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SOME CONTRIBUTIONS TO THE THEORY OF MULTISTAGE YOUDEN DESIGN[1]

BY K. AFSARINEJAD AND A. HEDAYAT

Florida State University and University of Illinois

It is shown that complete sets of $(v-1)/2$ by v and $(v+1)/2$ by v multistage balanced Youden designs of type I and II can be constructed if v, the number of treatments, is a prime power of the form $4\lambda+3$. It is also proved that the existence of a difference set with certain properties implies the existence of a 2-stage balanced Youden design. The usefulness of this latter result is demonstrated for those experiments where the number of treatments is not a prime power.

1. Introduction and summary. The concept of multistage balanced Youden designs has recently been introduced by Hedayat, Seiden, and Federer (1972) (referred to hereafter as HSF (1972)), who have also proved a number of results concerning the existence and construction of these designs. These designs have many practical applications both as "experiment designs" and "treatment designs" as have been pointed out by HSF (1972). Our objectives in this paper are: (i) to answer some of the unsolved problems which have been stated by HSF (1972); (ii) to extend the domain of multistage Youden designs to cover some practically useful cases which have not been considered previously. Specifically, we have obtained the following results: A complete set of $(v-1)/2 \times v$ multistage balanced Youden design of Type I can be constructed if v, the number of treatments, is of the form $4\lambda+3=p^{\alpha}$. By the Type I balance we mean balanced for unordered pairs in the sense of HSF (1972). Type II balance is a useful extension of Type I in a sense that its statistical implication is similar to those of Type I and in addition it is a step closer to orthogonality. This type of balance is formally defined in Definition 2.2. In this regard we have shown that a complete set of $(v+1)/2 \times v$ multistage balanced Youden design of Type II can be constructed if v is of the form $4\lambda+3=p^{\alpha}$. Finally, we have proved that the existence of a difference set with certain properties implies the existence of a 2-stage balanced Youden design. The usefulness of this latter result is demonstrated for those experiments where the number of treatments is not a prime power.

2. Balanced multistage Youden designs of Types I and II. Let $\mathscr{D}=\{D_1, D_2, \cdots, D_t\}$ be a t-stage $k \times v$ Youden design on a set Σ containing v distinct treatments. Now we formally define two types of symmetries for $\mathscr{D}$.

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DEFINITION 2.1. $\mathscr{D}$ is said to be a t-stage $k \times v$ balanced Youden design of Type I if the superimposed form of D_i and D_j, $i \neq j$, have the property that for $w \in \Sigma$ there exist exactly $v - 1$ pairs of the form (w, y) or (z, w), such that the collection of y's and z's exhausts the set $\Sigma - \{w\}$, $a \in D_i$ and $b \in D_j$ in the ordered pair (a, b), $i, j = 1, 2, \cdots, t$.

DEFINITION 2.2. $\mathscr{D}$ is said to be a t-stage $k \times v$ balanced Youden design of Type II if the superimposed form of D_i and D_j, $i \neq j$, have the property that for $w \in \Sigma$ there exist exactly v pairs of the form (w, y) or (z, w) such that the collection of y's and z's exhausts the set Σ, where the pair (a, b) is defined as in Definition 2.1.

Note that our definition of Type I balance is the same as the balance for unordered pairs defined by HSF (1972). Type II balance is a natural extension of Type I because: (i) Its statistical implications are exactly the same as those of Type I. First, the design is Youden in each stage. Second, the sets of treatments in different stages are factorially balanced in the sense of Definition 3.3 of HSF (1972). (ii) These designs can accommodate more levels and are one step closer to orthogonality, as can be seen from the definition.

The demand for symmetry naturally imposes restrictions on the form of k, v and t, as has been demonstrated in the following lemmas:

LEMMA 2.1. *If $\mathscr{D}$ is a t-stage $k \times v$ balanced Youden design of Type* I *then* $v \equiv 3 (\mathrm{mod}\, 4)$, $k = (v - 1)/2$ *and* $t \leqq v - 1$.

For a proof see HSF (1972).

LEMMA 2.2. *If $\mathscr{D}$ is a t-stage $k \times v$ balanced Youden design of Type* II, *then* $v \equiv 3 (\mathrm{mod}\, 4)$, $k = (v + 1)/2$ *and* $t \leqq v - 1$.

The proof is analogous to the proof of Lemma 2.1.

DEFINITION 2.3. A t-stage $k \times v$ balanced Youden design of Type I or II is said to be complete if $t = v - 1$.

Now the question is that: Is it possible to construct $(v - 1)$-stage balanced $k \times v$ Youden designs of Type I or II whenever k and v satisfy the restrictions of Lemma 2.1. or Lemma 2.2? Limited by mathematical tools available we cannot answer this question in general. HSF (1972) have shown that if in addition $v = p^\alpha$, p a prime and α a positive integer, then there exist $(v - 1)/2$—balanced Youden design of Type I. Shortly, we show that one can embed the $(v - 1)/2$—balanced Youden design constructed by HSF (1972) into a complete set. We also construct a complete set of multistage balanced Youden design of Type II whenever $v = p^\alpha$.

3. Existence and construction of balanced multistage Youden designs of Types I and II when the number of treatments is a prime power.

THEOREM 3.1. *A complete set of $(v - 1)/2 \times v$ multistage balanced Youden design of Type I exists whenever $v = 4\lambda + 3 = p^\alpha$.*

PROOF. By construction. Identify the $4\lambda + 3$ treatments with the elements of the $GF(p^\alpha)$ with x as a primitive element. Let $\mathscr{D} = \{D_0, D_1, \cdots, D_{4\lambda+1}\}$, where D_r is the $(2\lambda + 1) \times (4\lambda + 3)$ array with the entry

$$x^{r+2i} + \delta(j)x^j\,, \qquad i = 0, 1, \cdots, 2\lambda; j = 0, 1, \cdots, 4\lambda + 2 \text{ in its } (i, j)$$

cell. Here $\delta(j) = 0$ for $j = 0$ and 1 otherwise. We shall now prove that $\mathscr{D}$ is the desired multistage design. Clearly D_r is a $(2\lambda + 1) \times (4\lambda + 3)$ Youden design. Now consider the superimposed form of D_r and D_s and let $w = x^k \in GF(p^\alpha)$. By examining the $2\lambda + 1$ cells of D_r which contain w, we see that w is in row i and column j where j satisfies

$$\delta(j)x^j = x^k - x^{r+2i}\,, \qquad i = 0, 1, \cdots, 2\lambda\,.$$

The corresponding entries in the cells of D_s are:

$$U_i = x^{2i}(x^s - x^r) + w\,, \qquad i = 0, 1, \cdots, 2\lambda\,.$$

Similarly, the entries in $2\lambda + 1$ cells of D_r corresponding to those cells of D_s containing w are:

$$V_i = x^{2i}(x^r - x^s) + w\,, \qquad i = 0, 1, \cdots, 2\lambda\,.$$

Now $\{U_i, i = 0, 1, \cdots, 2\lambda\} \cup \{V_i, i = 0, 1, \cdots, 2\lambda\} = GF(p^\alpha) - \{w\}$ and thus the proof.

REMARKS. Note that the set of designs constructed by Theorem 4.4 of HSF (1972) is embedded in the set of designs constructed by Theorem 3.1 above.

THEOREM 3.2. *A complete set of* $(v + 1)/2 \times v$ *multistage balanced Youden design of Type* II *exists whenever* $v = 4\lambda + 3 = p^\alpha$.

PROOF. By construction. First construct $\mathscr{D} = \{D_0, D_1, \cdots, D_{4\lambda+1}\}$ the complete set of $(v - 1)/2 \times v$ multistage balanced Youden design of Type I by the algorithm of Theorem 3.1. Second, augment D_i in $\mathscr{D}$ with the following $(2\lambda + 1)$th row $0, x^1, x^2, \cdots, x^{4\lambda+2}$ to obtain the desired design.

THEOREM 3.3. *The two sets of treatments in any pair of Youden designs belonging to the* $(4\lambda + 2)$*-stage balanced Youden designs of Theorems* 3.1 *and* 3.2 *are balanced in the sense of Definition* 3.3 *of* HSF (1972).

The proof is analogous to the proof of Theorem 4.5 of HSF (1972).

4. Existence and construction of balanced multistage Youden designs of Types I and II when the number of treatments is not a prime power. Unfortunately, not every $v = 4\lambda + 3$ is of the form p^α. For instance, in the range of $v \leqq 100$ the following integers are such examples: 15, 35, 39, 51, 55, 63, 87 and 95. Therefore, there is a need for an algorithm for constructing multistage balanced designs for those orders which are not covered by Theorems 3.1 and 3.2. Before proceeding further we need the following definition.

DEFINITION 4.1. Let $D = \{d_1, d_2, \cdots, d_k\}$ be a subset of cardinality k in a group G of order v. We say D has property P_1 if there exist two permutations of the elements of D, say $A = (a_1, a_2, \cdots, a_k)$ and $B = (b_1, b_2, \cdots, b_k)$ such that $c_i c_j \neq e$ and $c_i c_j^{-1} \neq e$ where $c_l = a_l b_l^{-1}$ for all $i, j = 1, 2, \cdots, k$, and e is the identity element in G.

THEOREM 4.1. *The existence of a* $(4\lambda + 3, 2\lambda + 1, \lambda)$ *difference set with property* P_1 *implies the existence of a* 2-*stage* $(2\lambda + 1) \times (4\lambda + 3)$ *balanced Youden design of Type* I.

PROOF. By construction. Consider the following two $(2\lambda + 1) \times (4\lambda + 3)$ matrices D_1 and D_2 with $a_i g_j$ and $b_i g_j$ in their (i, j) entries respectively, $i = 1, 2, \cdots, 2\lambda + 1$; $j = 1, 2, \cdots, 4\lambda + 3$, $g_j \in G$. Clearly D_1 and D_2 are Youden designs. Now consider $2\lambda + 1$ cells of D_1 which contain a fixed element of G, say g. Then there exists a subset $\{h_i, i = 1, 2, \cdots, 2\lambda + 1\}$ in G such that $g = a_i h_i$, $i = 1, 2, \cdots, 2\lambda + 1$. The corresponding entries in the cells of D_2 will be $F = \{f_i = c_i^{-1} g, i = 1, 2, \cdots, 2\lambda + 1\}$. Similarly, the entries in $2\lambda + 1$ cells of D_1 corresponding to those cells of D_2 containing g will be $T = \{t_i = c_i g, i = 1, 2, \cdots, 2\lambda + 1\}$. Therefore $FUT = G - \{g\}$.

DEFINITION 4.2. Let $D = \{d_1, d_2, \cdots, d_k\}$ be a subset of cardinality k in G. We say D has property P_1^* if there exists a permutation of D, say $(d_1^*, d_2^*, \cdots, d_k^*)$ such that $w_1 = d_2^* d_1^{*-1}$, $w_2 = d_3^* d_2^{*-1}$, $\cdots$, $w_k = d_1^* d_k^{*-1}$ are all different, and moreover, $w_i w_j \neq e$ for all i and j.

It is clear that if D has property P_1^* then it has property P_1. This is because if $D = \{d_1, d_2, \cdots, d_k\}$ has property P_1^* then $A = (d_1, d_2, \cdots, d_k)$ together with $B = (d_2, d_3, \cdots, d_1)$ satisfies the requirement of Definition 4.1. Thus we have:

THEOREM 4.2. *The existence of a difference set* $(4\lambda + 3, 2\lambda + 1, \lambda)$ *with property* P_1^* *implies the existence of a* 2-*stage* $(2\lambda + 1) \times (4\lambda + 3)$ *balanced Youden design of Type* I.

EXAMPLE 4.1. Let $v = 15$. Then $\{1, 2, 4, 0, 8, 5, 10\}$ is a difference set with property P_1^* in $G = \{0, 1, 2, \cdots, 14\}$ with the addition mod 15 as the binary operation.

Several parallel results can be obtained for multistage balanced Youden design of Type II. These have been demonstrated below.

DEFINITION 4.3. Let $D = \{d_1, d_2, \cdots, d_k\}$ be a subset of cardinality k in G. We say D has property $P_2(P_2^*)$ if there is a subset of cardinality $k - 1$ in D, say $\bar{D} = \{\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_{k-1}\}$ which has property $P_1(p_1^*)$.

Similarly, we have:

THEOREM 4.3. *The existence of a* $(4\lambda + 3, 2\lambda + 2, \lambda + 1)$ *difference set with property* $P_2(P_2^*)$ *implies the existence of a* 2-*stage* $(2\lambda + 2) \times (4\lambda + 3)$ *balanced Youden design of Type* II.

PROOF. By construction. Suppose $D = \{d_1, d_2, \cdots, d_{2\lambda+2}\}$ is a $(4\lambda + 3, 2\lambda + 2, \lambda + 1)$ difference set with property P_2 (the proof for P_2^* is similar) in G.

Write D as $\{\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_{2\lambda+1}\} \cup \{\bar{d}_{2\lambda+2}\}$. Let $A = (a_1, a_2, \cdots, a_{2\lambda+1})$ and $B = (b_1, b_2, \cdots, b_{2\lambda+1})$ be the two corresponding permutations associated with $\bar{D}$ as defined in Definition 4.1. Then by an analogous proof to the proof of Theorem 3.3 it can be shown that the following two designs form a 2-stage $(2\lambda + 2) \times (4\lambda + 3)$ balanced Youden design of Type II.

$$D_1 = [d_{ij}^{(1)}] \quad \text{with} \quad d_{ij}^{(1)} = a_i g_j \quad \text{and} \quad d_{2\lambda+2,j}^{(1)} = \bar{d}_{2\lambda+2} g_j$$
$$D_2 = [d_{ij}^{(2)}] \quad \text{with} \quad d_{ij}^{(2)} = b_i g_j \quad \text{and} \quad d_{2\lambda+2,j}^{(2)} = \bar{d}_{2\lambda+2} g_j$$
$$i = 1, 2, \cdots, 2\lambda + 1; j = 1, 2, \cdots, 4\lambda + 3 .$$

EXAMPLE 4.2. $D = \{3, 6, 7, 9, 11, 12, 13, 14\}$ is a $(15, 8, 2)$ difference set in $G = \{0, 1, 2, \cdots, 14\}$ with addition mod 15 as the binary operation. D can be written as $\{3, 6, 7, 9, 11, 12, 13\} \cup \{14\}$ where $\{3, 6, 7, 9, 11, 12, 13\}$ has property P_1^*. The desired permutation in this example is $(3, 12, 7, 6, 9, 13, 11)$.

REFERENCE

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THE FREE UNIVERSITY OF IRAN
P.O. BOX 11-1962
101 KAKH ST.-SHAHREZA AVE.
TEHRAN 14
IRAN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
P.O. BOX 4348
CHICAGO, ILLINOIS 60680