P}$.

By (3.7) the distribution P_0 is in $\mathscr{P}_0$ and hence in $\mathscr{P}$. If P is in $\mathscr{P}$, so is $Q = \frac{1}{2}(P_0 + P)$, due to the convexity of $\mathscr{P}$. Hence $\int |g| \, dQ^n < \infty$. But $\int |g| \, dQ^n$ can be written as a linear combination with positive coefficients of the integrals in (3.16). Thus (3.16) is true. This completes the proof under the assumption that the origin $\mathbf{0}$ is in the interior of the convex hull of $\mathscr{U}$.

Now suppose that the origin is a boundary point of the convex hull of $\mathscr{U}$. Then there are real numbers $b_1, \cdots, b_k$, not all zero, such that $b_1 u_1(x) + \cdots + b_k u_k(x) = 0$ for all $x \in \mathscr{X}$. Therefore one of the conditions (3.1) is implied by the others. In this way the problem can be reduced to one of these two: (I) a problem of the same structure, with k replaced by k', $1 \leqq k' < k$, such that the origin of k'-space is in the interior of the convex hull of the set corresponding to $\mathscr{U}$; (II) the same kind of problem but with no restrictions (3.1) present. In case (I), the conclusion of the theorem follows from the first part of the proof. In case II, equality (3.6) with $N = n$ and arbitrary $(x_1, \cdots, x_n) \in \mathscr{X}^n$ holds for all positive $p_1, \cdots, p_n$, so that $g(x_1, \cdots, x_n) = 0$. (This is, essentially, Halmos' Lemma 2 in [4].) Theorem 1A is proved.

4. Proof of Theorem 2A. Let the conditions of Theorem 1A be satisfied, and suppose that g is bounded while every nontrivial linear combination of $u_1, \cdots, u_k$ is unbounded. We must show that $g(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots x_n) \in \mathscr{X}^n$.

We again assume that $c_1 = \cdots = c_k = 0$. Since every nontrivial linear combination of $u_1, \cdots, u_k$ is unbounded, there exist k points $z_1, \cdots, z_k$ in $\mathscr{X}$ such that the $k \times k$ matrix $(\mathbf{u}(z_1), \cdots, \mathbf{u}(z_k))$ is nonsingular. Hence, by Theorem 1A and Lemma 1A, we have for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$

$$g(x_1, \cdots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} T_{n,m}(x_1, \cdots, x_n; g) , \tag{4.1}$$

where (we now exhibit the dependence of $T_{n,m}$ on g)

$$T_{n,m}(x_1, \cdots, x_n; g) = \sum_{m,n-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-m}=1}^{k} g(x_{j_1}, \cdots, x_{j_m}, z_{i_1}, \cdots, \\ z_{i_{n-m}}) \, v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{n-m}}(x_{j_n}) . \tag{4.2}$$

Here each of $v_1, \cdots, v_k$ is a nontrivial linear combination of $u_1, \cdots, u_k$ and hence is unbounded.

The theorem will be proved by induction on k and, for each k, by induction on n.

For $n = 1$ and k arbitrary we have, by Theorem 1A, $g(x) = h_1 u_1(x) + \cdots + h_k u_k(x)$, where $h_1, \cdots, h_k$ are constants. The right side is bounded only if $h_1 = \cdots = h_k = 0$, so that the theorem is true in this case.

Now let $k = 1$. By (4.2),

$$(4.3) \qquad T_{n,m}(x_1, \cdots, x_n; g) = \textstyle\sum_{m,n-m} g(x_{j_1}, \cdots, x_{j_m}, z, \cdots, z)\, v(x_{j_{m+1}}) \cdots v(x_{j_n}),$$

where $z = z_1$, and $v = v_1$ is unbounded. There is a sequence $\{y_N\}$ in $\mathscr{X}$ such that

$$|v(y_N)| \to +\infty \qquad \text{as} \quad N \to \infty .$$

Divide both sides of (4.3) by $v(x_n)$, set $x_n = y_N$ and let $N \to \infty$. The terms on the right of (4.3) with $j_m = n$, divided by $v(x_n) = v(y_N)$, converge to zero, and we obtain

$$\begin{aligned} \lim_{N\to\infty} & T_{n,m}(x_1, \cdots, x_{n-1}, y_N; g)/v(y_N) \\ &= \textstyle\sum_{m,n-1-m} g(x_{j_1}, \cdots, x_{j_m}, z, \cdots, z) v(x_{j_{m+1}}) \cdots v(x_{j_{n-1}}) \\ &= T_{n-1,m}(x_1, \cdots, x_{n-1}; g^{(1)}) , \end{aligned}$$

where $g^{(1)}(x_1, \cdots, x_{n-1}) = g(x_1, \cdots, x_{n-1}, z)$, for $m = 0, \cdots, n-2$. For $m = n-1$, the limit is $g^{(1)}(x_1, \cdots, x_{n-1})$. Thus if we set $x_n = y_N$ in (4.1), divide by $v(y_N)$ and let $N \to \infty$, we obtain

$$g^{(1)}(x_1, \cdots, x_{n-1}) = \textstyle\sum_{m=0}^{n-2} (-1)^{n-m-2} T_{n-1,m}(x_1, \cdots, x_{n-1}; g^{(1)}) .$$

It follows by induction on n that the theorem is true for $k = 1$.

Now let $k \geqq 2$, and suppose that the theorem is true with k replaced by $k - 1$. Since v_k is unbounded, there is a sequence $\{y_N\}$ in $\mathscr{X}$ such that $|v_k(y_N)| \to \infty$ as $N \to \infty$. There is a subsequence $\{y_{N'}\}$ of $\{y_N\}$ such that $v_1(y_{N'})/v_k(y_{N'})$ tends to a limit λ_1, $-\infty \leqq \lambda_1 \leqq \infty$. Repeating this argument, we see that there is a sequence $\{y_N\}$ in $\mathscr{X}$ such that $|v_k(y_N)| \to \infty$ and $v_i(y_N)/v_k(y_N) \to \lambda_i$, $i = 1, \cdots, k$, as $N \to \infty$, where $-\infty \leqq \lambda_i \leqq \infty$ for $i = 1, \cdots, k-1$. Suppose that $\lambda_1, \cdots, \lambda_{k-1}$ are not all finite, say $|\lambda_i| = \infty$ for $i = 1, \cdots, r$; $|\lambda_i| < \infty$ for $i \geqq r+1$. Then $v_k(y_N)/v_r(y_N) \to 0$, hence $|v_r(y_N)| \to \infty$ and $v_i(y_N)/v_r(y_N) \to \lambda_i'$ with $\lambda_i' = 0$ or 1, for $i \geqq r$. Also, there is a subsequence $\{y_{N'}\}$ of $\{y_N\}$ such that $v_i(y_N')/v_r(y_N') \to \lambda_i'$, with $-\infty \leqq \lambda_i' \leqq \infty$, for $i \leqq r-1$. It now follows by induction that there is an index j, $1 \leqq j \leqq k$, and a sequence $\{y_N\}$ in $\mathscr{X}$ such that $|v_j(y_N)| \to \infty$ and $v_i(y_N)/v_j(y_N) \to \lambda_i$, $i = 1, \cdots, k$, where $\lambda_1, \cdots, \lambda_k$ are all finite. We may assume that $j = k$, so that

$$(4.4) \qquad \lim_{N\to\infty} |v_k(y_N)| = \infty , \qquad \lim_{N\to\infty} v_i(y_N)/v_k(y_N) = \lambda_i , \qquad |\lambda_i| < \infty,\ i = 1, \cdots, k .$$

After dividing both sides of (4.2) by $v_k(x_n)$, setting $x_n = y_N$, and letting $N \to \infty$, we obtain

$$\lim_{N\to\infty} T_{n,m}(x_1, \cdots, x_{n-1}, y_N; g)/v_k(y_N) = T_{n-1,m}(x_1, \cdots, x_{n-1}; g^{(1)}) ,$$

where

$$g^{(1)}(x_1, \cdots, x_{n-1}) = \sum_{i=1}^{k} \lambda_i g(x_1, \cdots, x_{n-1}, z_i) . \tag{4.5}$$

Combined with (4.1) this yields

$$g^{(1)}(x_1, \cdots, x_{n-1}) = \sum_{m=0}^{n-2} (-1)^{n-m-2} T_{n-1,m}(x_1, \cdots, x_{n-1}; g^{(1)}) . \tag{4.6}$$

It follows in the same way that if we define $g^{(1)}, g^{(2)}, \cdots, g^{(n-1)}$ by (4.5) and

$$g^{(s+1)}(x_1, \cdots, x_{n-s-1}) = \sum_{i=1}^{k} \lambda_i g^{(s)}(x_1, \cdots, x_{n-s-1}, z_i) , \qquad s = 1, \cdots, n-2 , \tag{4.7}$$

then

$$g^{(n-s)}(x_1, \cdots, x_s) = \sum_{m=0}^{s-1} (-1)^{s-1-m} T_{s,m}(x_1, \cdots, x_s; g^{(n-s)}) \tag{4.8}$$

for $s = n-1, n-2, \cdots, 1$. In particular,

$$g^{(n-1)}(x) = T_{1,0}(x; g^{(n-1)}) = \sum_{i=1}^{k} g^{(n-1)}(z_i) v_i(x) .$$

Hence

$$g^{(n-1)}(x) = 0 , \qquad \text{all} \quad x \in \mathscr{X} . \tag{4.9}$$

We now show that $g^{(n-s+1)}(x_1, \cdots, x_{s-1}) = 0$ for all $(x_1, \cdots, x_{s-1}) \in \mathscr{X}^{s-1}$ implies $g^{(n-s)}(x_1, \cdots, x_s) = 0$ for all $(x_1, \cdots, x_s) \in \mathscr{X}^s$, $s = 2, \cdots, n$. Suppose that

$$g^{(n-s+1)}(x_1, \cdots, x_{s-1}) = 0 \qquad \text{for} \quad (x_1, \cdots, x_{s-1}) \in \mathscr{X}^{s-1} . \tag{4.10}$$

From (4.10) and (4.7)

$$g^{(n-s)}(x_1, \cdots, x_{s-1}, z_k) = -\sum_{i=1}^{k-1} \lambda_i g^{(n-s)}(x_1, \cdots, x_{s-1}, z_i) . \tag{4.11}$$

By (4.8), $g^{(n-s)}(x_1, \cdots, x_s)$ is a sum involving the terms

$$T_{s,m}(x_1, \cdots, x_s; g^{(n-s)}) = \sum_{m,s-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{s-m}=1}^{k} g^{(n-s)}(x_{j_1}, \cdots, x_{j_m}, z_{i_1}, \cdots, z_{i_{s-m}}) v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{s-m}}(x_{j_s}) \tag{4.12}$$

with $m = 0, 1, \cdots, s-1$.

Let

$$w_i(x) = v_i(x) - \lambda_i v_k(x) , \qquad i = 1, \cdots, k-1 , \tag{4.13}$$

and let $T^*_{n,m}(x_1, \cdots, x_n; g)$ be defined as $T_{n,m}(x_1, \cdots, x_n; g)$, but with $k, v_1(\cdot), \cdots, v_k(\cdot)$ replaced by $k-1, w_1(\cdot), \cdots, w_{k-1}(\cdot)$. If we eliminate z_k from the right side of (4.12) by using (4.11), we obtain

$$T_{s,m}(x_1, \cdots, x_s; g^{(n-s)}) = T^*_{s,m}(x_1, \cdots, x_s; g^{(n-s)}) \tag{4.14}$$

for $m = 0, 1, \cdots, s-1$. Note that any nontrivial linear combination of $w_1, \cdots, w_{k-1}$ is unbounded. It now follows from (4.8), (4.14) and the induction hypothesis that $g^{(n-s)}(x_1, \cdots, x_s) = 0$ for all $(x_1, \cdots, x_s)$ in $\mathscr{X}^s$. Thus $g(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots, x_n) \in \mathscr{X}^n$.

5. Proof of Theorem 1B. We again assume that $c_1 = \cdots = c_k = 0$. Let μ

be a σ-finite measure on the measurable space $(\mathscr{X}, \mathscr{A})$, let $\mathscr{P}$ be a convex family of distributions which are absolutely continuous with respect to μ and satisfy conditions (3.1), and let $\mathscr{P}_0(\mu) \subset \mathscr{P}$. Let g be a symmetric $\mathscr{A}^{(n)}$-measurable real-valued function such that $\int g\, dP^n = 0$ for all $P \in \mathscr{P}$. We must show that there exist k symmetric $\mathscr{A}^{(n-1)}$-measurable real-valued functions $h_1, \cdots, h_k$ such that the integrability conditions (3.2) are satisfied and the representation (3.3) of $g(x_1, \cdots, x_n)$ holds a.e. $(\mathscr{P}^{(n)})$.

Let A_0 be the class of all sets A in $\mathscr{A}$ such that

$$(5.1) \qquad \mu(A) + \textstyle\sum_{i=1}^k \int_A |u_i|\, d\mu < \infty\ .$$

Let N be a positive integer, $A_1, \cdots, A_N$ be sets in $\mathscr{A}_0$, and $a_1, \cdots, a_N$ be non-negative numbers such that

$$(5.2) \qquad \textstyle\sum_{j=1}^N a_j \mu(A_j) = 1\ ,$$

$$(5.3) \qquad \textstyle\sum_{j=1}^N a_j \int_{A_j} u_i\, d\mu = 0\ , \qquad i = 1, \cdots, k\ .$$

Then $p(x) = \sum_{j=1}^N a_i I_{A_j}(x)$, where I_A denotes the indicator function of set A, is the probability density with respect to μ of a distribution in $\mathscr{P}_0(\mu)$ and therefore in $\mathscr{P}$. Hence conditions (5.2) and (5.3) imply

$$(5.4) \qquad \textstyle\sum_{j_1=1}^N \cdots \sum_{j_n=1}^N a_{j_1} \cdots a_{j_n} G(A_{j_1}, \cdots, A_{j_n}) = 0\ ,$$

where

$$(5.5) \qquad G(A_1, \cdots, A_n) = \int_{A_1 \times \cdots \times A_n} g\, d\mu^n\ .$$

(The existence of the integrals in (5.5) is guaranteed by the assumption that $\int g\, dP^n$ exists for $P \in \mathscr{P}$.)

Conditions (5.2) and (5.3) also imply that the origin $\mathbf{0}$ of R^k is in the convex hull of the set

$$\mathscr{U}(\mu) = \{\textstyle\int_A \mathbf{u}\, d\mu / \mu(A) \colon A \in \mathscr{A}_0,\ \mu(A) > 0\}\ .$$

We first assume that $\mathbf{0}$ is in the interior of the convex hull of $\mathscr{U}(\mu)$. Then there exist $k+1$ sets $B_1, \cdots, B_{k+1}$ in $\mathscr{A}_0$ of positive μ-measure such that $\mathbf{0}$ is an inner point of the polytype whose vertices are $\int_{B_j} u\, d\mu/\mu(B_j)$, $j = 1, \cdots, k+1$.

Let $A_1, \cdots, A_n$ be any n sets in $\mathscr{A}_0$ and let $A_{n+i} = B_i$, $i = 1, \cdots, k+1$. We now use the argument in the proof of Theorem 1A, with $\mathscr{X}$, $\mathbf{u}(x)$, $g(x_1, \cdots, x_n)$ and p_j replaced by $\mathscr{A}_0$, $\int_A \mathbf{u}\, d\mu$, $G(A_1, \cdots, A_n)$ and $a_j \mu(A_j)$, respectively, to infer that there exist symmetric real-valued functions $H_1, \cdots, H_k$ on $\mathscr{A}_0^{n-1}$ such that, for every $(A_1, \cdots, A_n) \in \mathscr{A}_0^n$,

$$(5.6) \qquad G(A_1, \cdots, A_n) = \textstyle\sum_{i=1}^k \sum_{j=1}^n \int_{A_j} u_i\, d\mu\, H_i(A_1, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n)\ .$$

By the first part of Lemma 2, with $\mathscr{X}$, $u_i(x)$, z_i replaced by $\mathscr{A}_0$, $\int_A u_i\, d\mu$, B_i, respectively, the functions H_i can be so chosen that each $H_i(A_1, \cdots, A_{n-1})$ is a finite linear combination of terms of the form

$$G(A_{j_1}, \cdots, A_{j_m}, B_{i_1}, \cdots, B_{i_{n-m}}) \int_{A_{j_{m+1}}} u_{r_1}\, d\mu \cdots \int_{A_{j_{n-1}}} u_{r_{n-m-1}}\, d\mu\ ,$$

where $(j_1, \cdots, j_{n-1})$ is a permutation of $(1, \cdots, n-1)$. Define $h_i(x_1, \cdots, x_{n-1})$ as the same linear combination of the terms

$$g(x_{j_1}, \cdots, x_{j_m}, B_{i_1}, \cdots, B_{i_{n-m}})\, u_{r_1}(x_{j_{m+1}}) \cdots u_{r_{n-m-1}}(x_{j_{n-1}}) ,$$

where $g(\cdots, B, \cdots) = \int_B g(\cdots, x, \cdots)\, d\mu$. Then

$$H_i(A_1, \cdots, A_{n-1}) = \int_{A_1 \times \cdots \times A_{n-1}} h_i \, d\mu^{n-1}$$

and, by (5.6),

$$\int_C \{g(x_1, \cdots, x_n) - \sum_{i=1}^k \sum_{j=1}^n u_i(x_j) h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)\} d\mu^n = 0 \tag{5.7}$$

for all sets $C = A_1 \times \cdots \times A_n$ in $\mathscr{A}_0^n$.

Let $w(x_1, \cdots, x_n)$ denote the integrand in (5.7). The integral $J(C) = \int_C w \, d\mu^n$ is zero for $C \in \mathscr{A}_0^n$. Let B be a set in $\mathscr{A}_0$. By a standard argument, $J(E \cap B^n) = 0$ for all E in the σ-field $\mathscr{A}^{(n)}$ and hence $w(x_1, \cdots, x_n) = 0$ a.e. (μ^n) on B^n.

For every $P \in \mathscr{P}$ the set $B_\varepsilon = \{x : (dP/d\mu)(x) > \varepsilon\}$ is in $\mathscr{A}_0$ for all $\varepsilon > 0$. This implies $w(x_1, \cdots, x_n) = 0$ a.e. $(\mathscr{P}^{(n)})$, proving that g has the representation (3.3) a.e. $(\mathscr{P}^{(n)})$. The proof that the h_i, as here defined, satisfy the integrability conditions (3.2) is similar to the corresponding proof in Section 4. So is the proof in the case where the origin is not an inner point of the convex hull of $\mathscr{U}(\mu)$. This proves Theorem 1B.

6. Proof of Theorem 2B. Let the conditions of Theorem 1B be satisfied, and suppose that g is bounded while every nontrivial linear combination of $u_1, \cdots, u_k$ is $\mathscr{P}$-unbounded. We must show that $g(x_1, \cdots, x_n) = 0$ a.e. $(\mathscr{P}^{(n)})$.

We again assume that $c_1 = \cdots = c_k = 0$.

First it will be shown that there exists a measure ν on $(\mathscr{X}, \mathscr{A})$ which is (i) equivalent to the family $\mathscr{P}$, (ii) finite, and (iii) satisfies

$$\int |u_i| \, d\nu < \infty , \qquad i = 1, \cdots, k .$$

Since the family $\mathscr{P}$ is dominated by a σ-finite measure, it contains a countable equivalent subset (Halmos and Savage (1949), Lemma 7). Let the sequence $P_1, P_2, \cdots$ of distributions in $\mathscr{P}$ be equivalent to $\mathscr{P}$ (so that $P_j(A) = 0$ for all j implies $P(A) = 0$ for all P in $\mathscr{P}$). Let $d_j = \sum_{i=1}^k \int |u_i| \, dP_j$, $b_j = 2^{-j}(1 + d_j)^{-1}$, $\nu = \sum_{j=1}^\infty b_j P_j$. The numbers b_j are strictly positive and $\sum b_j$ is finite. Hence ν is a finite measure equivalent to $\mathscr{P}$. Also, $\sum_{i=1}^k \int |u_i| \, d\nu = \sum_{j=1}^\infty b_j d_j < \sum_{j=1}^\infty 2^{-j} < \infty$, so that ν satisfies conditions (i), (ii), (iii).

Since ν is equivalent to $\mathscr{P}$, we have that if u is a nontrivial linear combination of $u_1, \cdots, u_k$ then $\nu(|u(x)| > c) \neq 0$ for all real c. Let $\mathscr{A}_+$ denote the class of sets A in $\mathscr{A}$ such that $\nu(A) \neq 0$. For $A \in \mathscr{A}_+$ define the set functions $U_1, \cdots, U_k$ by

$$U_i(A) = \int_A u_i \, d\nu / \nu(A) , \qquad i = 1, \cdots, k .$$

Then every nontrivial linear combination of $U_1, \cdots, U_k$ is unbounded on $\mathscr{A}_+$. Hence there exist k sets $B_1, \cdots, B_k$ in $\mathscr{A}_+$ such that the matrix

$$\mathbf{U}_\nu = (\int_{B_1} \mathbf{u} \, d\nu, \cdots, \int_{B_k} \mathbf{u} \, d\nu)$$

is nonsingular.

By Theorem 1B, the conditions of Lemma 1B (last paragraph) are satisfied. ($\int |g|\, d\nu^n$ is finite since g is bounded.) Hence the representation (2.11) of $g(x_1, \cdots, x_n)$ holds a.e. $(\nu^{(n)})$. Let $A_1, \cdots, A_n$ be n sets in A_+. Integrating both sides of (2.11) over the product set $A_1 \times \cdots \times A_n$ in $\mathscr{A}_+{}^n$ with respect to ν^n, we obtain

$$G^\dagger(A_1, \cdots, A_n) = \sum_{m=0}^{n-1}(-1)^{n-m-1}T^\dagger_{n,m}(A_1, \cdots, A_n) \tag{6.1}$$

where

$$G^\dagger(A_1, \cdots, A_n) = \int_{A_1\times\cdots\times A_n} g\, d\nu^n / \prod_{j=1}^n \nu(A_j) ,$$

$$T^\dagger_{n,m}(A_1, \cdots, A_n) = \sum_{m,n-m} \sum_{i_1=1}^k \cdots \sum_{i_{n-m}=1}^k G^\dagger(A_{j_1}, \cdots, A_{j_m}, B_{i_1}, \cdots, B_{i_{n-m}}) V^\dagger_{i_1}(A_{j_{m+1}}) \cdots V^\dagger_{i_{n-m}}(A_{j_n}) ,$$

$$V_i{}^\dagger(A) = \nu(B_i) \int_A v_i\, d\nu/\nu(A) , \qquad \mathbf{v}(x) = \mathbf{U}_\nu{}^{-1}\mathbf{u}(x) .$$

The representation (6.1) of the set function $G^\dagger(A_1, \cdots, A_n)$ is strictly analogous to the representation (4.1) of $g(x_1, \cdots, x_n)$. Since g is bounded, $G^\dagger$ is bounded on $\mathscr{A}_+{}^n$, and the $V_i{}^\dagger(A)$ are unbounded on $\mathscr{A}_+$. Thus the proof of Theorem 2A implies that $G^\dagger(A_1, \cdots, A_n) = 0$ on $\mathscr{A}_+{}^n$. Therefore

$$\int_C g\, d\nu^n = 0$$

for all cylinder sets $C = A_1 \times \cdots \times A_n$ in $\mathscr{A}^n$. Hence $g(x_1, \cdots. x_n) = 0$ a.e. (ν^n), and thus a.e. $(\mathscr{P}^{(n)})$.

Note added in proof. Some extensions of the theorems of the present paper are considered in [7].

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