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ADMISSIBILITY RESULTS FOR GENERALIZED BAYES ESTIMATORS OF COORDINATES OF A LOCATION VECTOR[1]

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Let X be an n-dimensional random vector with density $f(x-\theta)$. It is desired to estimate θ_1, under a strictly convex loss $L(\delta-\theta_1)$. If F is a generalized Bayes prior density, the admissibility of the corresponding generalized Bayes estimator, δ_F, is considered.

An asymptotic approximation to δ_F is found. Using this approximation, it is shown that if (i) f has enough moments, (ii) L and F are smooth enough, and (iii) $F(\theta) \leqq K(|\theta_1| + \sum_{i=2}^n \theta_i^2)^{(3-n)/2}$, then δ_F is admissible for estimating θ_1. For example, assume that $F(\theta) \equiv 1$ and that L is squared error loss. Under appropriate conditions it can be shown that $\delta_F(x) = x_1$, and that δ_F is the best invariant estimator. If, in addition, f has 7 absolute moments and $n \leqq 3$, it can be concluded that δ_F is admissible.

1. Introduction.

1.1. *Summary of results.* Considerable study has been given to the question of admissibility of estimators of location vectors. The results obtained have, for the most part, dealt solely with estimating the full location vector. The question of the admissibility of estimators of coordinates of a location vector has been long outstanding.

In this paper, admissibility results are developed for generalized Bayes estimators of one coordinate of a location vector. The paper deals with a wide class of loss functions, densities and generalized priors. The analysis in this general setting is accomplished by using a new and powerful method of admissibility analysis, developed by Brown (1974c). A heuristic discussion of this method (as it applies to the problem in this paper) is given in Section 1.4, after the necessary notation has been developed. For a more comprehensive discussion of this method, see Brown (1974c). In Berger (1974), the "other half" of the problem is considered; namely, the question of when a generalized Bayes estimator of one coordinate of a location vector is inadmissible.

It should be noted that the problem considered here has recently been partially answered by Portnoy (1975). He has obtained results for the best invariant estimator, using squared error loss, and for a class of distributions with mass on the $(n-1)$ planes $x_1 = \pm m$ (where x_1 is the first coordinate of the random variable and m is an arbitrary integer).

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Section 1.2 develops the necessary notation, and clearly defines the problem. In Section 1.3, the problem is reduced to a relatively simple canonical form, which is dealt with in the remainder of the paper. Section 1.5 summarizes the various assumptions and results. In Chapter 2, approximations to generalized Bayes estimators are developed. These may be of some intrinsic interest. Chapter 3 deals with the major admissibility results. Chapter 4 discusses possible generalizations.

1.2. *Preliminaries.* Let $X = (X_1, X_2, \cdots, X_n)$ be an n-dimensional random variable with density $f(x - \theta)$ with respect to Lebesgue measure ($\theta \in R^n$). Let F be a bounded generalized prior density w.r.t. Lebesgue measure (i.e. $0 \leqq F(\theta) \leqq B < \infty$, while possibly $\int F(\theta)\, d\theta = \infty$).

It is desired to estimate θ_1, under the strictly convex loss $L(\delta - \theta_1)$. (The more general problem of estimating a linear combination of the θ_i can be reduced to the above problem by a simple linear transformation.) Assume that L is nonnegative and that $L(0) = 0$.

For convenience, the notation $h^{(i)}(x) = (\partial/\partial x_i)h(x)$, $h^{(i,j)}(x) = (\partial^2/\partial x_i\, \partial x_j)h(x)$, etc., will be adopted for any function h with the appropriate number of derivatives. (For simplicity in stating assumptions, let $h^{(0,j)} = h^{(j)}$, etc.) Since L is a function on R^1, derivatives of L will be denoted by $L'(y) = (d/dy)L(y)$, $L''(y) = (d^2/dy^2)L(y)$, and $L'''(y) = (d^3/dy^3)L(y)$.

For convenience, K will be used as a generic constant. E_θ will stand for the expectation under θ. Let $|\xi|$ denote the usual Euclidean norm of the vector ξ. If $x \in R^n$, define $||x|| = |x_1| + \sum_{i=2}^n x_i^2$. Note that this is not a norm. Considerable subscripting will be saved by this somewhat unusual notation, however. Finally, define the differential operators $\mathscr{D}^*_{r_1,r_2}$ and $\bar{\mathscr{D}}$, on twice differentiable functions $G: R^n \to R^1$, by

$$\mathscr{D}^*_{r_1,r_2} G(x) = -[G^{(1)}(x) + \tfrac{1}{2} \textstyle\sum_{i=2}^{(r_1+1)} G^{(i,i)}(x) - \tfrac{1}{2} \sum_{i=(r_1+2)}^{(r_1+r_2+1)} G^{(i,i)}(x)]\,, \qquad r_1 + r_2 + 1 \leqq n\,,$$

$$\bar{\mathscr{D}} G(x) = |G^{(1)}(x)| + \tfrac{1}{2} \textstyle\sum_{i=2}^{n} |G^{(i,i)}(x)|\,.$$

When clear from context, the subscripts r_1 and r_2 will be dropped from $\mathscr{D}^*_{r_1,r_2}$.

The following conditions on L, f and F will be needed throughout the paper:

(i) All third order derivatives of L and F exist.

(ii) For every $c \in R^1$, $E_0 L(X_1 + c) < \infty$ and $E_0|L'(X_1 + c)| < \infty$.

(iii) $E_0 L'(X_1) = 0$.

(iv) $\int f(x - \theta)F(\theta)\, d\theta > 0$, for every $x \in R^n$.

(If no region of integration for an integral is given, it is to be understood to be R^n.) Assumption (iii) can really be made without loss of generality. Using (i), (ii) and the convexity of L, it is easy to see that there exists a number c for which $E_0 L'(X_1 + c) = 0$. A simple translation of the density f will now ensure that (iii) is satisfied, while leaving admissibility considerations unchanged.

Denote the generalized Bayes estimator of θ, with respect to F, by δ_F. Under the above assumptions, it can easily be checked that δ_F satisfies $\int L'(\delta_F(x) - \theta_1)f(x-\theta)F(\theta)\,d\theta = 0$, and is unique.

For an estimator δ, define $\gamma(x) = \delta(x) - x_1$. Thus, for example, $\gamma_F(x) = \delta_F(x) - x_1$.

As usual, define the risk of an estimator δ by $R(\delta, \theta) = \int L(\delta(x) - \theta_1)f(x - \theta)\,dx$. (Only estimators for which the risk is defined and finite for every θ will be considered.) Also, define $\Delta_\delta^F(\theta) = R(\delta_F, \theta) - R(\delta, \theta)$. The estimator δ_F is said to be admissible if $\Delta_\delta^F(\theta) \geqq 0$ for all θ, implies that $\Delta_\delta^F(\theta) \equiv 0$. Thus δ_F is inadmissible if there exists an estimator δ such that $\Delta_\delta^F(\theta) \geqq 0$ for all θ, with strict inequality for some θ. Finally, δ_F is said to be admissible with respect to F if $\Delta_\delta^F(\theta) \geqq 0$ for all θ, implies that $\Delta_\delta^F(\theta) = 0$ for θ in the support of F.

The following quantities play a crucial and relatively unheralded role in questions of admissibility. Define $b = E_0 L''(X_1)$, and $m_{j(1),j(2),\cdots,j(k)} = E_0[(\prod_{i=1}^k X_{j(i)})L'(X_1)]$. These quantities will be called the "moment structure" of the problem.

1.3. *Reduction to canonical form.* The purpose of this section is to reduce the moment structure to a relatively simple canonical form. Consider first the terms $m_i = E_0[X_i L'(X_1)]$. It is clear that $m_1 > 0$ (since L is strictly convex). Multiplying L by a constant does not affect admissibility considerations. Hence, assume $m_1 = 1$. Finally, consider the linearly transformed problem defined by $Y_1 = X_1$, $Y_i = X_i - m_i X_1$ for $i \geqq 2$. It is easy to check for this transformed problem (i.e. the Y problem with induced transformations on L, f, and F) that

$$m_1 = 1\,, \qquad \text{and} \qquad m_i = 0 \quad \text{for} \quad i \geqq 2\,. \tag{1.3.1}$$

Clearly, admissibility in the transformed problem is equivalent to admissibility in the original problem, since all that has been done is a change of variables in the integrals for the risks.

The reduction must be carried one step farther. (We revert to the usual (X, L, f, F) notation, but assume (1.3.1) holds.) Let M be the $(n-1) \times (n-1)$ matrix with (i, j) element $m_{i+1,j+1} = E_0[X_{i+1}X_{j+1}L'(X_1)]$, $1 \leqq i \leqq n-1$ and $1 \leqq j \leqq n-1$. It is clear that M is a symmetric matrix. Hence, there exists a non-singular $(n-1) \times (n-1)$ matrix P for which PMP^t is a diagonal matrix with diagonal elements $d_i = -1$ for $1 \leqq i \leqq r_1$, $d_i = 1$ for $r_1 + 1 \leqq i \leqq r_1 + r_2$, and $d_i = 0$ for $r_1 + r_2 + 1 \leqq i \leqq n - 1$. Let Q be the $n \times n$ matrix with elements $q_{1,1} = 1$, $q_{i,1} = q_{1,i} = 0$ for $2 \leqq i \leqq n$, and $q_{i,j} = p_{(j-1),(i-1)}$ for $2 \leqq i \leqq n$ and $2 \leqq j \leqq n$. (The $p_{i,j}$ are, of course, the elements of P.) Consider the transformed problem defined by $Y = XQ$. It is straightforward to check that for the transformed problem

$$\begin{aligned} m_{i,j} &= -1 \quad &&\text{if} \quad 2 \leqq i = j \leqq r_1 + 1 \\ &= 1 \quad &&\text{if} \quad r_1 + 2 \leqq i = j \leqq r_1 + r_2 + 1 \\ &= 0 \quad &&\text{otherwise, for} \quad 2 \leqq i \quad \text{and} \quad 2 \leqq j\,. \end{aligned} \tag{1.3.2}$$

Without loss of generality, it can thus be assumed that the moment structure of the original problem satisfies (1.3.1) and (1.3.2).

1.4. *Heuristic approach to results.* As the technical aspects of the theoretical development are somewhat involved, it seems desirable to briefly sketch the important arguments. The main ideas are patterned after Brown (1974c), to which the reader is referred for further discussion.

The first item of importance is to obtain an approximation to γ_F. Chapter 2 deals with this problem. The basic idea used is as follows.

Under the assumptions of Section 1.2,

$$\int L'(\gamma_F(x) + x_1 - \theta_1) f(x - \theta) F(\theta) \, d\theta = 0 \, . \tag{1.4.1}$$

Expanding $L'(\gamma_F(x) + x_1 - \theta_1)$ in a Taylor series about $(x_1 - \theta_1)$ gives

$$L'(\gamma_F(x) + x_1 - \theta_1) = L'(x_1 - \theta_1) + \gamma_F(x) L''(x_1 - \theta_1) + \text{remainder.} \tag{1.4.2}$$

If $F(x)$ is "smooth" and "flat" in the neighborhood of x, it can be shown that $\gamma_F(x)$ is very small, and that the remainder term in (1.4.2) is $o(\gamma_F(x))$. Thus, ignoring the remainder and using (1.4.1) and (1.4.2) gives

$$\gamma_F(x) \cong \frac{-\int L'(x_1 - \theta_1) f(x - \theta) F(\theta) \, d\theta}{\int L''(x_1 - \theta_1) f(x - \theta) F(\theta) \, d\theta} \, . \tag{1.4.3}$$

Next, expand $F(\theta)$ in a Taylor series about x (up to third order terms). Again, it will be possible to ignore the remainder term. Using the assumptions (1.3.1), (1.3.2) and $E_0 L'(X_1) = 0$, it is easy to see that

$$\int L'(x_1 - \theta_1) f(x - \theta) F(\theta) \, d\theta \cong \mathscr{D}^*_{r_1, r_2} F(x) \, . \tag{1.4.4}$$

(The other third order terms of the Taylor expansion can be ignored under appropriate conditions.)

Similarly,

$$\int L''(x_1 - \theta_1) f(x - \theta) F(\theta) \, d\theta \cong bF(x) \, . \tag{1.4.5}$$

Combining (1.4.3), (1.4.4) and (1.4.5) gives

$$\gamma_F(x) \cong -\mathscr{D}^* F(x) / (bF(x)) \, . \tag{1.4.6}$$

This is the desired approximation. Again, it will be valid whenever F is appropriately "smooth" and "flat" near x.

In Chapter 3, conditions are given under which δ_F is admissible. The proof makes use of a standard statistical argument for proving admissibility. A sequence of bounded, finite mass priors, g_R, is found, such that $\lim_{R \to \infty} g_R(\theta) = 1$ for every $\theta \in R^n$, and such that $\lim_{R \to \infty} \int \Delta^F_{\delta_R}(\theta) F(\theta) g_R(\theta) \, d\theta = 0$. (Here δ_R is the generalized Bayes estimator for $(F g_R)$.) It can then be concluded that δ_F is admissible w.r.t. F. The analysis proceeds as follows.

For simplicity, consider the case $F(\theta) \equiv 1$. (Thus the admissibility of the best invariant estimator is being investigated.) Expanding $L(\delta_R(x) - \theta_1) = L(\gamma_R(x) + x_1 - \theta_1)$ in a Taylor series about $(x_1 - \theta_1)$, gives for appropriate g_R

$$\begin{aligned} \Delta^F_{\delta_R}(\theta) &= R(\delta_F, \theta) - R(\delta_R, \theta) \\ &= \int [L(x_1 - \theta_1) - L(\delta_R(x) - \theta_1)] f(x - \theta)\, dx \\ &\cong \int -[\gamma_R(x) L'(x_1 - \theta_1) + \tfrac{1}{2}\gamma_R^2(x) L''(x_1 - \theta_1)] f(x - \theta)\, dx \,. \end{aligned} \tag{1.4.7}$$

Interchanging orders of integration (valid since the g_R are finite priors), and using (1.4.7) gives

$$\begin{aligned} \int \Delta^F_{\delta_R}(\theta) g_R(\theta)\, d\theta \cong & -\int \gamma_R(x) \int L'(x_1 - \theta_1) f(x - \theta) g_R(\theta)\, d\theta\, dx \\ & - \int \tfrac{1}{2}[\gamma_R(x)]^2 \int L''(x_1 - \theta_1) f(x - \theta) g_R(\theta)\, d\theta\, dx \,. \end{aligned} \tag{1.4.8}$$

Using (1.4.4), (1.4.5) and (1.4.6), with F replaced by g_R, thus gives

$$\int \Delta^F_{\delta_R}(\theta) g_R(\theta)\, d\theta \cong \int ([\mathscr{D}^* g_R(x)]^2/[2b g_R(x)])\, dx \,.$$

To complete the admissibility argument, a sequence of finite priors, g_R, must be found, such that $\lim_{R\to\infty} g_R(\theta) = 1$ for every θ, and such that

$$\lim_{R\to\infty} \int ([\mathscr{D}^* g_R(x)]^2/[2b g_R(x)])\, dx = 0 \,. \tag{1.4.9}$$

Such a sequence is given in Chapter 3 (where also the details of the argument are filled in).

1.5. *Summary of assumptions, results and examples.* The assumptions that will be needed for the results are first discussed. The conditions given are not the most general possible, but they do include a wide variety of interesting cases. Furthermore, their "relative" simplicity and ease of verification make them desirable.

For use in the first assumption, let Ω be a subset of R^n, and define $d(x) = \inf_{\xi \in \Omega^c} |x - \xi|$. (Define $d(x) = |x|$ if $\Omega = R^n$.) Thus $d(x)$ is the distance from x to Ω^c. The region Ω will be said to be * *unbounded* if $\sup_{x\in\Omega} d(x) = \infty$. Thus if Ω is * unbounded, it is possible to get arbitrarily far from Ω^c. For notational convenience, the following modification of "o" notation will be adopted:

$$r(x) \quad \text{is} \quad \bar{o}(1)\,, \quad \text{if} \quad \lim_{T\to\infty} \sup_{\{x: d(x) > T\}} |r(x)| = 0 \,.$$

ASSUMPTION 1. (Conditions on the generalized prior F.)

(i) F is absolutely continuous w.r.t. Lebesgue measure.

(ii) $0 \leqq F(\theta) \leqq B < \infty$.

(iii) There exists a * unbounded region Ω, such that if $x \in \Omega$, then the following conditions hold:

(a) $F(x) > 0$.

(b) F has continuous third order partial derivatives at x.

(c) $|F^{(i)}(x)| + |F^{(i,j)}(x)| + |F^{(i,j,k)}(x)| = \bar{o}(1) F(x)$.
$|F^{(1,j)}(x)| + |F^{(i,j,k)}(x)| = \bar{o}(1) \bar{\mathscr{D}} F(x)$.
$|F^{(i,j)}(x)| = \bar{o}(1) \sum_{i=1}^n |F^{(i)}(x)|$.

(d) There exists a positive increasing function $h^*: R^1 \to R^1$, for which the following hold:

1. $[d - h^*(d)]$ is increasing in d and positive for some d.
2. There exists $q > 0$ such that $[h(x)]^{-q} = \bar{o}(1)F(x)$, where $h(x) = h^*(d(x))$.
3. If $d(x) - h(x) > 0$, then for $i \geqq 0$, $j \geqq 0$, $k \geqq 1$, $\sup_{\{\xi : |\xi| < h(x)\}} |F^{(i,j,k)}(x + \xi)| \leqq K|F^{(i,j,k)}(x)|$.

ASSUMPTION 2. (Conditions on f and L.)

(i) L is strictly convex, $L \geqq 0$ and $L(0) = 0$.
(ii) The third derivative of L exists.
(iii) $\int f(x - \theta)F(\theta)\, d\theta > 0$ for every $x \in R^n$.
(iv) $E_0[|X|^5 L(X_1)] < \infty$, $E_0|X|^\alpha < \infty$, and $E_0[|X|^\alpha L^{<i>}(X_1)] < \infty$, where $\alpha = \max(q + 4, 5)$, $L^{<i>}$ is the ith derivative of L, $1 \leqq i \leqq 3$, and q is from Assumption 1 (iii) d. 2.
(v) If $|\xi| < D$, then there exist $K_D > 0$ and $C_D > 0$ such that $|L^{<i>}(x_1 + \xi)| \leqq K_D|L^{<i>}(x_1)| + C_D$, $0 \leqq i \leqq 3$.
(vi) $E_0 L'(X_1) = 0$.
(vii) (1.3.1) and (1.3.2) hold.

DISCUSSION. Though numerous, these assumptions are often quite easy to verify. Assumption 1 (iii) c is a "flatness" assumption, guaranteeing that lower order derivatives "dominate" higher order derivatives. Assumption 1 (iii) d is mainly technical, but it is usually quite easy to check. For example, if $F(x) = (1 + |x|^2)^{-1}$, choose $\Omega = R^n$ (so $d(x) = |x|$), $h(x) = |x|^{\frac{3}{4}}$, and $q = 3$. The verification of 1 (iii) d is then trivial.

The need for so many moments in Assumption 2 (iv) arises mainly for technical reasons. For a rough idea of how many moments are needed, note that if $L(x_1) = x_1^2$ and $F(\theta) \equiv 1$, it is possible to choose h so that $q = 1$. Hence, f must have 7 absolute moments. Assumption 2 (v) is mainly technical, but again is usually easy to verify. Recall that 2 (vi) and 2 (vii) can really be made without loss of generality, since the problem can be transformed to ensure their validity.

Two particular regions Ω are of interest:

(1) $\Omega_1 = R^n - \mathscr{K}$, where $\mathscr{K}$ is a compact set. Note that in Ω_1, if $|x| > T$, then $d(x) > T - c$, where $c = \sup_{\xi \in \mathscr{K}} |\xi|$. Hence $d(x)$ and $|x|$ are equivalent for asymptotic purposes. Chapter 3 will deal entirely with this region.

(2) $\Omega_2 = \{x \in R^n : |x_1| > c\}$. This region is of interest when the prior $F(\theta)$ depends only on θ_1. (Recall it is θ_1 that is to be estimated.) Again, it is easy to see that for asymptotic purposes, $|x_1|$ is equivalent to $d(x)$.

In Chapter 2, the following result is established.

THEOREM A. *Assume Assumptions* 1 *and* 2 *hold. Assume also that the following* *strengthening of* 1 (iii) d. 2 *holds*: *there exists* $q > 0$, *such that* $[h(x)]^{-q} = \bar{o}(1)\mathscr{D}F(x)$. *Then*,

$$\gamma_F(x) = \frac{-\mathscr{D}^*F(x)}{bF(x)} + \varepsilon(x)\,, \qquad \text{where} \quad |\varepsilon(x)| = \frac{\bar{o}(1)\bar{\mathscr{D}}F(x)}{bF(x)}\,.$$

COMMENTS. This asymptotic approximation to γ_F is important in admissiblity studies. Besides its use in this paper, it is needed to verify an inadmissibility condition in Berger (1974).

As an example of the application of this theorem, let

$$\begin{aligned} F(\theta) &= 1 && \text{if} \quad |\theta_1| \leqq 1 \\ &= |\theta_1|^{-a} && \text{if} \quad |\theta_1| > 1\,,\ a > 0\,. \end{aligned}$$

It is easy to check that Assumption 1 can be satisfied with $\Omega = \Omega_2$. $(c = 1)$, $h^*(d) = d/2$ and $q > a$. To satisfy the additional assumption of Theorem A, choose $q = a + 1 + \varepsilon$. If f and L are such that Assumption 2 is valid, Theorem A thus gives

$$\gamma_F(x) = \frac{-\mathscr{D}^*F(x)}{bF(x)} + \bar{o}(1)\frac{\bar{\mathscr{D}}F(x)}{bF(x)} = \frac{-a}{bx_1}(1 + \bar{o}(1))\,.$$

The following additional assumption is needed to prove the admissibility results.

ASSUMPTION 3. (Further conditions on F.)

(i) F is such that $\Omega = \Omega_1$ can be chosen in Assumption 1.

(ii) There exist positive numbers q, T and C_1, such that if $|x| > T$, then $[h(x)]^{-q} \leqq C_1 F(x)||x||^{-\frac{1}{2}}$. (Here $h(x)$ is as in Assumption 1.) Note that this condition is stronger than 1 (iii) d. 2. Thus the q found to satisfy this assumption will be the q used in Assumptions 1 and 2.

(iii) There exist $T > 0$ and $C_2 > 0$, such that if $|x| > T$, then

(a) $|F^{(i)}(x)|/F(x) \leqq C_2||x||^{-\frac{1}{2}}$.

(b) $F(x) \leqq C_2||x||^{(3-n)/2}$.

DISCUSSION. Assumptions 3 (i), 3 (ii) and 3 (iii) a are mainly technical and are quite easy to verify. Note that q should be chosen as small as possible in 3 (ii), in order to make the number moments needed in Assumption 2 (iv) as small as possible. Observe, also, that the faster F decreases in the tails, the larger q must be, and hence the larger the number of moments needed by Assumption 2. This is due to the method of analysis; it is easier to approximate a flat F than a quickly decreasing F. In fact, for many finite priors F, such as those with exponential tails, the theory will not even apply. (Of course it is already known that such F give admissible estimators.)

Assumption 3 (iii) b is the central condition on F, necessary to ensure admissibility. In Berger (1974) it is shown that if F is much bigger than the given bound, then δ_F is inadmissible.

Chapter 3 proves the following theorem.

THEOREM B. *Under Assumptions* 1, 2 *and* 3, δ_F *is admissible with respect to* F. *If* $F(\theta) > 0$ *everywhere or* $f(x) > 0$ *everywhere, then* δ_F *is admissible.*

An application of Theorem B is next given, in order to demonstrate the applicability to the assumptions, and to give the reader a more concrete situation to think of.

Assume $L(x_1) = |x_1|^a$, where $a = 2$ or $a \geqq 3$. Consider the generalized prior $F(\theta) = (1 + |\theta|^r)^{-1}$, $r \geqq 0$. Assumptions 1 (i) and 1 (ii) are clearly satisfied. For 1 (iii), let $\Omega = R^n - \{0\}$ and $d(x) = |x|$. It is then trivial to verify a, b and c of Assumption 1 (iii). Choose $h(x) = |x|/2$. Assumptions 1 (iii) d. 1 and 1 (iii) d. 3 are then easy to check. Assumption 1 (iii) d. 2 will be verified when 3 (ii) is. Assumption 2 is easy to check for the above L and F. 2 (iv) says that max $(5 + a, q + 3 + a)$ moments of f are needed.

Finally, consider Assumption 3. Assumption 3 (i) is clearly satisfied. Using the obvious fact that $||x||^{\frac{1}{2}} < |x| + 1 < 2|x|$ for $|x| > 1$, it is easy to see that if $|x| > 1$, then

$$[h(x)]^{-(r+1)} = 2^{(r+1)}|x|^{-(r+1)} \leqq 2^{(r+3)}(|x|^r + 1)^{-1}||x||^{-\frac{1}{2}}\,.$$

Hence, choosing $C_1 = 2^{(r+3)}$, $T = 1$, and $q = r + 1$, Assumption 3 (ii) is verified. A similar calculation will verify Assumption 3 (iii) a. Finally, it is necessary to check 3 (iii) b. If $n \leqq 3$, F clearly satisfies the assumption. If $n > 3$ and $|x| > 1$, then $|x|^{-(n-3)} < 2^{(n-3)}||x||^{(3-n)/2}$. Thus 3 (iii) b is satisfied if $n - 3 \leqq r$. Theorem B thus implies

COROLLARY B1. *Assume*

(i) $L(x_1) = |x_1|^a$, *where* $a = 2$, *or* $a \geqq 3$.
(ii) $F(\theta) = (1 + |\theta|^r)^{-1}$, *where* $r \geqq 0$.
(iii) f *has* max $(5 + a, r + 4 + a)$ *absolute moments.*
(iv) $n \leqq r + 3$.

Then the generalized Bayes estimator, δ_F, *is admissible for estimating* θ_1.

COROLLARY B2. *For squared error loss, the best invariant estimator of* θ_1 *is admissible if* f *has* 7 *absolute moments and* $n \leqq 3$.

PROOF. Obvious, noting that $a = 2$ and $r = 0$. □

2. Generalized Bayes estimators.

2.1. *General result.* In this chapter, an approximation to generalized Bayes estimators is developed. The results will be applied to the priors (Fg_R), as well as to F. Hence, for this section, consider an arbitrary generalized prior density H on R^n. If $\delta_H(x) = \gamma_H(x) + x_1$ is the generalized Bayes estimator for θ_1, it is desired to find an approximation to $[\gamma_H(x) - a(x)]$, where $a(x)$ is a measurable function such that $|a(x)| < A < \infty$. The approximation will be established locally, at a fixed point x. For this section, assume only that Assumption 2 holds and that $\delta_H(x)$ satisfies

$$\int L'(\gamma_H(x) + x_1 - \theta_1) f(x - \theta) H(\theta)\, d\theta = 0 . \tag{2.1.1}$$

Expanding $L'(\gamma_H(x) + x_1 - \theta_1)$ in a Taylor expansion about $[a(x) + x_1 - \theta_1]$ gives

$$\begin{aligned} L'(\gamma_H(x) + x_1 - \theta_1) = L'(a(x) + x_1 - \theta_1) + [\gamma_H(x) - a(x)]L''(a(x) \\ + x_1 - \theta_1) + \mathscr{R}(\gamma_H(x), x_1, \theta_1) , \end{aligned} \tag{2.1.2}$$

where

$$\mathscr{R}(\xi, x_1, \theta_1) = \int_{(a(x)+x_1-\theta_1)}^{(\xi+x_1-\theta_1)} L'''(\eta)[\xi + x_1 - \theta_1 - \eta]\, d\eta .$$

Define

$$\begin{aligned} U_H(x) &= \int L'(a(x) + x_1 - \theta_1) f(x - \theta) H(\theta)\, d\theta , \\ V_H(x) &= \int L''(a(x) + x_1 - \theta_1) f(x - \theta) H(\theta)\, d\theta , \\ W_H(\xi, x) &= \int \mathscr{R}(\xi, x_1, \theta_1) f(x - \theta) H(\theta)\, d\theta . \end{aligned}$$

Note that $V_H(x) \geqq 0$. Combining (2.1.1) and (2.1.2) gives

$$U_H(x) + [\gamma_H(x) - a(x)]V_H(x) + W_H(\gamma_H(x), x) = 0 . \tag{2.1.3}$$

LEMMA 2.1.1. *If* $|\xi - a(x)| < 1$, *then* $W_H(\xi, x)$ *is continuous in* ξ.

PROOF. Straightforward, using Assumptions 2 (iv) and 2 (v), the boundedness of $a(x)$ and the dominated convergence theorem. □

THEOREM 1. *Assume* $V_H(x) > 0$. *Assume also that there exists* $C > 1$, *such that if* $|\xi| < A + 1$, *then*

(a) $\int |L'''(\xi + x_1 - \theta_1)| f(x - \theta) H(\theta)\, d\theta < C V_H(x)/2$, *and*

(b) $|U_H(x)|/V_H(x) < 1/(10C)$.

Under these assumptions, it can be concluded that

$$\begin{aligned} &[\gamma_H(x) - a(x)] \\ &\quad = -(1 + \varepsilon(x)) \frac{U_H(x)}{V_H(x)} , \qquad \textit{where} \quad |\varepsilon(x)| < \frac{C|U_H(x)|}{V_H(x)} < \tfrac{1}{10} . \end{aligned} \tag{2.1.4}$$

PROOF. In this and following proofs, the notational dependence of U_H, V_H and W_H on the prior is dropped. Recall, x is considered fixed. Define

$$r(\varepsilon) = -(1 + \varepsilon)U(x)/V(x) + a(x) .$$

Clearly, (2.1.4) will be formally satisfied if ε can be found so that $r(\varepsilon) = \gamma_H(x)$. Using (2.1.3), this is equivalent to finding ε for which

$$W(r(\varepsilon), x) = \varepsilon U(x) . \tag{2.1.5}$$

Consider first the case $U(x) = 0$. Clearly $\varepsilon = 0$ is then a solution to (2.1.5), since $r(0) = a(x)$ and $W(a(x), x) = 0$.

If $U(x) \neq 0$, a solution ε must be found to

$$W(r(\varepsilon), x)/U(x) = \varepsilon . \tag{2.1.6}$$

Note first that

$$(2.1.7) \qquad |\mathscr{R}(\xi, x_1, \theta_1)| \leq \int_{(a(x)+x_1-\theta_1)}^{(\xi+x_1-\theta_1)} |L'''(\eta)|[\xi + x_1 - \theta_1 - \eta]\, d\eta$$
$$\leq (\xi - a(x)) \int_{(a(x)+x_1-\theta_1)}^{(\xi+x_1-\theta_1)} |L'''(\eta)|\, d\eta\,.$$

Using Assumption (b), it is clear that $|r(\varepsilon)| < (1+\varepsilon)|U(x)|/V(x) + |a(x)| < 1 + A$. This, together with the definition of W, (2.1.7), and Assumption (a) gives

$$\begin{aligned} |W(r(\varepsilon), x)| &\leq [r(\varepsilon) - a(x)] \iint_{(a(x)+x_1-\theta_1)}^{(r(\varepsilon)+x_1-\theta_1)} |L'''(\eta)| f(x-\theta)H(\theta)\, d\eta\, d\theta \\ &\leq |r(\varepsilon) - a(x)|\, |\textstyle\int_{a(x)}^{r(\varepsilon)} (CV(x)/2)\, d\eta| \\ &= C|r(\varepsilon) - a(x)|^2 V(x)/2 \\ &= C(1+\varepsilon)^2 |U(x)|^2/(2V(x))\,. \end{aligned}$$

Assume for the moment that $|\varepsilon| < \frac{1}{10}$. Then

$$|W(r(\varepsilon), x)|/|U(x)| \leq C(1+\varepsilon)^2|U(x)|/(2V(x)) \leq (.7)C|U(x)|/V(x)\,.$$

Thus the left-hand side of (2.1.6) has the above bound. By Lemma 2.1.1, it is continuous in $r(\varepsilon)$ and hence in ε. But then for some ε, where $|\varepsilon| < C|U(x)|/V(x)$, (2.1.6) must be satisfied. By Assumption (b), $|\varepsilon| < C|U(x)|/V(x) < \frac{1}{10}$. Hence the above result is valid and the proof is complete. □

2.2. *Asymptotic approximation of* δ_F. Theorem 1 will be very important in the admissibility proof. In this section a more direct application of the theorem is considered; namely, proving Theorem A (see Section 1.5). The result obtained will itself be essential in proving the main admissibility theorem.

Throughout this section, assume Assumptions 1 and 2 of Section 1.5 hold. Section 2.1 will be applied with $H = F$ and $a(x) = 0$. Thus,

$$\begin{aligned} U_F(x) &= \int L'(x_1 - \theta_1) f(x-\theta)F(\theta)\, d\theta\,, \\ V_F(x) &= \int L''(x_1 - \theta_1) f(x-\theta)F(\theta)\, d\theta\,, \\ W_F(\xi, x) &= \int \left(\int_{(x_1-\theta_1)}^{(\xi+x_1-\theta_1)} L'''(\eta)[\xi + x_1 - \theta_1 - \eta]\, d\eta\right) f(x-\theta)F(\theta)\, d\theta\,. \end{aligned}$$

LEMMA 2.2.1. *There exists* $T > 0$, *such that if* $d(x) > T$, *then*

$$U_F(x) = \mathscr{D}^*F(x) + \varepsilon(x)\,, \qquad \textit{where} \quad |\varepsilon(x)| < \bar{o}(1)\bar{\mathscr{D}}F(x) + K[h(x)]^{-q}\,.$$

PROOF. Let $Q = \{\theta : |x - \theta| < h(x)\}$. Clearly

$$(2.2.1) \qquad U(x) = \int_Q L'(x_1 - \theta_1) f(x-\theta)F(\theta)\, d\theta + \int_{Q^c} L'(x_1 - \theta_1) f(x-\theta)F(\theta)\, d\theta\,.$$

A simple Chebyshev argument and Assumption 2 (iv) give

$$|\textstyle\int_{Q^c} L'(x_1 - \theta_1) f(x-\theta)F(\theta)\, d\theta|$$
$$\leq B[h(x)]^{-q} \int |x-\theta|^q |L'(x_1 - \theta_1)| f(x-\theta)\, d\theta \leq KB[h(x)]^{-q}\,.$$

Thus it is only necessary to consider the integral in (2.2.1) over Q. Now if $|x - \theta| < h(x)$, then $d(\theta) > d(x) - h(x)$. (Recall $d(x)$ is just the distance from x to Ω^c.) By Assumption 1 (iii)d, it is clear that there exists a T such that if

$d(x) > T$, then $d(x) - h(x) > 0$. Hence, $d(\theta) > d(x) - h(x) > 0$ and $\theta \in \Omega$. Since this holds for every $\theta \in Q$, it is clear that $F(\theta)$ can be expanded in a Taylor expansion about x. Using the expansion (up to fourth order terms) gives

$$\int_Q L'(x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta = I_1 + I_2 ,$$
$$I_1 = \int_Q L'(x_1 - \theta_1) f(x - \theta)[F(x) + \sum_{i=1}^n (\theta_i - x_i) F^{(i)}(x) + \tfrac{1}{2} \sum_i \sum_j (\theta_i - x_i)(\theta_j - x_j) F^{(i,j)}(x)]\, d\theta ,$$
$$I_2 = \int_Q L'(x_1 - \theta_1) f(x - \theta)[\tfrac{1}{6} \sum_i \sum_j \sum_k (\theta_i - x_i)(\theta_j - x_j)(\theta_k - x_k) \times F^{(i,j,k)}(t(x, \theta)x + (1 - t(x, \theta))\theta)]\, d\theta ,$$

where $t(x, \theta)$ is a measurable function such that $0 \leqq t(x, \theta) \leqq 1$. Let φ denote the expression within the brackets in the definition of I_1. Clearly

$$I_1 = \int L'(x_1 - \theta_1) f(x - \theta)[\varphi]\, d\theta - \int_{Q^c} L'(x_1 - \theta_1) f(x - \theta)[\varphi]\, d\theta . \tag{2.2.2}$$

Using Assumptions 1 (iii)c, 2 (iv), 2 (vi), and 2 (vii), it is clear that

$$\begin{aligned} &\int L'(x_1 - \theta_1) f(x - \theta)[\varphi]\, d\theta \\ &\quad = \mathscr{D}^* F(x) + \int L'(x_1 - \theta_1) f(x - \theta)(\theta_1 - x_1) \\ &\qquad \times [\tfrac{1}{2}(\theta_1 - x_1) F^{(1,1)}(x) + \sum_{i=2}^n (\theta_i - x_i) F^{(1,i)}(x)]\, d\theta \\ &\quad = \mathscr{D}^* F(x) + \bar{o}(1) \bar{\mathscr{D}} F(x) . \end{aligned}$$

The second integral in (2.2.2) is bounded by $KB[h(x)]^{-q}$ using another Chebyshev argument and Assumption 1 (iii)c.

Finally, consider the integral I_2. By Assumption 1 (iii)d. 3, which is applicable since $d(x) - h(x) > 0$,

$$\sup_{\theta \in Q} |F^{(i,j,k)}(tx + (1 - t)\theta)| = \sup_{\{\xi : |\xi| < h(x)\}} |F^{(i,j,k)}(x + \xi)| \leqq K |F^{(i,j,k)}(x)| .$$

Assumptions 1 (iii) c and 2 (iv) can thus be used to conclude that $I_2 = \bar{o}(1) \bar{\mathscr{D}} F(x)$. □

LEMMA 2.2.2. *There exists $T > 0$, such that if $d(x) > T$, then*

$$V_F(x) = bF(x)(1 + \bar{o}(1)) .$$

PROOF. By an argument similar to that in Lemma 2.2.1, it can be shown that $V_F(x) = bF(x)(1 + \bar{o}(1)) + K[h(x)]^{-q}$. ($F(x)$ will be the dominant term in the Taylor expansion, since its coefficient $b = E_0 L''(X_1) > 0$.) Assumption 1 (iii)d 2 can be applied to complete the proof. □

LEMMA 2.2.3. *There exists $T > 0$, such that if $d(x) > T$ and $|c| < 1$, then*

$$\int |L'''(c + x_1 - \theta_1)| f(x - \theta) F(\theta)\, d\theta \leqq K_1 V_F(x) .$$

PROOF. As in Lemma 2.2.2, it can be shown that $\int |L'''(c + x_1 - \theta_1)| f(x - \theta) F(\theta)\, d\theta \leqq KF(x)$. But from Lemma 2.2.2, it is clear that for large enough T, $F(x) < 2V_F(x)/b$. The result follows. □

THEOREM 2. *Under Assumptions 1 and 2 of Section 1.5, there exists $T > 0$ such* *that if* $d(x) > T$, *then*

$$\gamma_F(x) = -\mathscr{D}^* F(x)/(bF(x)) + \varepsilon(x) ,$$

where $|\varepsilon(x)| < [\bar{o}(1)\mathscr{D}F(x) + K(h(x))^{-q}]/(bF(x))$.

PROOF. Lemmas 2.2.1 and 2.2.2 give that if $d(x) > T$, then

$$|U_F(x)|/V_F(x) < K[\mathscr{D}F(x) + (h(x))^{-q}]/F(x) .$$

By Assumptions 1 (iii)c and 1 (iii)d. 2, this is $\bar{o}(1)$. Thus T can be chosen so that $|U_F(x)|/V_F(x) < 1/(20K_1)$, where K_1 is from Lemma 2.2.3. The conditions of Theorem 1 are then satisfied. It can thus be concluded that

$$\gamma_F(x) = -(1 + \varepsilon_1(x))U_F(x)/V_F(x) ,$$

where $|\varepsilon_1(x)| < 2K_1|U_F(x)|/V_F(x)$. Using Lemmas 2.2.1 and 2.2.2 again, and combining error terms, gives the desired result. □

COROLLARY 2.1. $\gamma_F(x) = \bar{o}(1)$.

PROOF. Obvious. □

Note that Theorem A follows immediately from Theorem 2 under the additional hypothesis.

3. Admissibility.

3.1. *Introduction.* In this chapter, it will be assumed that assumptions 1, 2 and 3 of Section 1.5 hold. It will then be shown that δ_F is admissible w.r.t. F for estimating θ_1. The proof will be based on the following theorem, which is basically Stein's sufficient condition for admissibility (Stein (1955)), but in the form given is due to Farrell (1964).

THEOREM 3. *Suppose* g_R *is a sequence of generalized priors such that* $\int g_R(\theta)\, d\theta < \infty$ *for every* R, *and such that* $\lim_{R\to\infty} g_R(\theta) = 1$ *for every* θ. *If* δ_R *is the generalized Bayes estimator for the prior* $G_R(\theta) = F(\theta)g_R(\theta)$, *assume*

$$\lim_{R\to\infty} \int [R(\delta_F, \theta) - R(\delta_R, \theta)]F(\theta)g_R(\theta)\, d\theta = 0 . \tag{3.1.1}$$

Then δ_F *is admissible with respect to* F. *Furthermore, if* $F(\theta) > 0$ *everywhere or* $f(x) > 0$ *everywhere then* δ_F *is admissible.*

PROOF. The proof is exactly analogous to the proof in Farrell (1964). □

In order to proceed, a suitable sequence of priors, g_R, must be found. The heuristic results of Section 1.4 indicate that the sequence g_R should satisfy (1.4.9). It can be checked that the following sequence of priors does this.

$$\begin{aligned} g_R(\theta) &= 1 && \text{if } \|\theta\| \leqq 1 \\ &= [1 - (\ln\|\theta\|)/(\ln R)]^{23} && \text{if } 1 \leqq \|\theta\| \leqq R \\ &= 0 && \text{if } \|\theta\| > R . \end{aligned} \tag{3.1.2}$$

Perhaps a word is in order as to how the above sequence of priors was found.

First, roughly speaking, the flatter the prior is, the smaller $\mathscr{D}^* g_R(x)$ will be in (1.4.9), and hence the larger the chance of success. Priors of the form

$$\begin{aligned} g_R(\theta) &= 1 && \text{if } \quad |\theta| < 1 \\ &= [1 - (\ln|\theta|)/\ln R] && \text{if } \quad 1 \leqq |\theta| \leqq R \\ &= 0 && \text{if } \quad |\theta| > R\,, \end{aligned}$$

are about as flat as can be obtained (subject of course to the restrictions that $g_R(\theta) \to 1$ and $\int g_R(\theta)\,d\theta < \infty$). A quick check shows that the priors must be functions of $||\theta||$, rather than $|\theta|$, in order for (1.4.9) to be satisfied. (As an intuitive guide, the components of $\mathscr{D}^* g_R(x)$ should be of comparable magnitude. Using $||\theta||$ ensures this.) Choosing the 23rd power of $[1 - (\ln||\theta||)/\ln R]$ is basically a necessary technicality, although at least the fourth power is necessary to satisfy (1.4.9).

It is clear that if g_R is given by (3.1.2), then $\int g_R(\theta)\,d\theta < \infty$ for every R and $\lim_{R\to\infty} g_R(\theta) = 1$ for every θ. Hence, by Theorem 3, it is only necessary to show that (3.1.1) is satisfied to prove that δ_F is admissible w.r.t. F. Verifying (3.1.1) is the goal of the remainder of the chapter.

3.2. *Preparatory lemmas.*

LEMMA 3.2.1. $|\gamma_F(x)| \leqq A < \infty$.

PROOF. An argument analogous to that in Farrell (1964) shows that γ_F is continuous. By Corollary 2.1, there exists $T > 0$ such that if $|x| > T$, then $|\gamma_F(x)| < 1$. Since γ_F is continuous, the lemma follows. □

LEMMA 3.2.2. *If* $|c| \leqq A$ *and* $|x| \leqq T$, *then*

(i) $\int L''(c + x_1 - \theta_1) f(x - \theta) F(\theta)\,d\theta$ *is continuous in* c *and* x_1.
(ii) $\int L''(c + \theta_1) f(\theta)\,d\theta$ *is continuous in* c.
(iii) *There exists* $\varepsilon_{T,A} > 0$ *such that*

$$\int L''(c + x_1 - \theta_1) f(x - \theta) F(\theta)\,d\theta \geqq \varepsilon_{T,A}\,.$$

PROOF. Straightforward, using Assumption 2. □

LEMMA 3.2.3. *There exists* $T > 0$, *such that if* $|x| > T$, *then*

(i) $\int L''(\gamma_F(x) + x_1 - \theta_1) f(x - \theta) F(\theta)\,d\theta = bF(x)(1 + \varepsilon(x))$, *where* $|\varepsilon(x)| < \frac{1}{2}$.
(ii) $\int |x - \theta|^i |L^{<j>}(c + x_1 - \theta_1)| f(x - \theta) F(\theta)\,d\theta \leqq KF(x)$, *where* $|c| < D < \infty$, $i \leqq 4$, *and* $1 \leqq j \leqq 3$.
(iii) $|\int (\theta_i - x_i) L'(\gamma_F(x) + x_1 - \theta_1) f(x - \theta) F(\theta)\,d\theta| \leqq K(\sum_{j=1}^n |F^{(j)}(x)| + [h(x)]^{-q})$, $2 \leqq i \leqq n$.
(iv) $h(x) < ||x|| + c_1$, *where* $c_1 < \infty$.

PROOF.

(i) By Lemma 3.2.2 (ii), there exists $\lambda > 0$ such that if $|c| < \lambda$, then

$\int L''(c+\theta_1)f(\theta)\,d\theta \geqq 3b/4$. (Recall that $b = \int L''(\theta_1)f(\theta)\,d\theta > 0$.) An argument identical to that of Lemma 2.2.2 thus gives

$$\int L''(c+x_1-\theta_1)f(x-\theta)F(\theta)\,d\theta = \left(\int L''(c+\theta_1)f(\theta)\,d\theta\right)F(x)(1+\bar{o}(1)) \geqq (3b/4)F(x)(1+\bar{o}(1))\,.$$

Choosing T large enough so that if $|x| > T$, then $|\gamma_F(x)| < \lambda$, the conclusion follows.

(ii) The argument here is similar to previous ones. In the appropriate region, $F(\theta)$ is expanded in a Taylor series about x. The $F(x)$ term will dominate the derivative terms. Note that it is here that all the moments of Assumption 2 (iv) are needed. Assumption 2 (v) must also be used.

(iii) Expanding $L'(\gamma_F(x)+x_1-\theta_1)$ in a Taylor series about $(x_1-\theta_1)$ gives

$$\left|\int (\theta_i - x_i)L'(\gamma_F + x_1 - \theta_1)f(x-\theta)F(\theta)\,d\theta\right| \leqq I_1 + I_2\,,$$

where

$$I_1 = \left|\int (\theta_i - x_i)L'(x_1-\theta_1)f(x-\theta)F(\theta)\,d\theta\right|\,,$$
$$I_2 = \left|\int \left(\int_{(x_1-\theta_1)}^{(\gamma_F+x_1-\theta_1)} L''(\eta)\,d\eta\right)(\theta_i - x_i)f(x-\theta)F(\theta)\,d\theta\right|\,.$$

I_1 is analyzed as in Lemma 2.2.1, since by (1.3.1) the coefficient of $F(x)$ in the Taylor expansion of $F(\theta)$ is $\int (\theta_i - x_i)L'(x_1-\theta_1)f(x-\theta)\,d\theta = 0$. Thus the secondary terms $F^{(j)}(x)$ are dominant. The error terms which arise can be bounded by $[h(x)]^{-q}$, using the familiar Chebyshev argument.

To handle I_2, note that a change of variables gives

$$I_2 \leqq \left|\int_0^{\gamma_F}\int L''(x_1-\theta_1+\eta)(\theta_i-x_i)f(x-\theta)F(\theta)\,d\theta\,d\eta\right|\,.$$

Since $|\gamma_F|$ is bounded, part (ii) of the lemma can be applied to give $I_2 \leqq |\gamma_F(x)|KF(x)$. Theorem 2 and Assumption 1 (iii)c finally give the desired bound.

(iv) Recall $\Omega = \Omega_1 = R^n - \mathscr{K}$ (where $\mathscr{K}$ is a compact set). It is thus clear that there exists $c > 0$, such that $d(x) < |x| + c$. By Assumption 1 (iii)d. 1, there exists a $T > 0$ such that if $d(x) > T$, then $h(x) < d(x)$. Finally, note that $|x| + c \leqq \sum_{i=1}^n |x_i| + c \leqq ||x|| + c + (n-1)$. Hence if $d(x) > T$, then $h(x) < d(x) < |x| + c \leqq ||x|| + c_1$. □

The proofs of the next three lemmas are straightforward and are omitted.

LEMMA 3.2.4.

(a) $||x||^{\frac{1}{2}}$ *is subadditive.*
(b) $||x-\theta|| \geqq (||x||^{\frac{1}{2}} - ||\theta||^{\frac{1}{2}})^2$.
(c) *If* $||x-\theta|| \leqq C \leqq ||x||$, *then* $||\theta|| \geqq [||x||^{\frac{1}{2}} - C^{\frac{1}{2}}]^2$.
(d) $|x-\theta|^2 \geqq ||x-\theta|| - 1$.

LEMMA 3.2.5.

$$\int_{\{x:a<||x||<b\}} f(||x||)\,dx = \rho_{n-1}\int_a^b u^{(n-1)/2}f(u)\,du\,,$$

where ρ_{n-1} *is the volume of the unit* $(n-1)$ *sphere.*

LEMMA 3.2.6. *If* $q(R) > 0$ *and* $\lim_{R\to\infty}[q(R)/R] = 0$, *then*

(i) $[R - q(R)]^{\frac{1}{2}} = R^{\frac{1}{2}} - (1 + o(1))q(R)/(2R^{\frac{1}{2}})$,
(ii) $\ln [R/(R - q(R))] = (1 + o(1))q(R)/R$.

Some information will be needed about the derivatives of g_R. Note that in $\Gamma_R = \{x : 1 < ||x|| < R,\ x_1 \neq 0\}$, $g_R(x)$ is infinitely differentiable. For notational convenience, let $s = s(||x||) = \ln (R/||x||)$. Note that if $x \in \Gamma_R$, then

$$g_R(x) = [1 - (\ln ||x||)/\ln R]^{23} = s(||x||)^{23}/(\ln R)^{23} .$$

Note also that $s(||x||)$ is a decreasing function of $||x||$.

LEMMA 3.2.7. *If* $x \in \Gamma_R$, *then*

$$|g_R^{(1)}(x)| = \frac{23s^{22}}{(\ln R)^{23}||x||}, \qquad |g_R^{(i)}(x)| \leqq \frac{Ks^{22}}{(\ln R)^{23}||x||^{\frac{1}{2}}},$$

$$|g_R^{(i,j)}(x)| \leqq \frac{Ks^{21}(1+s)}{(\ln R)^{23}||x||}, \qquad |g_R^{(1,j)}(x)| \leqq \frac{Ks^{21}(1+s)}{(\ln R)^{23}||x||^{\frac{3}{2}}},$$

$$|g_R^{(i,j,k)}(x)| \leqq \frac{Ks^{20}(1+s+s^2)}{(\ln R)^{23}||x||^{\frac{3}{2}}}, \qquad |g_R^{(i,j,k,l)}(x)| \leqq \frac{Ks^{19}(1+s+s^2+s^3)}{(\ln R)^{23}||x||^2} .$$

PROOF. Straightforward. Note that every time a derivative w.r.t. x_1 is taken, a factor of $||x||$ occurs in the denominator. A differentiation w.r.t. x_i, $i > 1$, only gives rise to a $||x||^{\frac{1}{2}}$ in the denominator. □

LEMMA 3.2.8. *Assume that* $\ln \ln R \leqq ||x|| \leqq R - R^{\frac{11}{12}}$. *Then uniformly in* x,

(i) $\lim_{R\to\infty} \dfrac{(\ln R)^{23}}{s^{23}||x||^{\frac{47}{24}}} = 0$,

(ii) $\lim_{R\to\infty} \dfrac{\ln R}{s||x||^{\frac{1}{2}}} = 0$.

PROOF. It is easy to check that $s^{-a}||x||^{-b}$, $a > 0$ and $b > 0$, is maximized at the end points $\ln \ln R$ and $R - R^{\frac{11}{12}}$ of the given interval.

To prove (i), note that $s(\ln \ln R) = \ln R - \ln \ln \ln R$. Clearly

$$\lim_{R\to\infty} \frac{(\ln R)^{23}}{(\ln R - \ln \ln \ln R)^{23}(\ln \ln R)^{\frac{47}{24}}} = 0 .$$

Also, by Lemma 3.2.6 (ii), $s(R - R)^{\frac{11}{12}} = (1 + o(1))R^{-\frac{1}{12}}$. Hence,

$$\lim_{R\to\infty} \frac{(\ln R)^{23}}{[s(R - R^{\frac{11}{12}})]^{23}[R - R^{\frac{11}{12}}]^{\frac{47}{24}}} = \lim_{R\to\infty} \frac{(\ln R)^{23}}{R^{\frac{1}{24}}} = 0 .$$

Thus part (i) is established. Part (ii) is verified in a similar manner. □

3.3. *Approximations to* $\delta_R - \delta_F$. In this section, estimates and bounds are developed for $[\gamma_R(x) - \gamma_F(x)]$. (Recall δ_R is the generalized Bayes estimator for $G_R(\theta) = F(\theta)g_R(\theta)$.) The results of Section 2.1 will be applied with $H = G_R$ and $a(x) = \gamma_F(x)$. By Lemma 3.2.1, $|\gamma_F(x)| \leqq A$. Also, results will be obtained only for the region $\{x : ||x|| \leqq R - R^{\frac{11}{12}}\}$. It is easy to check that for large enough R, and x in this set, $\delta_R(x)$ is given by $\int L'(\delta_R(x) - \theta_1)f(x - \theta)G_R(\theta)\, d\theta = 0$. Hence the results of Section 2.1 can be used. As in Section 2.1, define

$$U_R(x) = \int L'(\gamma_F(x) + x_1 - \theta_1) f(x - \theta) F(\theta) g_R(\theta)\, d\theta\,,$$
$$V_R(x) = \int L''(\gamma_F(x) + x_1 - \theta_1) f(x - \theta) F(\theta) g_R(\theta)\, d\theta\,,$$
$$W_R(\xi, x) = \int \left(\int_{(\gamma_F + x_1 - \theta_1)}^{(\xi + x_1 - \theta_1)} L'''(\eta)[\xi + x_1 - \theta_1 - \eta]\, d\eta\right) f(x - \theta) F(\theta) g_R(\theta)\, d\theta\,.$$

As usual, estimates must first be found for these quantities.

Because of the nondifferentiability of $g_R(x)$ at $x_1 = 0$, results must be obtained separately for $x_1 > 0$ and for $x_1 < 0$. The analysis in each case is identical, however. For simplicity, in the rest of this section assume that $x_1 > 0$ and $||x|| < R - R^{11/12}$. (Note that $\{x : x_1 = 0\}$ has measure 0 and can hence be ignored.)

Assume, finally, that T is a number chosen to satisfy Lemma 3.2.3, and that $\varepsilon_{T,A}$ is the number from Lemma 3.2.2 (where A is the bound on $|\gamma_F|$).

Results will be developed first for the case $||x|| < \ln \ln R$.

LEMMA 3.3.1. *There exists* $R_0 > 0$, *such that if* $R > R_0$ *and* $||x|| < \ln \ln R$, *then*

(i) $|U_R(x)| \leqq K(\ln R)^{-\frac{3}{4}}$,

(ii) $V_R(x) \geqq \varepsilon_{T,A}/2$ *for* $|x| < T$, $V_R(x) = bF(x)(1 + \varepsilon(x))$ *for* $|x| > T$, *where* $|\varepsilon(x)| < \frac{2}{3}$,

(iii) $\int |L'''(\xi + x_1 - \theta_1)| f(x - \theta) F(\theta) g_R(\theta)\, d\theta < K V_R(x)$ *for* $|\xi| < A + 1$.

PROOF. (i) Using the definition of g_R and the fact that $\int L'(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta = 0$, it is clear that

$$|U_R(x)| = \Big|\int_{\{\theta : 1 \leqq ||\theta|| \leqq R\}} L'(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)(1 - g_R(\theta))\, d\theta + \int_{\{\theta : ||\theta|| > R\}} L'(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta\Big|\,.$$

Define $P = \{\theta : 1 \leqq ||\theta|| \leqq \ln R\}$ and $Q = \{\theta : ||\theta|| > \ln R\}$. Noting that $g_R(\theta) \leqq 1$, it is clear that

$$|U_R(x)| \leqq \int_P |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta) F(\theta)(1 - g_R(\theta))\, d\theta + \int_Q |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta) F(\theta)\, d\theta\,. \tag{3.3.1}$$

Using the monotonicity of g_R, Assumptions 2 (iv) and 2 (v), and Lemma 3.2.1, it is clear that for large enough R_0,

$$\begin{aligned} &\int_P |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta) F(\theta)(1 - g_R(\theta))\, d\theta \\ &\quad\leqq K[1 - g_R(\ln R)] = K\left[1 - \left(1 - \frac{\ln \ln R}{\ln R}\right)^{23}\right] \\ &\quad\leqq \frac{K' \ln \ln R}{\ln R} \leqq (\ln R)^{-\frac{3}{4}}\,. \end{aligned} \tag{3.3.2}$$

Finally, consider the second integral in (3.3.1). Since $||\theta|| > \ln R$ for $\theta \in Q$, and since $||x|| < \ln \ln R$, Lemma 3.2.4 (b) and (d) establish that for large R

$$|x - \theta|^2 \geqq (||x||^{\frac{1}{2}} - ||\theta||^{\frac{1}{2}})^2 - 1 \geqq ([\ln \ln R]^{\frac{1}{2}} - [\ln R]^{\frac{1}{2}})^2 - 1 \geqq \ln R/2\,.$$

Thus $Q \subset \{\theta : |x - \theta|^2 \geqq \ln R/2\}$. A simple Chebyshev argument will thus show that the second integral in (3.3.1) is bounded by $K/\ln R$.

(ii) Since $L'' > 0$, it is clear that

$$V_R(x) \geqq \int_{Q^c} L''(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta) g_R(\theta)\, d\theta .$$

For $\theta \in Q^c$ and large enough R_0, it is clear from (3.3.2) that $g_R(\theta) \geqq 1 - (\ln R)^{-\frac{3}{4}}$. Hence

$$V_R(x) \geqq (1 - (\ln R)^{-\frac{3}{4}}) \int_{Q^c} L''(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta . \tag{3.3.3}$$

If $|x| < T$, then Lemma 3.2.2 (iii) and a Chebyshev argument give

$$\begin{aligned} \int_{Q^c} L''(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta &= \int L''(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta \\ &\qquad - \int_Q L''(\gamma_F + x_1 - \theta_1) f(x - \theta) F(\theta)\, d\theta \\ &\geqq \varepsilon_{T,A} - K/\ln R. \end{aligned}$$

The conclusion follows for $|x| < T$.

If $|x| > T$, a similar analysis, using Lemma 3.2.3 (i), shows that $V_R(x) \geqq bF(x)(1 + \varepsilon_1(x)) + \varepsilon_2(x)$, where $|\varepsilon_1(x)| < \frac{1}{2}$, and $|\varepsilon_2(x)| < K/\ln R$. Using Lemma 3.2.3 (iv) and Assumption (1 (iii)d. 2) gives for large R

$$|\varepsilon_2(x)| < K/\ln R \leqq (||x|| + c_1)^{-q} < [h(x)]^{-q} = \bar{o}(1) F(x) . \tag{3.3.4}$$

The second conclusion in part (ii) follows.

(iii) The result follows immediately for $|x| > T$ by Lemma 3.2.3 (ii) and part (ii) of this lemma. If $|x| \leqq T$, the integral is uniformly bounded in ξ and x, by assumptions 2 (iv) and 2 (v). The conclusion then follows from part (ii). □

THEOREM 4. *There exists $R_0 > 0$, such that if $R > R_0$ and $||x|| < \ln \ln R$, then*

(i) $|U_R(x)|/V_R(x) \leqq K(\ln R)^{-\frac{1}{2}}$,
(ii) $U_R^2(x)/V_R(x) \leqq K/\ln R$,
(iii) $[\gamma_R(x) - \gamma_F(x)] = -(1 + \varepsilon(x)) U_R(x)/V_R(x)$, *where* $|\varepsilon(x)| < \frac{1}{10}$.

PROOF. (i) The result is obvious for $|x| < T$, using Lemma 3.3.1. If $|x| > T$, then by Lemma 3.3.1, $|U_R(x)|/V_R(x) \leqq K(\ln R)^{-\frac{3}{4}}/F(x)$. As in (3.3.4) it can be shown that $(\ln R)^{-\frac{1}{4}} = \bar{o}(1) F(x)$. The result follows.

(ii) The proof goes as in (i).

(iii) A simple application of Theorem 1, using (i) and Lemma 3.3.1 (iii). □

It remains to consider the region $\{x : \ln \ln R < ||x|| < R - R^{\frac{11}{12}} \text{ and } x_1 > 0\}$. If x is in this region, it will be desired to expand $g_R(\theta)$ in a Taylor expansion about x. For this purpose define

$$\begin{aligned} x^- &= (-x_1, x_2, \cdots, x_n) \\ B_x^+ &= \{\theta : \theta_1 > 0 , ||x - \theta|| < ||x||/3, \text{ and } ||\theta|| < R\} . \\ B_x^- &= \{\theta : \theta_1 < 0 , ||x^- - \theta|| < ||x||/3, \text{ and } ||\theta|| < R\} . \end{aligned}$$

It is easy to check that B_x^+ and B_x^- are convex. By Lemma 3.2.4 (c) (with $C = ||x||/3$), it is clear that $||\theta|| \geqq [||x||^{\frac{1}{2}} - ||x||^{\frac{1}{2}} 3^{-\frac{1}{2}}]^2 > 1$ for large enough R.

Hence neither B_x^+ or B_x^- intersects $\{\theta : \|\theta\| = R, \text{ or } \theta_1 = 0, \text{ or } \|\theta\| = 1\}$. Therefore, all derivatives of g_R exist and are continuous in B_x^+ and B_x^-. Clearly $x \in B_x^+$ and $x^- \in B_x^-$. Hence $g_R(\theta)$ can be expanded in a Taylor series about x in B_x^+, and about x^- in B_x^-. Note that

$$g_R(x) = g_R(x^-) \quad \text{and} \quad g_R^{(i)}(x) = g_R^{(i)}(x^-) \quad \text{for} \quad i > 1 . \tag{3.3.5}$$

The next few lemmas in some sense give the heart of the theory, since they deal with the important region where both g_R and F are "smooth" and "flat."

LEMMA 3.3.2. *There exists* $R_0 > 0$, *such that if* $R > R_0$, $\ln \ln R \leqq \|x\| \leqq R - R^{1\frac{1}{2}}$, *and* $x_1 > 0$, *then*

$$|U_R(x)| \leqq KF(x)\left[\frac{s(\|x\|)^{21}(1 + s(\|x\|))}{(\ln R)^{23}\|x\|} + \|x\|^{-2}\right]. \tag{3.3.6}$$

PROOF. For notational convenience, define $P = \{\theta : \|x - \theta\| > \|x\|/3\}$ and $Q = \{\theta : \|\theta\| > R\}$. Clearly

$$\begin{aligned} |U_R(x)| \leqq {} & |\textstyle\int_P L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta| \\ & + |\textstyle\int_{P^c} L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta| . \end{aligned}$$

Call the integrals above I_1 and I_2 respectively.

To deal with I_1, note that by Lemma 3.2.4 (d) and the definition of P, $|x - \theta|^2 > \|x - \theta\| - 1 > \|x\|/4$ (for large enough R_0). Since $g_R(\theta) \leqq 1$, a simple Chebyshev argument and Lemma 3.2.3 (ii) thus give $I_1 \leqq KF(x)\|x\|^{-2}$.

Consider next I_2. Noting that $g_R(\theta) = 0$ for $\theta \in Q$, and observing that $P^c \cap Q^c = B_x^+ \cup (B_x^- \cap P^c)$, it is clear that

$$\begin{aligned} I_2 = {} & |\textstyle\int_{B_x^+} L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta \\ & + \textstyle\int_{(B_x^- \cap P^c)} L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta| . \end{aligned}$$

Expanding $g_R(\theta)$ about x for $\theta \in B_x^+$, and about x^- for $\theta \in B_x^- \cap P^c$, and using (3.3.5) gives

$$\begin{aligned} I_2 \leqq {} & |g_R(x) \textstyle\int_{P^c \cap Q^c} L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)\, d\theta| \\ & + |\textstyle\sum_{i=2}^n g_R^{(i)}(x) \int_{P^c \cap Q^c} (\theta_i - x_i)L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)\, d\theta| \\ & + |g_R^{(1)}(x) \textstyle\int_{B_x^+} (\theta_1 - x_1)L'fF\, d\theta + g_R^{(1)}(x^-) \int_{(B_x^- \cap P^c)} (\theta_1 + x_1)L'fF\, d\theta| \\ & + \text{similar terms in} \quad g_R^{(i,j)}(x) \quad \text{and} \quad g_R^{(i,j,k)}(x) \\ & + K \textstyle\sum_{i,j,k,l} \int_{B_x^+} |\theta - x|^4 |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta)F(\theta) \\ & \times |g_R^{(i,j,k,l)}(tx + (1 - t)\theta)|\, d\theta \\ & + \text{similar remainder term over} \quad (B_x^- \cap P^c) . \end{aligned} \tag{3.3.7}$$

The terms in (3.3.7) will be bounded one at a time.

(i) Noting that $\int L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)\, d\theta = 0$, it is clear that

$$\begin{aligned} & |g_R(x) \textstyle\int_{P^c \cap Q^c} L'(\gamma_F + x_1 - \theta_1)f(x - \theta)F(\theta)\, d\theta| \\ & \quad \leqq \textstyle\int_P |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta)F(\theta)\, d\theta \\ & \qquad + g_R(x) \textstyle\int_Q |L'(\gamma_F + x_1 - \theta_1)| f(x - \theta)F(\theta)\, d\theta . \end{aligned}$$

The first of these two integrals can be bounded as I_1 was. For the second, note that since $||x|| < R - R^{11/12}$ and $||\theta|| > R$, Lemma 3.2.4 (b) and (d) and Lemma 3.2.6 give

$$|x-\theta|^2 \geqq ||x-\theta|| - 1 \geqq [(R - R^{11/12})^{1/2} - R^{1/2}]^2 - 1 \geqq [(1+o(1))R^{5/12}/2]^2 - 1 \geqq R^{10/12}/9\,. \tag{3.3.8}$$

Hence a Chebyshev argument and Lemma 3.2.3 (ii) give

$$\int_Q |L'(\gamma_F + x_1 - \theta_1)| f(x-\theta)F(\theta)\,d\theta \leqq KF(x)R^{-15/12}\,.$$

Noting that $s(||x||) < \ln R$ and $1 < ||x|| < R$, it follows that if R_0 is large enough, then

$$g_R(x)F(x)R^{-15/12} \leqq F(x)s^{21}(1+s)||x||^{-1}(\ln R)^{-23}\,.$$

(ii) Using the same types of arguments and Lemma 3.2.3 (iii), the second term of (3.3.7) can be bounded by

$$K \sum_{i=2}^{n} g_R^{(i)}(x)\,[\sum |F^{(j)}(x)| + [h(x)]^{-q} + F(x)||x||^{-1} + F(x)R^{-15/12}]\,.$$

Using Lemma 3.2.7 to bound $g_R^{(i)}(x)$, Assumption 3 (iii)a to bound $|F^{(j)}(x)|$, and Assumption 3 (ii) to bound $[h(x)]^{-q}$, the desired result is obtained.

(iii) Noting that $|\theta_1 + x_1| < |\theta_1 - x_1|$ for $\theta \in B_x^-$, it is clear that the third term of (3.3.7) is bounded by

$$|g_R^{(1)}(x)| \int |\theta_1 - x_1|\,|L'(\gamma_F + x_1 - \theta_1)| f(x-\theta)F(\theta)\,d\theta\,.$$

The result now follows from Lemma 3.2.3 (ii) and Lemma 3.2.7.

(iv) The bounds for the terms involving $g_R^{(i,j)}(x)$ are obtained in a similar fashion. For the terms involving $g_R^{(i,j,k)}(x)$, Lemma 3.2.8 (ii) must be used in conjunction with Lemma 3.2.7 to obtain the desired bound.

(v) and (vi). The bounds for the remaining terms of (3.3.7) follow from the following observation. Consider the integral over B_x^+ for simplicity. Since $\theta \in B_x^+$, $x \in B_x^+$ and B_x^+ is convex, it is clear that $(tx + (1-t)\theta) \in B_x^+$. (Of course, $0 \leqq t(x,\theta) \leqq 1$.) By Lemma 3.2.4 (c) (with $C = ||x||/3$), it is clear that if $||x-\theta|| < ||x||/3$, then $||\theta|| > ||x||/6$. Letting $M = \{\theta : ||\theta|| > ||x||/6\}$, it follows that $B_x^+ \subset M$. Hence by Lemma 3.2.7 and the fact that $s(||x||) < \ln R$, it follows that

$$\begin{aligned}|g_R^{(i,j,k,l)}(tx + (1-t)\theta)| &\leqq \sup_{\theta \in M} |g_R^{(i,j,k,l)}(\theta)| \\ &\leqq \sup_{\theta \in M} K||\theta||^{-2} = 36K||x||^{-2}\,.\end{aligned}$$

Applying Lemma 3.2.3 (ii) to the remaining integral completes the proof. □

LEMMA 3.3.3. *There exists* $R_0 > 0$, *such that if* $R > R_0$, $\ln\ln R \leqq ||x|| \leqq R - R^{11/12}$, *and* $x_1 > 0$, *then*

$$V_R(x) = bg_R(x)F(x)(1 + \varepsilon(x))\,, \qquad \textit{where} \quad |\varepsilon(x)| < \tfrac{2}{3}\,.$$

PROOF. Similar to that of Lemma 3.3.2. Lemmas 3.2.7 and 3.2.8 are used to show that $g_R(x)$ dominates the higher order derivatives of g_R in the Taylor expansion of $g_R(\theta)$. Lemma 3.2.3 (i) is also needed, to handle the coefficient of $g_R(x)$. □

LEMMA 3.3.4. *There exists* $R_0 > 0$, *such that if* $R > R_0$, $\ln \ln R \leqq ||x|| \leqq R - R^{\frac{11}{12}}$, $x_1 > 0$, *and* $|\xi| < A + 1$, *then*

$$\int |L'''(\xi + x_1 - \theta_1)| f(x - \theta)F(\theta)g_R(\theta)\, d\theta < KV_R(x)\,.$$

PROOF. Similar to preceding proofs. □

THEOREM 5. *There exists* $R_0 > 0$, *such that if* $R > R_0$, $\ln \ln R \leqq ||x|| \leqq R - R^{\frac{11}{12}}$ *and* $x_1 > 0$, *then*

(i) $\dfrac{|U_R(x)|}{V_R(x)} \leqq K\left[\dfrac{1 + s(||x||)}{||x||s(||x||)^2} + \dfrac{(\ln R)^{23}}{||x||^2 s(||x||)^{23}}\right] \to 0$ *uniformly in x as* $R \to \infty$,

(ii) $U_R^2(x)/V_R(x) \leqq KF(x)(\ln R)^{-2}||x||^{-2}$,

(iii) $[\gamma_R(x) - \gamma_F(x)] = -(1 + \varepsilon(x))U_R(x)/V_R(x)$, *where* $|\varepsilon(x)| < \frac{1}{10}$.

PROOF. (i) The bound follows immediately from Lemmas 3.3.2 and 3.3.3. The convergence to zero follows from Lemma 3.2.8.

(ii) Another simple application of Lemmas 3.3.2, 3.3.3 and 3.2.8.

(iii) A simple application of Theorem 1, using (i) and Lemma 3.3.4. □

3.4. *Verification of* (3.1.1). The following theorem completes the admissibility argument.

THEOREM 6. *If Assumptions* 1, 2 *and* 3 *of Section* 1.5 *hold, and if* g_R *is given by* (3.1.2), *then*

$$\lim_{R\to\infty} \int [R(\delta_F, \theta) - R(\delta_R, \theta)]F(\theta)g_R(\theta)\, d\theta = 0\,.$$

PROOF. Let $D = \{x : ||x|| \geqq R - R^{\frac{11}{12}}\}$. Interchanging orders of integration (legal since the g_R have finite mass) gives

$$\int [R(\delta_F, \theta) - R(\delta_R, \theta)]F(\theta)g_R(\theta)\, d\theta = I_R^1 + I_R^2\,,$$

where

$$I_R^1 = \int_D \int [L(\gamma_F(x) + x_1 - \theta_1) - L(\gamma_R(x) + x_1 - \theta_1)]f(x - \theta)F(\theta)g_R(\theta)\, d\theta\, dx\,,$$

$$I_R^2 = \int_{D^c} \int [L(\gamma_F(x) + x_1 - \theta_1) - L(\gamma_R(x) + x_1 - \theta_1)]f(x - \theta)F(\theta)g_R(\theta)\, d\theta\, dx\,.$$

Part (i). It is desired to show that $I_R^1 \to 0$. Consider first

$$I_R^3 = \int_D \int L(\gamma_R + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta\, dx\,.$$

By definition, $\int L(c + x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta$ is minimized at $c = \gamma_R(x)$. Hence, defining $E = \{\theta : ||\theta|| \leqq R - 2R^{\frac{11}{12}}\}$, it is clear that

$$\text{(3.4.1)} \qquad I_R^3 \leqq \int_D \int_E L(x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta\, dx + \int_D \int_{E^c} L(x_1 - \theta_1)f(x - \theta)F(\theta)g_R(\theta)\, d\theta\, dx\,.$$

These integrals will be considered separately. For $\theta \in E$ and $x \in D$, an argument like that in line (3.3.8) shows that $|x - \theta| \geqq R^{\frac{5}{12}}/3$. A simple Chebyshev argument and Assumption 2 (iv) thus give

$$(3.4.2)\qquad \int_D \int_E L(x_1-\theta_1)f(x-\theta)F(\theta)g_R(\theta)\,d\theta\,dx \leqq KR^{-\frac{25}{12}} \int_E F(\theta)\,d\theta\ .$$

Using Assumption 3 (iii) b and Lemma 3.2.5 gives

$$\int_E F(\theta)\,d\theta \leqq \int_E ||\theta||^{(3-n)/2}\,d\theta < \int_0^R u\,du = R^2/2\ .$$

This, combined with (3.4.2), shows that

$$(3.4.3)\qquad \int_D \int_E L(x_1-\theta_1)f(x-\theta)F(\theta)g_R(\theta)\,d\theta\,dx \leqq KR^{-\frac{1}{12}} \to 0\ .$$

Next, consider the second integral in (3.4.1). Note first that $g_R(\theta)=0$ for $||\theta||>R$. On the remaining part of E^c, $g_R(\theta)$ is decreasing in $||\theta||$. This, together with Lemma 3.2.6 (ii), shows that if $R-2R^{\frac{11}{12}} \leqq ||\theta|| \leqq R$, then

$$g_R(\theta) \leqq g_R(R-2R^{\frac{11}{12}}) \leqq KR^{-\frac{23}{12}}(\ln R)^{-23}\ .$$

A simple calculation (again using Assumption 3 (iii) b and Lemma 3.2.5) thus gives

$$\begin{aligned}\int_D \int_{E^c} L(x_1-\theta_1)f(x-\theta)F(\theta)g_R(\theta)\,d\theta\,dx &\leqq KR^{-\frac{23}{12}}(\ln R)^{-23} \int_{R-R^{\frac{11}{12}}}^{R} u\,du \\ &= K(\ln R)^{-23}(1-R^{-\frac{1}{12}}/2) \to 0\ .\end{aligned}$$

Combining this with (3.4.3) and (3.4.1) shows that $I_R^3 \to 0$.

Next, consider the other component of I_R^1, namely $\int_D \int L(\gamma_F+x_1-\theta_1)f(x-\theta)F(\theta)g_R(\theta)\,d\theta\,dx$. Since $|\gamma_F|<A<\infty$, Assumption 2 (v) can be used to reduce the situation to that considered above. It can thus be concluded that $I_R^1 \to 0$.

Part (ii). It is desired to show that $I_R^2 \to 0$. Expanding $L(\gamma_R+x_1-\theta_1)$ in a Taylor series about $(\gamma_F+x_1-\theta_1)$ gives

$$\begin{aligned}I_R^2 = \int_{D^c} \int [&(\gamma_F-\gamma_R)L'(\gamma_F+x_1-\theta_1) \\ &-\tfrac{1}{2}(\gamma_F-\gamma_R)^2L''(\gamma_F+x_1-\theta_1)-\mathscr{R}]f(x-\theta)F(\theta)g_R(\theta)\,d\theta\,dx\ ,\end{aligned}$$

where

$$\mathscr{R}(x,\theta)=\tfrac{1}{2}\int_{(\gamma_F+x_1-\theta_1)}^{(\gamma_R+x_1-\theta_1)} L'''(\eta)[\gamma_R+x_1-\theta_1-\eta]^2\,d\eta\ .$$

By Theorems 4 and 5, it is clear that if $x\in D^c$ and R is large enough, then $|\gamma_R(x)|<|\gamma_F(x)|+1<A+1$. Using Lemma 3.3.1 (iii) and Lemma 3.3.4, it is therefore straightforward to verify that for large enough R

$$(3.4.4)\qquad \left|\int \mathscr{R}(x,\theta)f(x-\theta)F(\theta)g_R(\theta)\,d\theta\right| \leqq K|\gamma_F(x)-\gamma_R(x)|^3V_R(x)\ .$$

Theorems 4 and 5 also show that if $R>R_0$ and $x\in D^c$, then

$$(3.4.5)\qquad \gamma_F(x)-\gamma_R(x)=(1+\varepsilon(x))U_R(x)/V_R(x)\ ,\qquad \text{where}\quad |\varepsilon(x)|<\tfrac{1}{10}\ .$$

Using (3.4.4) and (3.4.5) and recalling the definitions of $V_R(x)$ and $U_R(x)$, it is clear that

$$\begin{aligned}(3.4.6)\qquad I_R^2 &\leqq K\int_{D^c}[U_R^2(x)/V_R(x)]\,dx \\ &= K\int_{B_1}[U_R^2(x)/V_R(x)]\,dx + K\int_{B_2}[U_R^2(x)/V_R(x)]\,dx\ ,\end{aligned}$$

where $B_1=\{x: ||x||<\ln\ln R\}$ and $B_2=\{x: \ln\ln R \leqq ||x|| \leqq R-R^{\frac{11}{12}}\}$.

By Theorem 4 (ii) and Lemma 3.2.5,

(3.4.7) $$K \int_{B_1} [U_R^2(x)/V_R(x)]\, dx \leqq K'(\ln R)^{-1} \int_0^{\ln \ln R} u^{(n-1)/2}\, du = K''(\ln R)^{-1}(\ln \ln R)^{(n+1)/2} \to 0 .$$

By Theorem 5 (ii), Assumption 3 (iii) b, and Lemma 3.2.5,

(3.4.8) $$\begin{aligned} K \int_{B_2} [U_R^2(x)/V_R(x)]\, dx &\leqq K'(\ln R)^{-2} \int_{B_2} F(x)\|x\|^{-2}\, dx \\ &\leqq K''(\ln R)^{-2} \int_{\ln \ln R}^{R} u^{-1}\, du \\ &= K''[(\ln R)^{-1} - (\ln R)^{-2} \ln \ln \ln R] \to 0 . \end{aligned}$$

Combining (3.4.6), (3.4.7) and (3.4.8) completes the proof. □

4. Generalizations.

1. It should be possible to generalize the problem to the multiobservational situation by conditioning on the maximal invariant. (See Farrell (1964) or Brown (1966) for typical such arguments.)

2. Most of the results should carry over to the case where the distribution of X does not have a density w.r.t. Lebesgue measure.

3. The assumption of convex loss made life easier but is probably not absolutely essential. The crucial features of the loss function were that L satisfy Assumption 2 (ii), (iv), and (v), that $b = E_0 L''(X_1) > 0$, and that the loss be such that δ_F is unique. (Note that results of Farrell (1964) indicate that δ_F must be unique in order for it to be admissible.) It should be possible to obtain similar results for L satisfying these properties.

4. The method of proof necessitated the large number of assumptions. It would obviously be desirable to weaken many of these assumptions. In particular, the assumption that $\Omega = \Omega_1 = R^n - \mathscr{K}$ in Assumption 3 is unattractive. A prior such as $(1 + \theta_1^2)^{-1}$ does not have $\Omega = \Omega_1$ (where Ω is chosen to satisfy Assumption 1). Hence the admissibility results will not apply to such a prior.

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