The Annals of Statistics
1975, Vol. 3, No. 1, 132–154

INVARIANT NORMAL MODELS

BY STEEN ANDERSSON

University of Copenhagen

Many hypotheses in the multidimensional normal distribution are given or can be given by symmetries or, in other words, invariance. This means that the variances are invariant under a given subgroup of the general linear group in the vector space of observations. In this paper we define a class of hypotheses, the Invariant Normal Models, including all symmetry hypotheses. We derive the maximum likelihood estimator of the mean and variance and its distribution under the hypothesis. The value of the paper lies in the mathematical formulation of the theory and in the general results about hypotheses given by symmetries. Especially the formulation gives an easy simultaneous derivation of the real, complex and quaternion version of the Wishart distribution. Furthermore, we show that every invariant normal model with mean-value zero can be obtained by a symmetry.

1. Introduction and summary. In the statistical theory for the multidimensional normal distribution symmetry hypotheses in the covariance play a fundamental role. Anderson [1], Consul [8], James [10], Votaw [16], Wilks [15], Arnold [2], and Olkin and Press [14] have investigated some special cases, but a general theory seems not to exist. (Compare however Maclaren [13]). In this paper we define a class of statistical models for the multidimensional normal distribution in terms of conditions on its mean and covariance. These models will be called invariant normal models and include in particular all models specified by symmetries of the covariance, as well as the linear models for the mean.

Let $\mathscr{R}$ be the real field and let $GL_{\mathscr{R}}(\mathscr{R}^N)$ denote the group of all regular $N \times N$ matrices. For $g \in GL_{\mathscr{R}}(\mathscr{R}^N)$, g' denotes the transposed matrix. $\mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r$ denotes the set of $N \times N$ regular covariance matrices (positive definite matrices). $\mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r$ is closed under addition and multiplication by positive constants. For a family $P \subseteq \mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r$ we define $\mathscr{G}(P) = \{g \in GL_{\mathscr{R}}(\mathscr{R}^N) \mid \forall \theta \in P: g'\theta g = \theta\}$, $\mathscr{G}(P)$ is a subgroup in $GL_{\mathscr{R}}(\mathscr{R}^N)$. If I_N denotes the identity $N \times N$ matrix, then $\mathscr{G}(\{I_N\})$ is the orthogonal group O_N in $GL_{\mathscr{R}}(\mathscr{R}^N)$. Conversely, for a family $G \subseteq GL_{\mathscr{R}}(\mathscr{R}^N)$ we define $\mathscr{P}(G) = \{\theta \in \mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r \mid \forall g \in G: g'\theta g = \theta\}$. $\mathscr{P}(G)$ is also closed under addition and multiplication by positive constants. Now we can establish the following definition. A family $P \subseteq \mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r$ is said to be *reflexive* if $\mathscr{P}(\mathscr{G}(P)) = P$. It is trivial to see that every family of the form $\mathscr{P}(G)$ for a family $G \subseteq GL_{\mathscr{R}}(\mathscr{R}^N)$ is reflexive. The family of covariance matrices arising from n independent observations from a normal distribution on $\mathscr{R}^p$ with mean value zero and unknown regular covariance is a well-known example of a

Received September 1972; revised November 1973.

AMS 1970 *subject classifications*. Primary 62H05; Secondary 62H10.

Key words and phrases. Multivariate statistical analysis, maximum likelihood estimation, real, complex and quaternion Wishart distribution, hypothesis given by symmetries in the variance.

reflexive family on $\mathscr{R}^{pn}$. Such a family is said to have a real covariance structure and the corresponding model is called a mean-zero real normal model. The real covariance structure on $\mathscr{R}^{pn}$ is given by

$$P_{\mathscr{R}} = \{\Sigma \otimes_{\mathscr{R}} I_n \in \mathscr{S}_{\mathscr{R}}(\mathscr{R}^{pn})_r \mid \Sigma \in \mathscr{S}_{\mathscr{R}}(\mathscr{R}^p)_r\} ,$$

where

$$\Sigma \otimes_{\mathscr{R}} I_n = \begin{bmatrix} \Sigma & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & \Sigma \end{bmatrix}.$$

$\mathscr{G}(P_{\mathscr{R}}) = \{I_p \otimes_{\mathscr{R}} g \in GL_{\mathscr{R}}(\mathscr{R}^{pn}) \mid g \in O_n\}$, where $g = (g_{ij})$ and

$$I_p \otimes_{\mathscr{R}} g = \begin{bmatrix} g_{11} I_p & \cdot & \cdot & \cdot & g_{1n} I_p \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ g_{n1} I_p & \cdot & \cdot & \cdot & g_{nn} I_p \end{bmatrix}.$$

Clearly $\mathscr{P}(\mathscr{G}(P_{\mathscr{R}})) = P_{\mathscr{R}}$.

The group homomorphism $\Pi : O_n \to GL_{\mathscr{R}}(\mathscr{R}^{pn})$ defined by $\Pi(g) = I_p \otimes_{\mathscr{R}} g$ is an example of a group representation of the group O_n on the vector space $\mathscr{R}^{pn}$ over $\mathscr{R}$. $P_{\mathscr{R}}$ is precisely the family of regular covariance matrices invariant under Π, which is thus another way of characterizing the families we are investigating.

We shall now define two more examples of reflexive families. First let $\mathscr{C}$ denote the field of complex numbers represented by $\{x + iy \mid x, y \in \mathscr{R}\}$ and $i^2 = -1$. Define $\mathrm{Re}(x + iy) = x$ and $I(x + iy) = y$.

Let $\mathscr{S}_{\mathscr{C}}(\mathscr{C}^p)_r$ denote the set of conjugate symmetrical positive definite $p \times p$ matrices of complex numbers. For $A \in \mathscr{S}_{\mathscr{C}}(\mathscr{C}^p)_r$ the real $(2p) \times (2p)$ matrix

$$\begin{bmatrix} \mathrm{Re}(A) & I(A) \\ -I(A) & \mathrm{Re}(A) \end{bmatrix}$$

is symmetric and positive definite. Let $\mathscr{T}_{\mathscr{C}}(\mathscr{R}^{2p})_r$ denote the subset in $\mathscr{S}_{\mathscr{R}}(\mathscr{R}^{2p})_r$ of elements constructed in this way from elements in $\mathscr{S}_{\mathscr{C}}(\mathscr{C}^p)_r$.

Similarly, let $\mathscr{H}$ denote the division algebra over $\mathscr{R}$ of quaternion numbers represented by $\{x + iv + jw + ku \mid x, v, w, u \in \mathscr{R}\}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i$ and $ki = -ik = j$. Define $\mathrm{Re}(x + iv + jw + ku) = x$, $I(x + iv + jw + ku) = v$, $J(x + iv + jw + ku) = w$ and $K(x + iv + jw + ku) = u$. $\mathscr{S}_{\mathscr{H}}(\mathscr{H}^p)_r$ is defined in the same manner as $\mathscr{S}_{\mathscr{C}}(\mathscr{C}^p)_r$. For $A \in \mathscr{S}_{\mathscr{H}}(\mathscr{H}^p)_r$ the $(4p) \times (4p)$ matrix

$$\begin{bmatrix} \mathrm{Re}(A) & I(A) & J(A) & K(A) \\ -I(A) & \mathrm{Re}(A) & K(A) & -J(A) \\ -J(A) & -K(A) & \mathrm{Re}(A) & I(A) \\ -K(A) & J(A) & -I(A) & \mathrm{Re}(A) \end{bmatrix}$$

will be symmetric and positive definite. $\mathscr{F}_{\mathscr{H}}(\mathscr{R}^{4p})_r$ is defined in the same manner as $\mathscr{F}_{\mathscr{C}}(\mathscr{R}^{4p})_r$.

The family of covariance matrices arising from n independent observations from a normal distribution on $\mathscr{R}^{2p}[\mathscr{R}^{4p}]$ with an unknown covariance in the family $\mathscr{F}_{\mathscr{C}}(\mathscr{R}^{2p})_r[\mathscr{F}_{\mathscr{H}}(\mathscr{R}^{4p})_r]$ and mean value zero is also an example of a reflexive family on $\mathscr{R}^{2np}[\mathscr{R}^{4np}]$. Such a family is said to have a complex [quaternion] covariance structure and the corresponding normal model is called a mean-zero complex [quaternion] normal model.

In this paper it will be shown that a family of regular covariance matrices is reflexive if and only if it is a finite product of families of regular covariance matrices with each factor possessing a real, complex or quaternion structure. A mean-zero normal model is called reflexive if the family of regular covariance matrices is reflexive. We can then state the result above as: A mean-zero normal model is reflexive if and only if it is a finite product of models with each model being a real, complex or quaternion normal model. For a reflexive normal model we derive the maximum likelihood estimator and its distribution.

The normal models arising from a family of normal distributions on $\mathscr{R}^N$ which are invariant under a subgroup G in the symmetric group $\mathscr{S}_N$ of order N are in the literature called symmetry models. Some examples are the following:

(1) $G = \mathscr{S}_N$ gives the normal model for complete symmetry, Wilks [15].

(2) G the cyclic subgroup in $\mathscr{S}_N$, gives the circular stationary normal model, Olkin and Press, [14].

(3) $G = \mathscr{S}_{N_1} \times \cdots \times \mathscr{S}_{N_h}$ where $\mathscr{R}^N = \mathscr{R}^{N_1} \times \cdots \times \mathscr{R}^{N_h}$ is called the compound symmetry normal model, Votaw [16],
and

(4) $G = S_n$, where $\mathscr{R}^N = \mathscr{R}^n \otimes \mathscr{R}^h$ is another normal model for compound symmetry.

All symmetry normal models are special cases of the following general formulation of a mean-zero normal model.

Let G be a group and $\Pi: G \to GL_{\mathscr{R}}(\mathscr{R}^N)$ a group representation. The mean-zero normal model arising from the family $\mathscr{P}(\Pi(G)) \subseteq \mathscr{S}_{\mathscr{R}}(\mathscr{R}^N)_r$ is called a mean-zero invariant model. A mean-zero invariant model is reflexive and therefore a product of real models, complex models and quaternion models. Since the representation theory plays a fundamental role, we shall make a few comments about that. A representation Π on $\mathscr{R}^N$ is called irreducible if $\{0\}$ and $\mathscr{R}^N$ are the only Π-invariant subspaces in $\mathscr{R}^N$ and Π is called reducible if $\mathscr{R}^N$ is a direct sum of Π-invariant subspaces and the restrictions of Π to the subspaces are irreducible.

Since the decomposition of a reducible representation in irreducible parts is not unique, it is a better idea first to decompose the representation in the isotropic representations (a representation is isotropic if it is a direct sum of equivalent irreducible representations). This decomposition is unique. An isotropic representation decomposes in a unique way in tensor products (with respect to different division algebras) of the identity representation and an irreducible representation. This decomposition of a reducible representation in a direct sum of tensor products is unique. The decomposition of Π is not easily expressed in terms of the canonical basis in $\mathscr{R}^N$ and it is therefore more convenient to replace $\mathscr{R}^N$ by an abstract finite dimensional vector space E over $\mathscr{R}$. This formulation seems to be unusual in multivariate analysis. Usually one works with a vector space with a fixed basis ($\mathscr{R}^N$) or with a fixed inner product. In such a formulation the positive forms and the positive linear mappings are usually confused. But these two objects transform in a different way if the transformation of the underlying space is given by a nonorthogonal linear mapping. Also one cannot distinguish between results which depend on the choice of inner product and those that do not. This is one of the reasons that a general theory for covariance models seems to be lacking.

The existing literature contains many examples of normal models and it is the purpose of this paper to give a general formulation of a family of statistical models containing these examples as special cases in which the estimation problem can be solved. We will give a formulation, using simple algebra methods and the theory of invariant measures.

Now we can reformulate the definitions. Let E be a finite dimensional vector space over $\mathscr{R}$ and let $GL_{\mathscr{R}}(E)$ be the group of regular linear mappings on E. $\mathscr{S}_{\mathscr{R}}(E)_r$ is the positive definite forms on E. Notice that it does not make sense to talk about the unit form in $\mathscr{S}_{\mathscr{R}}(E)_r$, the eigenvalues of a positive form in $\mathscr{S}_{\mathscr{R}}(E)_r$ and the orthogonal subgroup in $GL_{\mathscr{R}}(E)$, since no basis or inner product has been chosen. Further, one cannot identify a positive form and a linear mapping.

We shall now give a short resumé of the paper. In the second section we give the basic notations and definitions. Section 3 contains a careful treatment of the more and less well-known theory of sesquilinear functionals on vectorspaces over $\mathscr{R}$, $\mathscr{C}$ or $\mathscr{H}$ and the relation to the positive semi-definite forms. In particular we define a general tensor product of positive forms. The representation theory and the reflexive families of positive forms are surveyed in the next section. In Section 5 we define the normal distribution and the invariant normal model. Further, we find the maximum likelihood estimator and its distribution. Finally, in the last section we give the results in matrix formulation and illustrate the theory with some examples.

The general idea of an invariant formulation of the multi-dimensional normal models arose from H. Brøns, one of the authors of [7]. In the paper [7], Brøns, Henningsen and Jensen give an invariant definition and treatment of a canonical hypothesis, which is a class of normal models. This class is with respect to the covariance structure, precisely all finite products of real normal models. The canonical hypothesis includes all models treated in Anderson [1], Consul [8], Olkin and Press [14], Votaw [16] and Wilks [15]. The present paper arose from the supposition that all hypotheses given by symmetries of the covariance matrix were canonical. It turns out that this supposition is wrong. Nevertheless, if one includes the division algebras $\mathscr{C}$ and $\mathscr{H}$ and generalizes the definition of the canonical hypothesis in a straightforward manner, one obtains the invariant normal model that includes all symmetry models. Section 5 is therefore only a generalization of the content of [7]. The examples in Section 6 were already treated in a thesis by Jensen.

2. Notation. Let $\mathscr{R}$ denote the real field, $\mathscr{C}$ the complex field and $\mathscr{H}$ the division algebra over $\mathscr{R}$ of quaternions. $\mathscr{D}$ denotes a division algebra over $\mathscr{R}$ isomorphic to $\mathscr{R}$, $\mathscr{C}$ or $\mathscr{H}$. There exists a unique conjugation in $\mathscr{D}$ (Bourbaki, [3], Chapter III, Section 2, No. 4, Proposition 4). This will be written $d \to \bar{d}$, $d \in \mathscr{D}$. We define $\operatorname{Re}(d) = \frac{1}{2}(d + \bar{d})$, $d \in \mathscr{D}$. Note that $\operatorname{Re}(d_1 d_2) = \operatorname{Re}(d_2 d_1)$, $d_1, d_2 \in \mathscr{D}$. We use the abbreviation left [right] $\mathscr{D}$-space for a finite dimensional left [right] vector space over $\mathscr{D}$. The scalar multiplication in a left [right] $\mathscr{D}$-space E will be denoted $(d, x) \to dx$ $[(d, x) \to xd]$, $x \in E$, $d \in \mathscr{D}$. $L_{\mathscr{D}}(E, F)$ denotes the linear mappings from the left [right] $\mathscr{D}$-space E to the left [right] $\mathscr{D}$-space F. Since $\mathscr{D}$ is an algebra over $\mathscr{R}$, every left [right] $\mathscr{D}$-space E is, by restricting the scalars to $\mathscr{R} \subseteqq \mathscr{D}$, also in a natural way an $\mathscr{R}$ space. $L_{\mathscr{D}}(E, F)$ can in a natural way be organized as a $C(\mathscr{D})$-space, where $C(\mathscr{D})$ is the center of $\mathscr{D}$. We always consider $L_{\mathscr{D}}(E, F)$ as an $\mathscr{R}$-space, since $\mathscr{R} \subseteqq C(\mathscr{D})$. $L_{\mathscr{D}}(E, E)$ is an algebra over $\mathscr{R}$. $GL_{\mathscr{D}}(E)$ is the group of regular elements in $L_{\mathscr{D}}(E, E)$. 1_E denotes the identity mapping of E. We define $E^* = L_{\mathscr{D}}(E, D)$ and $(E_{\mathscr{R}})^* = L_{\mathscr{R}}(E, \mathscr{R})$; E^* and $(E_{\mathscr{R}})^*$ are both right [left] $\mathscr{D}$-spaces by the definitions $(d, x^*) \to (x \to x^*(x)d)$ $[(x \to dx^*(x))]$ and $(d, x_{\mathscr{R}}^*) \to (x \to x_{\mathscr{R}}^*(dx))$ $[(x \to x_{\mathscr{R}}^*(xd))]$. The conjugate $\mathscr{D}$-space $\bar{E}$ is a right [left] $\mathscr{D}$-space by the definition on $(d, x) \to \bar{d}x[x\bar{d}]$. For further details see Bourbaki [3] and [5].

3. Positive forms. Let E be a left [right] $\mathscr{D}$-space.

3.1. DEFINITION. *A right [left] sesquilinear symmetric and positive functional* is a mapping $B\colon E \times E \to \mathscr{D}$ with the properties:

(i) $\forall x, x', y \in E\colon B(x + x', y) = B(x, y) + B(x', x)$.

(ii) $\forall x, y, y' \in E\colon B(x, y + y') = B(x, y) + B(x, y')$.

(iii) $\forall x, y \in E, \forall d \in \mathscr{D}\colon B(dx, y) = dB(x, y)$.

[(iii′) $\forall x, y \in E, \forall d \in \mathscr{D}\colon B(xd, y) = \bar{d}B(x, y)$.]

(iv) $\forall x, y \in E, \forall d \in \mathscr{D}\colon B(x, dy) = B(x, y)\bar{d}$.

[(iv′) $\forall x, y \in E, \forall d \in \mathscr{D}\colon B(x, yd) = B(x, y)d$.]

(v) $\forall x, y \in E\colon B(x, y) = \overline{B(y, x)}$.

(vi) $\forall x, y \in E\colon B(x, x) \geqq 0$.

A sesquilinear, symmetric and positive functional is called *definite* if

(vii) $\forall x \in E\colon x \neq 0 \Rightarrow B(x, x) > 0$.

In the case $\mathscr{D} = \mathscr{R}$, we call B a *positive (definite) form*.

$\mathscr{S}_{\mathscr{D}}(E)$ denotes the semi-vector space over $\mathscr{R}_+$ of right [left] sesquilinear, symmetric and positive functionals. $\mathscr{S}_{\mathscr{D}}(E)_r$ denotes the definite elements in $\mathscr{S}_{\mathscr{D}}(E)$.

3.2. For $B \in \mathscr{S}_{\mathscr{D}}(E)$ we set $Q(x) = B(x, x)$, and we have the identity

$$\forall x, y \in E\colon 2(B(x, y) + B(y, x)) = Q(x + y) - Q(x - y)$$

which shows that B is determined by its values on the diagonal in $E \times E$. We have

for $\mathscr{D} = \mathscr{R}$; $\quad \forall x, y \in E\colon B(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y))$,

for $\mathscr{D} \cong \mathscr{C}$;

$\forall x, y \in E, \forall d \in \mathscr{D}$:

$$(d - \bar{d})B(x, y) = \tfrac{1}{2}(dQ(x + y) - dQ(x - y) - Q(x + dy) + Q(x - dy))$$

$$[(d - \bar{d})B(x, y) = \tfrac{1}{2}(dQ(x + y) - dQ(x - y) - dQ(x + y\bar{d}) + Q(x - y\bar{d}))]\,,$$

and

for $\mathscr{D} \cong \mathscr{H}$;

$\forall x, y \in E, \forall d, e \in \mathscr{D}$:

$$\begin{aligned} 2B(x, y)(\bar{d}\bar{e} - \bar{e}\bar{d}) &= Q(x - dey) - Q(x + dey) + dQ(x + ey) - dQ(x - ey) \\ &\quad + Q(x + dy)\bar{e} - Q(x - dy)\bar{e} + dQ(x - y)\bar{e} \\ &\quad - dQ(x + y)\bar{e}\,. \end{aligned}$$

$$\begin{aligned} [2(\bar{d}\bar{e} - \bar{e}\bar{d})B(y, x) &= Q(x - yde) - Q(x + yde) + \bar{d}Q(x + ye) - \bar{d}Q(x - ye) \\ &\quad + Q(x + yd)e - Q(x - yd)e + \bar{d}Q(x - y)e \\ &\quad - \bar{d}Q(x + y)e]\,. \end{aligned}$$

The first two formulas are well known and the last one is easily verified.

3.3. Remark, that $\mathscr{S}_{\mathscr{D}}(E) = \mathscr{S}_{\mathscr{D}}(\bar{E})$.

3.4. DEFINITION. An element $\phi \in \mathscr{S}_{\mathscr{R}}(E)$ is said to have *the conjugated $\mathscr{D}$-property* if $\forall x, y \in E, \forall d \in \mathscr{D}\colon \phi(dx, y) = \phi(x, \bar{d}y)$ $[\phi(xd, y) = \phi(x, y\bar{d})]$. Let $\mathscr{F}_{\mathscr{D}}(E)$ denote the subsemi-vector space in $\mathscr{S}_{\mathscr{R}}(E)$ of elements having the conjugated $\mathscr{D}$-property. Remark, $\mathscr{S}_{\mathscr{R}}(E) = \mathscr{F}_{\mathscr{R}}(E)$ and $\mathscr{F}_{\mathscr{D}}(E) = \mathscr{F}_{\mathscr{D}}(\bar{E})$, $\mathscr{F}_{\mathscr{D}}(E)_r$ denotes the definite elements in $\mathscr{F}_{\mathscr{D}}(E)$.

3.5. PROPOSITION. *Let $B \in \mathscr{S}_{\mathscr{D}}(E)$, then* $\mathrm{Re} \circ B \in \mathscr{F}_{\mathscr{D}}(E)$. *The mapping*

$$R\colon \mathscr{S}_{\mathscr{D}}(E) \to \mathscr{F}_{\mathscr{D}}(E)$$
$$B \to \mathrm{Re} \circ B$$

is a semilinear one-to-one correspondence. $R(\mathscr{S}_{\mathscr{D}}(E)_r) = \mathscr{F}_{\mathscr{D}}(E)_r$.

PROOF. Let E be a left $\mathscr{D}$-space. $\forall x, y \in E, \forall d \in \mathscr{D}$: $\mathrm{Re}(B(dx, y)) = \mathrm{Re}(dB(x, y)) = \mathrm{Re}(B(x, y)d) = \mathrm{Re}(B(x, \bar{d}y))$, which shows that $\mathrm{Re} \circ B \in \mathscr{F}_{\mathscr{D}}(E)$. Note, that in the case $\mathscr{D} \cong \mathscr{H}$ we do not have the identity $B(dx, y) = B(x, \bar{d}y)$. From the formulas in 3.2 it follows that R is a one-to-one correspondence. The rest of the proposition is trivial. The case of E being a right $\mathscr{D}$-space is analogous.

3.6. Let F be a left [right] $\mathscr{D}$-space. If $A \in \mathscr{S}_{\mathscr{D}}(E)$ and $B \in \mathscr{S}_{\mathscr{D}}(F)$, then $R(A \oplus B) = R(A) \oplus R(B)$.

3.7. The mapping

$$\theta: \bar{E}^* \to (E_{\mathscr{R}})^* , \qquad x^* \to \mathrm{Re} \circ x^*$$

gives a natural isomorphism between the $\mathscr{R}$-spaces $\bar{E}^*$ and $(E_{\mathscr{R}})^*$. Let now $B \in \mathscr{S}_{\mathscr{D}}(E)$. The mapping

$$f_B: E \to \bar{E}^*$$
$$y \to (x \to B(x, y))[y \to (x \to B(y, x))]$$

is linear and f_B is a bijection if and only if B is definite ([5], Section 7, No. 1, Proposition 2).

In the same way we can define the $\mathscr{R}$-linear mapping $f_{R(B)}: E \to (E_{\mathscr{R}})^*$. It is easy to see that $f_{R(B)} = \theta \circ f_B$.

3.8. DEFINITION. Let $B \in \mathscr{S}_{\mathscr{D}}(E)_r$. The element $B^{-1} \in \mathscr{S}_{\mathscr{D}}(\bar{E}^*)_r$ defined by

$$B^{-1}: \bar{E}^* \times \bar{E}^* \to \mathscr{D}$$
$$(x^*, y^*) \to B(f_B{}^{-1}(x^*), f_B{}^{-1}(y^*))$$

is called *the inverse to B.*

3.9. PROPOSITION. *Let $B \in \mathscr{S}_{\mathscr{D}}(E)_r$. Then $R(B^{-1}) = R(B)^{-1}$ if we identify $\bar{E}^*$ and $(E_{\mathscr{R}})^*$ through θ.*

PROOF. $R(B)^{-1}(\theta(x^*), \theta(y^*)) = R(B)(f_{R(B)}^{-1}(\theta(x^*)), f_{R(B)}^{-1}(\theta(y^*))) = R(B)(f_B{}^{-1}(x^*), f_B{}^{-1}(y^*)) = \mathrm{Re}(B(f_B{}^{-1}(x^*), f_B{}^{-1}(y^*)) = \mathrm{Re}(B^{-1}(x^*, y^*)) = R(B^{-1})(x^*, y^*)$.

3.10. Let now F be a right $\mathscr{D}$-space and L a left $\mathscr{D}$-space. The tensor product $F \otimes_{\mathscr{D}} L$ is not in general a $\mathscr{D}$-space, but only a vector space over the center of $\mathscr{D}$. We always consider $F \otimes_{\mathscr{D}} L$ as an $\mathscr{R}$-space, since $\mathscr{R}$ always is a subalgebra of the center of $\mathscr{D}$. Remark that $F \otimes_{\mathscr{D}} L \cong \bar{L} \otimes_{\mathscr{D}} \bar{F}$.

We shall say that a mapping $W: F \times L \to M$, where M is an $\mathscr{R}$-space, has the $\mathscr{D}$-property if W is $\mathscr{R}$-bilinear and $\forall f \in F, \forall l \in L, \forall d \in \mathscr{D}$: $W(fd, l) = W(f, dl)$. Every $\mathscr{R}$-bilinear mapping $W: F \times L \to M$ having the $\mathscr{D}$-property determines one and only one $\mathscr{R}$-linear mapping $W': F \otimes_{\mathscr{D}} L \to M$ with the property $\forall f \in F, \forall l \in L$: $W'(f \otimes_{\mathscr{D}} l) = W(f, l)$ ([3], Chapter 2, Section 3, No. 1, Proposition 1).

Let $\Sigma \in \mathscr{F}_{\mathscr{D}}(F)$ and $\phi \in \mathscr{F}_{\mathscr{D}}(L)$. Σ and ϕ determine $A \in \mathscr{S}_{\mathscr{D}}(F)$ and $B \in \mathscr{S}_{\mathscr{D}}(L)$ by $\Sigma = R(A)$ and $\phi = R(B)$ (Proposition 3.5).

The mapping

$$\delta: (F \times L) \times (F \times L) \to \mathscr{R}$$
$$(f, l, f', l') \to \mathrm{Re}\,(A(f, f')\overline{B(l, l')})$$

has the property, $\forall d, e \in \mathscr{D}, \forall f, f' \in F, \forall l, l' \in L: \delta(fd, l, f'e, l') = \delta(f, dl, f', el')$. From ([3], Chapter 2, Section 3, No. 9, Remarque 2) it follows that δ determines one and only one $\mathscr{R}$-bilinear functional ρ on $F \otimes_{\mathscr{D}} L$ with the property $\rho(f \otimes_{\mathscr{D}} l, f' \otimes_{\mathscr{D}} l') = \delta(f, l, f', l')$. It is easily seen that $\rho \in \mathscr{S}_{\mathscr{R}}(F \otimes_{\mathscr{D}} L)$.

3.11. DEFINITION. The positive form on the $\mathscr{R}$-space $F \otimes_{\mathscr{D}} L$ defined above is called the *$\mathscr{D}$-tensor product of Σ and ϕ*. We denote it $\Sigma \otimes_{\mathscr{D}} \phi$.

3.12. Remark for $\mathscr{D} = \mathscr{R}$ the above $\mathscr{R}$-tensor product is the usual tensor product of positive forms.

The mapping

$$\psi: \bar{F}^* \otimes_{\mathscr{D}} \bar{L}^* \to (F \otimes_{\mathscr{D}} L)^*$$
$$f^* \otimes_{\mathscr{D}} l^* \to (f \otimes_{\mathscr{D}} l \to \mathrm{Re}\,(l^*(l) f^*(f)))$$

defines an $\mathscr{R}$-isomorphism between the $\mathscr{R}$-spaces $\bar{F}^* \otimes_{\mathscr{D}} \bar{L}^*$ and $(F \otimes_{\mathscr{D}} L)^*$.

3.13. PROPOSITION. *Let $\Sigma \in \mathscr{F}_{\mathscr{D}}(F)$ and $\phi \in \mathscr{F}_{\mathscr{D}}(L)$. Then $\Sigma \otimes_{\mathscr{D}} \phi$ is definite if and only if Σ and ϕ are definite. In that case $(\Sigma \otimes_{\mathscr{D}} \phi)^{-1} = \Sigma^{-1} \otimes_{\mathscr{D}} \phi^{-1}$ if we identify $(F \otimes_{\mathscr{D}} L)^*$ and $\bar{F}^* \otimes_{\mathscr{D}} \bar{L}^*$ through ψ defined above.*

PROOF. Let A and B be defined by $\Sigma = R(A)$ and $\phi = R(B)$. The mapping

$$\delta_3: F \times L \to \bar{F}^* \otimes_{\mathscr{D}} \bar{L}^*$$
$$(f, l) \to f_A(f) \otimes_{\mathscr{D}} f_B(l)$$

has the $\mathscr{D}$-property and defines therefore a mapping $f_A \otimes_{\mathscr{D}} f_B: F \otimes_{\mathscr{D}} L \to \bar{F}^* \otimes_{\mathscr{D}} \bar{L}^*$. From the identity $\psi(f_A(f) \otimes_{\mathscr{D}} f_B(l)) = (f' \otimes_{\mathscr{D}} l' \to \mathrm{Re}\,(f_A(f)(f') f_B(l)(l'))) = (f' \otimes_{\mathscr{D}} l' \to \mathrm{Re}\,(A(f, f')\overline{B(l, l')}))$ it follows that $f_{\Sigma \otimes_{\mathscr{D}} \phi} = \psi \circ (f_A \otimes_{\mathscr{D}} f_B)$.

Now for $l^*, l_1^* \in \bar{L}^*$ and $f^*, f_1^* \in \bar{F}^*$ we have

$$
\begin{aligned}
&(\Sigma^{-1} \otimes_{\mathscr{D}} \phi^{-1})(f^* \otimes_{\mathscr{D}} l^*, f_1^* \otimes_{\mathscr{D}} l_1^*) \\
&\quad = \mathrm{Re}\,(A^{-1}(f^*, f_1^*)\overline{B^{-1}(l^*, l_1^*)}) \\
&\quad = \mathrm{Re}\,(A(f_A^{-1}(f^*), f_A^{-1}(f_1^*))\overline{B(f_B^{-1}(l^*), f_B^{-1}(l_1^*))}) \\
&\quad = (\Sigma \otimes_{\mathscr{D}} \phi)((f_A \otimes_{\mathscr{D}} f_B)^{-1}(f^* \otimes_{\mathscr{D}} l^*), (f_A \otimes_{\mathscr{D}} f_B)^{-1}(f_1^* \otimes_{\mathscr{D}} l_1^*)) \\
&\quad = (\Sigma \otimes_{\mathscr{D}} \phi)(f^{-1}_{\Sigma \otimes_{\mathscr{D}} \phi}(\psi(f^* \otimes_{\mathscr{D}} l^*)), f^{-1}_{\Sigma \otimes_{\mathscr{D}} \phi}(\psi(f_1^* \otimes_{\mathscr{D}} l_1^*))) \\
&\quad = (\Sigma \otimes_{\mathscr{D}} \phi)^{-1}(\psi(f^* \otimes_{\mathscr{D}} l^*), \psi(f_1^* \otimes_{\mathscr{D}} l_1^*)) .
\end{aligned}
$$

3.14. Let L_1 and L_2 be left $\mathscr{D}$-spaces and let $\Sigma \in \mathscr{F}_{\mathscr{D}}(F)$, $\phi_1 \in \mathscr{F}_{\mathscr{D}}(L_1)$ and $\phi_2 \in \mathscr{F}_{\mathscr{D}}(L_2)$.

Then

$$\Sigma \otimes_{\mathscr{D}} (\phi_1 \oplus \phi_2) = (\Sigma \otimes_{\mathscr{D}} \phi_1) \oplus (\Sigma \otimes_{\mathscr{D}} \phi_2)$$

if we identify $F \otimes_{\mathscr{D}} (L_1 \oplus L_2)$ and $(F \otimes_{\mathscr{D}} L_1) \oplus (F \otimes_{\mathscr{D}} L_2)$ through the natural $\mathscr{R}$-space isomorphism.

3.15. Let L_1, L_2, ϕ_1, ϕ_2 be as in 3.14, then $(\phi_1 \oplus \phi_2)^{-1} = \phi_1^{-1} \oplus \phi_2^{-1}$ if we identify $\overline{(L_1 \oplus L_2)^*}$ and $\bar{L}_1^* \oplus \bar{L}_2^*$ through the natural $\mathscr{R}$-space isomorphism.

4. Positive forms invariant under a representation.

4.1. DEFINITION. *A representation* of a group G on an $\mathscr{R}$-space E is a homomorphism π of the group G into the group $GL_{\mathscr{R}}(E)$. E will be called *the representation space for* π. *The dimension of* π is the dimension of E. If π is a representation of G, then E is in a natural way a left module over the algebra $\mathscr{R}^{(G)}$ over $\mathscr{R}$ ([4], Section 13, No. 1, Remarque).

4.2. Let π and ρ be two representations of G on the $\mathscr{R}$-spaces E_π and E_ρ.

DEFINITION. *A representation-morphism* from π to ρ is an $\mathscr{R}$-linear operator f from E_π to E_ρ with the property $\forall g \in G: f \circ \pi(g) = \rho(g) \circ f$. $L(\pi, \rho)$ denotes the space of representation-morphisms from π to ρ. $L(\pi, \rho)$ is an $\mathscr{R}$-space. In the case $\pi = \rho$, $L(\pi, \pi)$ will be an algebra over $\mathscr{R}$.

4.3. $L(\pi, \rho)$ is the same as the $\mathscr{R}^{(G)}$-modul homomorphism from the $\mathscr{R}^{(G)}$-modul E_π to the $\mathscr{R}^{(G)}$-modul E_ρ. This ensures the existence of the following ([4]): Isomorphic representation; sub- and quotient representation; kernel, cokernel, image, and co-image for a representation-morphism; direct sum and product; irreducible (simple), reducible (semi-simple), isotropic and disjoint representations. Note that: π and ρ disjoint is equivalent to $L(\pi, \rho) = 0$. ρ isotropic is equivalent to ρ isomorphic with a direct sum of isomorphic irreducible representations (not necessarily in a unique way).

4.4. PROPOSITION. *Let π and ρ be two representations (of G). For $f \in L(\pi, \rho)$ and $f \neq 0$ we have: π irreducible $\Rightarrow f$ injective, ρ irreducible $\Rightarrow f$ surjective, and π and ρ irreducible $\Rightarrow f$ is an isomorphism.*

The proof is trivial See [4], Section 4, No. 3, Lemma 2.

4.5. From 4.4 it follows that two irreducible representations are either disjoint or isomorphic. Further, if π is irreducible, $L(\pi, \pi)$ is a division algebra over $\mathscr{R}$. $L(\pi, \pi)$ is finite dimensional. From [4], Section 11, No. 2, Theorem 2 it follows that only the three cases below can occur.

(1) $L(\pi, \pi)$ is isomorphic to $\mathscr{R}$. $L(\pi, \pi) = \{\lambda 1_E \mid \lambda \in \mathscr{R}\}$.

(2) $L(\pi, \pi)$ is isomorphic to $\mathscr{C}$. There exists $I \in L(\pi, \pi)$ with the property $I^2 = -1_E$, so $L(\pi, \pi) = \{\lambda 1_E + \mu I \mid \lambda, \mu \in \mathscr{R}\}$.

(3) $L(\pi, \pi)$ is isomorphic to $\mathscr{H}$. There exists $I, J, K \in L(\pi, \pi)$ with the properties $I^2 = J^2 = K^2 = -1_E$, $IJ = -JI = K$, $JK = -KJ = I$, and $KI = -IK = J$, so $L(\pi, \pi) = \{\lambda 1_E + \mu I + \nu J + \gamma K \mid \lambda, \mu, \nu, \gamma \in \mathscr{R}\}$.

We shall say that an irreducible representation π is of type $\mathscr{D}$, if $L(\pi, \pi)$ is isomorphic with $\mathscr{D}$.

4.6. Let π be a finite dimensional irreducible representation on the $\mathscr{R}$-space

L. The mapping $(f, x) \to f(x)$ from $L(\pi, \pi) \times L$ into L gives L a $\mathscr{D}$-space structure, ($\mathscr{D} = L(\pi, \pi)$). Note that the restriction to the reals in $L(\pi, \pi)$ gives the original $\mathscr{R}$-space structure on L. Let F be a right $\mathscr{D}$-space. The homomorphism $1_F \otimes_{\mathscr{D}} \pi : G \to GL_{\mathscr{R}}(F \otimes_{\mathscr{D}} L)$ defined by $g \to 1_F \otimes_{\mathscr{D}} \pi(g)$ gives a new representation on the $\mathscr{R}$ space $F \otimes_{\mathscr{D}} L$. *The representation* $1_F \otimes_{\mathscr{D}} \pi$ *is* π*-isotropic*. The mapping $(d, f) \to f \circ d$ from $\mathscr{D} \times L(\pi, 1_F \otimes_{\mathscr{D}} \pi)$ into $L(\pi, 1_F \otimes_{\mathscr{D}} \pi)$ gives $L(\pi, 1_F \otimes_{\mathscr{D}} \pi)$ a right $\mathscr{D}$-space structure. The mapping $\alpha : F \to L(\pi, 1_F \otimes_{\mathscr{D}} \pi)$ defined by $h \to (x \to h \otimes_{\mathscr{D}} x)$ is the canonical $\mathscr{D}$-space isomorphism between F and $L(\pi, 1_F \otimes_{\mathscr{D}} \pi)$. If both F and F' are right $\mathscr{D}$-spaces, the mapping $S : L_{\mathscr{D}}(F, F') \to L(1_F \otimes_{\mathscr{D}} \pi, 1_{F'} \otimes_{\mathscr{D}} \pi)$ defined by $f \to f \otimes 1_L$ is an $\mathscr{R}$-space isomorphism. In the case $F = F'$ it is also an algebra isomorphism. On the other hand, let ρ be a π-isotrope representation on the $\mathscr{R}$-space E. Again $L(\pi, \rho)$ is a right $\mathscr{D}$-space. The mapping $\beta : L(\pi, \rho) \otimes_{\mathscr{D}} L \to E$ defined by $f \otimes x \to f(x)$ is the canonical representation isomorphism between $1_{L(\pi,\rho)} \otimes_{\mathscr{D}} \pi$ and ρ. If both ρ and ρ' are π-isotrope, the mapping $T : L(\rho, \rho') \to L_{\mathscr{D}}(L(\pi, \rho), L(\pi, \rho'))$ defined by $u \to (f \to u \circ f)$ is an $\mathscr{R}$-space isomorphism. In the case $\rho = \rho'$, it is also an algebra isomorphism. For further details see [4], Section 1. From this and [4], Section 3, No. 4, Proposition 9 follows:

4.7. PROPOSITION. *If ρ is a reducible representation of G on the $\mathscr{R}$-space E, then ρ is canonically isomorphic with a direct sum of disjoint isotropic representation and each isotrope representation is canonically isomorphic with a representation of the form $1_F \otimes_{\mathscr{D}} \pi$ on the $\mathscr{R}$-space $F \otimes_{\mathscr{D}} L$. Here π is an irreducible representation on the $\mathscr{R}$-space L of type $\mathscr{D}$. If ρ' is another reducible representation then $L(\rho, \rho')$ is $\mathscr{R}$-isomorphic with a direct sum of $\mathscr{R}$-space of the form $L_{\mathscr{D}}(F, F')$, where F and F' come from the isotropic parts of ρ and ρ' respectively.*

4.8. From this we conclude that it is enough to look at representations of the form

$$E = \oplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} L_q)$$
$$\rho = \oplus_{q \in Q} (1_{F_q} \otimes_{\mathscr{D}_q} \pi_q)$$

where F_q is a right $\mathscr{D}_q$-space, $\mathscr{D}_q$ isomorphic with $\mathscr{R}$, $\mathscr{C}$ or $\mathscr{H}$, and π_q an irreducible representation of type $\mathscr{D}_q$ on the $\mathscr{R}$-space L_q. Let

$$E = \oplus_q (\oplus_{t \in T_q} E_{qt})$$
$$\rho = \oplus_q (\oplus_{t \in T_q} \pi_{qt})$$

be a decomposition of ρ in irreducible parts, such that $\pi_{qt} \cong \pi_{q't'} \Leftrightarrow q = q'$. Hence $\dim_{\mathscr{D}} (F_q) =$ the number of elements in T_q.

4.9. DEFINITION. Let π be a representation of G on the $\mathscr{R}$-space E. *The dual of* π is a representation π^* of G on the $\mathscr{R}$-space E^* defined by $(\pi^*(g)(x^*))(x) = x^*(\pi(g^{-1})x) \cdot x \in E$, $x^* \in E^*$, $g \in G$. Remark: π is irreducible if type $\mathscr{D} \Leftrightarrow \pi^*$ is irreducible of type $\mathscr{D}$.

4.10. Definition. Let π be a representation of G on the $\mathscr{R}$-space E. A $\phi \in \mathscr{S}_{\mathscr{R}}(E)$ is said to be *π-invariant* if $\forall g \in G, \forall x, y \in E$: $\phi(\pi(g)x, \pi(g)y) = \phi(x, y)$. Remark: To ensure ϕ π-invariant it is enough that $\phi(\pi(g)x, \pi(g)x) = \phi(x, x)$. ϕ is π-invariant if and only if $f_\phi \in L(\pi, \pi^*)$.

4.11. Lemma. *The null space of a π-invariant positive form is π-invariant, i.e. defines a subrepresentation.*

The proof is trivial.

4.12. Note that the subrepresentation defined by the null space may not have a complement. Therefore we shall usually assume reducibility of π. Also note that the existence of a π-invariant positive definite form automatically gives reducibility of π.

4.13. Proposition. *Let π be a representation. There exists a π-invariant positive definite form if and only if $\pi(G)$ is relatively compact.*

For proof, see [6], Section 3, No. 1, Proposition 1.

4.14. Proposition. *Let π be an irreducible representation. Then either the null form is the only π-invariant positive form or all π-invariant positive forms different from the null form are definite and proportional.*

Proof. The first two properties follow from Lemma 4.11. To prove that all π-invariant positive definite forms are proportional let ϕ_1 and ϕ_2 be π-invariant positive definite forms. Choose a basis for E such that ϕ_1 is the identity matrix and ϕ_2 is a diagonal matrix. It is easy to see that there exists a $\lambda > 0$, such that $\lambda\phi_1 - \phi_2$ is positive but not definite. The result again follows from Lemma 4.11.

4.15. Let π be an irreducible representation on the $\mathscr{R}$-space L. Hence π is of type $L(\pi, \pi)$. If ϕ is a π-invariant positive definite form on L, the adjoint mapping with respect to ϕ determines an anti-isomorphism $d \to d^*$ of $L(\pi, \pi)$. The mapping $(l, l') \to \phi(dl, dl')$ defines for all $d \in \mathscr{D}$ a π-invariant positive definite form on L. From Proposition 4.14 it follows that $\forall d \in \mathscr{D}$: $dd^* \in \mathscr{R} \subseteq L(\pi, \pi)$. Since $\forall d \in \mathscr{D}$: $(d + 1)(d + 1)^* \in \mathscr{R}$, it also follows that $(d + 1)(d^* + 1) = dd^* + d + d^* + 1 \in \mathscr{R}$. But from this we have the property $\forall d \in \mathscr{D}$: $d + d^* \in \mathscr{R}$. Since the adjoint mapping on $L(\pi, \pi)$ has the properties $\forall d \in \mathscr{D}$: $d^*d \in \mathscr{R}$ and $d^* + d \in \mathscr{R}$, it follows from [4], Chapter III, Section 2, No. 4, Proposition 4 that $d^* = \bar{d}$, $\forall d \in \mathscr{D}$, and therefore also that $\phi \in \mathscr{F}_{L(\pi,\pi)}(L)$.

4.16. Proposition. *Let π_1 and π_2 be two disjoint reducible representations on the $\mathscr{R}$-spaces E_1 and E_2 respectively. Let ϕ be a positive form on the $\mathscr{R}$-space $E_1 \oplus E_2$. Then ϕ is $\pi_1 \oplus \pi_2$-invariant if and only if the restriction $\phi \mid E_i$ of ϕ to E_i is π_i-invariant, $i = 1, 2$ and $\phi = \phi \mid E_1 \oplus \phi \mid E_2$.*

Proof. "If" is trivial. "Only if": $\phi \mid E_i$ is clearly π_i-invariant $i = 1, 2$. ϕ is $\pi_1 \oplus \pi_2$-invariant $\Rightarrow f_\phi \in L(\pi_1 \oplus \pi_2, (\pi_1 \oplus \pi_2)^*)$. $(\pi_1 \oplus \pi_2)^*$ is isomorphic with $\pi_1^* \oplus \pi_2^*$ if we identified $(E_1 \oplus E_2)^*$ and $E_1^* \oplus E_2^*$ through the natural $\mathscr{R}$-space isomorphism ψ. Since π_1^* and π_2^* are disjoint we have $\psi \circ f_\phi = f_{\phi_1} \oplus f_{\phi_2}$ where $\phi_1 = \phi \mid E_1$ and $\phi_2 = \phi \mid E_2$.

4.17. PROPOSITION. *Let $\rho = 1_F \otimes_{\mathscr{D}} \pi$ be a π-isotropic representation of G on the $\mathscr{R}$-space $F \otimes_{\mathscr{D}} L$. Let Γ be a positive form on $F \otimes_{\mathscr{D}} L$. Then Γ is ρ-invariant if and only if there exists a $\Sigma \in \mathscr{F}_{\mathscr{D}}(F)$, such that $\Gamma = \Sigma \otimes_{\mathscr{D}} \phi$,* where ϕ is a π-*invariant positive form on L.*

PROOF. "If" is trivial. "Only if": If the only π-invariant positive form on L is the trivial one it follows that $\Gamma = 0$ and $\Gamma = \Sigma \otimes_{\mathscr{D}} 0$, for $\Sigma \in \mathscr{F}_{\mathscr{D}}(F)$. Next let ϕ be a positive definite π-invariant form on L. Γ is ρ-invariant $\Leftrightarrow$ $f_\Gamma \in L(\rho, \rho^*)$. Define $B \in \mathscr{S}_{\mathscr{D}}(L)$ by $\phi = R(B)$. We have $f_\phi = \theta \circ f_B$ (3.6). If π^* is the dual representation to π on the $\mathscr{R}$-space $(L_{\mathscr{R}})^*$ and we define the representation π_1^* on $\bar{L}^*$ by $\pi_1^* = \theta^{-1} \circ \pi \circ \theta$, π_1^* will be irreducible of type $\mathscr{D}$ and $f_B \in L(\pi, \pi_1^*)$. Since ρ^* is isomorphic with $1_{\bar{F}^*} \otimes_{\mathscr{D}} \pi_1^*$ through ψ (3.12) there exists an $h \in L_{\mathscr{D}}(F, \bar{F}^*)$ such that $f_\Gamma = \psi(h \otimes_{\mathscr{D}} f_B)$. It is easy to see that there exists an $A \in \mathscr{S}_{\mathscr{D}}(F)$ such that $h = f_A$. From this it follows that $\Gamma = \Sigma \otimes_{\mathscr{D}} \phi$, where $\Sigma = R(A)$.

4.18. THEOREM. *Let ρ be a reducible representation of G on the $\mathscr{R}$-space E. We identify ρ with its canonical decomposition $\rho = \bigoplus_{q \in Q} (1_{F_q} \otimes_{\mathscr{D}_q} \pi_q)$. Then a positive form Γ on E is ρ-invariant if and only if there for every $q \in Q$ exists a $\Sigma_q \in \mathscr{F}_{\mathscr{D}_q}(F_q)$, such that $\Gamma = \bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathscr{D}_q} \phi_q)$, where ϕ_q is π_q-invariant on L_q.*

The proof follows from 4.16 and 4.17.

4.19. COROLLARY. *If ρ is multiplicity free, i.e.* $\dim_{\mathscr{D}_q}(F_q) = 1$ *for every $q \in Q$ there exists $a_q \geqq 0$, such that $\Gamma = \bigoplus_{q \in Q} a_q \phi_q$.*

4.20. PROPOSITION. *Let ρ be a representation of G on the $\mathscr{R}$-space E. If there exists a ρ-invariant positive definite form on E, then ρ is reducible, i.e. $\rho = \bigoplus_{q \in Q} (1_{F_q} \otimes_{\mathscr{D}_q} \pi_q)$ and for all $q \in Q$, there exists a π_q-invariant positive definite form ϕ_q on L_q. Let $\mathscr{P}_1(\rho(G))$ denote the ρ-invariant positive forms on the $\mathscr{R}$-space E. The mapping*

$$\Theta: \bigtimes_{q \in Q} \mathscr{F}_{\mathscr{D}_q}(F_q) \to \mathscr{P}_1(\rho(G))$$
$$(\Sigma_q)_{q \in Q} \to \bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathscr{D}_q} \phi_q)$$

is one-to-one. The null space of $\Theta((\Sigma_q)_{q \in Q})$ is $\bigoplus_{q \in Q} (N_q \otimes_{\mathscr{D}_q} L_q)$, where N_q is the null space of Σ_q. $\Theta((\Sigma_q)_{q \in Q}) = \bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathscr{D}_q} \phi_q)$ is definite if and only if Σ_q is definite for all $q \in Q$.

The proof is trivial.

4.21. COROLLARY. *Let the situation be as in* 4.18. *Define $Q' = \{q \in Q \mid L_q$ has only a trivial π_q-invariant positive form$\}$. The null space of a ρ-invariant positive form is then $\bigoplus_{q \in Q'} (F_q \otimes_{\mathscr{D}_q} L_q) \oplus (\bigoplus_{q \in Q \setminus Q'} (N_q \otimes_{\mathscr{D}_q} L_q))$, where N_q is the null space of Σ_q.*

The proof is trivial.

4.22. For a family $P \subseteq \mathcal{S}_{\mathcal{R}}(E)_r$ and for a family $T \subseteq GL_{\mathcal{R}}(E)$ we define

$$\mathcal{G}(P) = \{g \in GL_{\mathcal{R}}(E) \mid \forall x, y \in E, \forall \theta \in P: \theta(g(x), g(y)) = \theta(x, y)\}$$

and

$$\mathcal{P}(T) = \{\theta \in \mathcal{S}_{\mathcal{R}}(E)_r \mid \forall x, y \in E, \forall g \in T: \theta(g(x), g(y)) = \theta(x, y)\} .$$

$\mathcal{G}(P)$ is a subgroup in $GL_{\mathcal{R}}(E)$ and $\mathcal{P}(T)$ is a subset in $\mathcal{S}_{\mathcal{R}}(E)_r$ closed under addition and multiplication by positive constants.

4.23. The following properties are trivial. $P_1 \subseteq P_2 \subseteq \mathcal{S}_{\mathcal{R}}(E)_r \Rightarrow \mathcal{G}(P_1) \supseteq \mathcal{G}(P_2)$, $T_1 \subseteq T_2 \subseteq GL_{\mathcal{R}}(E) \Rightarrow \mathcal{P}(T_1) \supseteq \mathcal{P}(T_2)$, $\mathcal{P}(GL_{\mathcal{R}}(E)) = \emptyset$, $\mathcal{G}(\mathcal{S}_{\mathcal{R}}(E)) = \{-1_E, 1_E\}$ and for $\forall \Sigma \in \mathcal{S}_{\mathcal{R}}(E)_r$: $\mathcal{G}(\{\Sigma\}) = \mathcal{G}(\{\lambda\Sigma \mid \lambda \in \mathcal{R}_+\})$. $P_1 \neq \emptyset \Rightarrow \mathcal{G}(P)$ is compact.

4.24. DEFINITION. A family $P \subseteq \mathcal{S}_{\mathcal{R}}(E)$ is called *reflexive* if $\mathcal{P}(\mathcal{G}(P)) = P$.

4.25. PROPOSITION. *$\mathcal{P}(T)$ is reflexive for all families $T \subseteq GL_{\mathcal{R}}(E)$.*

PROOF. $\mathcal{P}(T) \subseteq \mathcal{P}(\mathcal{G}(\mathcal{P}(T)))$ is trivial. Since $T \subseteq \mathcal{G}(\mathcal{P}(T))$ we also have $\mathcal{P}(T) \supseteq \mathcal{P}(\mathcal{G}(\mathcal{P}(T)))$.

4.26. PROPOSITION. *A nonempty family $P \subseteq \mathcal{S}_{\mathcal{R}}(E)_r$ is reflexive if and only if there exists a decomposition $E = \bigoplus_{q \in Q} (F_q \otimes_{\mathcal{D}_q} L_q)$ and for every $q \in Q$ a $\phi_q \in \mathcal{F}_{\mathcal{D}_q}(L_q)_r$, such that*

$$P = \{\textstyle\bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathcal{D}_q} \phi_q) \mid \forall q \in Q : \Sigma_q \in \mathcal{F}_{\mathcal{D}_q}(F_q)_r\} .$$

PROOF. For every $P \subseteq \mathcal{S}_{\mathcal{R}}(E)_r$, $\mathcal{G}(P)$ is a compact group. Therefore the representation of $\mathcal{G}(P)$ on E defined by the imbedding of $\mathcal{G}(P)$ in $GL_{\mathcal{R}}(E)$ is reducible. If P is reflexive then P is defined by invariants under $\mathcal{G}(P)$ and the result follows from 4.20. Conversely, let $P = \{\bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathcal{D}_q} \phi_q) \mid \forall q \in Q : \Sigma_q \in \mathcal{F}_{\mathcal{D}_q}(F_q)\}$. Let $B_q \in \mathcal{S}_{\mathcal{D}_q}(L_q)$, such that $\phi_q = \mathcal{R}(B_q)$. The orthogonal group $O(B_q)$ in the left $\mathcal{D}_q$-space L_q with respect to B_q is irreducible of type $\mathcal{D}_q$ in the $\mathcal{R}$-space L_q. Define

$$\begin{aligned} G_q &= O(B_q) && \text{if } \dim_{\mathcal{D}_q}(F_q) > 1 , \\ &= O(\phi_q) && \text{if } \dim_{\mathcal{D}_q}(F_q) = 1 . \end{aligned}$$

The group representation

$$\begin{aligned} &\textstyle\mathsf{X}_{q \in Q}\, G_q \to GL_{\mathcal{R}}(E) \\ &(g_q)_{q \in P} \to \textstyle\bigoplus_{q \in Q} (1_{F_q} \otimes_{\mathcal{D}_q} g_q) \end{aligned}$$

is reducible and defines the family P; therefore P is reflexive (4.25).

5. The invariant normal model.

5.1. Let E be left [right] $\mathcal{D}$-space. From 3.10 it follows that the $\mathcal{R}$-space of bilinear mappings from $E^* \times \bar{E}^*$ [$\bar{E}^* \times E^*$] with the $\mathcal{D}$-property can be identified with $(E^* \otimes_{\mathscr{D}} \bar{E}^*)^*$ $[(\bar{E}^* \otimes_{\mathscr{D}} E^*)^*]$. The mapping:

$$\sigma: \bar{E} \otimes_{\mathscr{D}} E \to (E^* \otimes_{\mathscr{D}} \bar{E}^*)^*, [E \otimes_{\mathscr{D}} \bar{E} \to (\bar{E}^* \otimes_{\mathscr{D}} E^*)^*]$$
$$x \otimes_{\mathscr{D}} y \to (x^* \otimes_{\mathscr{D}} y^* \to \operatorname{Re}(x^*(x)y^*(y)))$$

is an isomorphism between $\mathscr{R}$-spaces. This follows from (3.12) and the natural isomorphism $E \cong E^{**}(x \to (x^* \to x^*(x)))$. Define $P_{\mathscr{D}}(E) = \{\sum_{\nu=1}^{N} x_\nu \otimes_{\mathscr{D}} x_\nu \mid x_\nu \in E, N = 1, 2, \cdots\}$. For $\theta \in P_{\mathscr{D}}(E)$ it follows that $\sigma(\theta) \in \mathscr{T}_{\mathscr{D}}(E^*)$. On the other hand, let $\Sigma \in \mathscr{T}_{\mathscr{D}}(E^*)$. Let $e_1^*, \cdots, e_n^*$ be a $\mathscr{D}$-basis for E^*, such that Σ is a diagonal matrix with the nonnegative diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$. For $\theta = \sum_{\nu=1}^{n} \lambda_\nu^{\frac{1}{2}} e_\nu \otimes_{\mathscr{D}} \lambda_\nu^{\frac{1}{2}} e_\nu$, where $e_1, \cdots, e_n$ is the dual basis in the $\mathscr{D}$-space E to $e_1^*, \cdots, e_n^*$, we have $\sigma(\theta) = \Sigma$. Therefore $\sigma: P_{\mathscr{D}}(E) \to \mathscr{T}_{\mathscr{D}}(E^*)$ is a semilinear isomorphism between the two semi-vector spaces over $\mathscr{R}_+$. If F is another left [right] D-space and $f \in L_{\mathscr{D}}(E, F)$ we define the mapping $P_{\mathscr{D}}(f): P_{\mathscr{D}}(E) \to P_{\mathscr{D}}(F)$ by $P_{\mathscr{D}}(f)(\sum_{\nu=1}^{N} x_\nu \otimes_{\mathscr{D}} x_\nu) = \sum_{\nu=1}^{N} f(x_\nu) \otimes_{\mathscr{D}} f(x_\nu)$. In the case $E = F$, $GL_{\mathscr{D}}(E)$ defines a transitive left action on $P_{\mathscr{D}}(E)_r (= \sigma^{-1}(\mathscr{T}_{\mathscr{D}}(E^*)_r))$. The existence of a $GL_{\mathscr{D}}(E)$-invariant measure μ on $P_{\mathscr{D}}(E)_r$, unique up to a multiplicative positive constant follows from (3.5) and [6], Section 3, No. 3, Example 8.

5.2. Now let E be an $\mathscr{R}$-space, λ a Lebesgue measure on E, $\Gamma \in P_{\mathscr{R}}(E)_r$ and $\xi \in E$. The function

$$\phi_{\Gamma,\xi}: E \to \mathscr{R}$$
$$x \to \exp\{-\tfrac{1}{2}\Gamma^{-1}((x - \xi) \otimes_{\mathscr{R}} (x - \xi))\}$$

is positive, continuous and λ-integrable. We define $n(\Gamma) = \int_E \phi_{\Gamma,\xi}\, d\lambda$.

DEFINITION. *The regular normal distribution* with variance $\Gamma \in P_{\mathscr{R}}(E)_r$ and mean value $\xi \in E$ is the measure

$$N(\Gamma, \xi) = n(\Gamma)^{-1} \phi_{\Gamma,\xi} \lambda$$

on E. The set of regular normal distributions on E is denoted by $\mathscr{N}(E)$.

5.3. The following properties are well known. If $E = E_1 \oplus E_2$, $\Gamma = \Gamma_1 \oplus \Gamma_2$, where $\Gamma_1 \in P_{\mathscr{R}}(E_1)_r$ and $\Gamma_2 \in P_{\mathscr{R}}(E_2)_r$ and $\xi = \xi_1 \oplus \xi_2$, where $\xi_1 \in E_1$ and $\xi_2 \in E_2$, then

$$N(\Gamma_1 \oplus \Gamma_2, \xi_1 \oplus \xi_2) = N(\Gamma_1, \xi_2) \otimes N(\Gamma_2, \xi_2) .$$

In particular

$$n(\Gamma_1 \oplus \Gamma_2) = n(\Gamma_1) n(\Gamma_2) .$$

For $f \in GL_{\mathscr{R}}(E)$ we have

$$fN(\Gamma, \xi) = N(P_{\mathscr{R}}(f)\Gamma, f(\xi))$$

and in particular

$$n(P_{\mathscr{R}}(f)\Gamma) = |\det f| n(\Gamma) .$$

Further,

$$\int_E x\, dN(\Gamma, \xi)(x) = \xi$$

and

$$\int_E (x - \xi) \otimes_{\mathscr{R}} (x - \xi)\, dN(\Gamma, \xi)(x) = \Gamma.$$

5.4. DEFINITION. *A $\mathscr{D}$-normal model on E* is a decomposition $E = F \otimes_{\mathscr{D}} (L \oplus L_1)$ of E together with a parametrized family of regular normal distributions on E given by the injections

$$\begin{aligned} m : F \otimes_{\mathscr{D}} L_1 &\to F \otimes_{\mathscr{D}} (L \oplus L_1) \\ f \otimes_{\mathscr{D}} l_1 &\to f \otimes_{\mathscr{D}} (0, l_1) \end{aligned}$$

and

$$\begin{aligned} v : P_{\mathscr{D}}(F)_r &\to P_{\mathscr{R}}(F \otimes_{\mathscr{D}} (L \oplus L_1))_r \\ \Sigma &\to \Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1)\,, \end{aligned}$$

where $\phi \in P_{\mathscr{D}}(L)_r$ and $\phi_1 \in P_{\mathscr{D}}(L_1)$ are given.

5.5. Since $E = F \otimes_{\mathscr{D}} (L \oplus L_1) \cong (F \otimes_{\mathscr{D}} L) \oplus (F \otimes_{\mathscr{D}} L_1)$, $F \otimes_{\mathscr{D}} L$, is a subspace in E. The form of the positive definite form $\phi \oplus \phi_1$ on $L \oplus L_1$ ensures that L and L_1 are orthogonal with respect to $\phi \oplus \phi_1$.

In the case $L_1 = \{0\}$ we get a mean-zero $\mathscr{D}$-normal model.

In the case $\mathscr{D} = \mathscr{R}$, $F \otimes_{\mathscr{D}} L_1$ is usually called the regression (or mean) manifold. Usually one has chosen a basis in L and L_1 such that ϕ and ϕ_1, and therefore also $\phi \oplus \phi_1$ are represented by the identity matrix.

5.6. We shall now find the maximum likelihood estimator for the problem defined in 5.5., and its distribution.

The regular normal distribution $N(v(\Sigma), m(\xi))$ defined by the parameter $(\Sigma, \xi) \in P_{\mathscr{D}}(F)_r \times F \otimes_{\mathscr{D}} L_1$ has the density

$$\begin{aligned} &\phi_{\Sigma \otimes_{\mathscr{D}} \phi, 0} \cdot \phi_{\Sigma \otimes_{\mathscr{D}} \phi_1, \xi} : \\ &\qquad (F \otimes_{\mathscr{D}} L) \oplus (F \otimes_{\mathscr{D}} L_1) \to \mathscr{R} \\ &\qquad (x, y) \to n(\Sigma \otimes_{\mathscr{D}} \phi)^{-1} \exp\{-\tfrac{1}{2}(\Sigma^{-1} \otimes_{\mathscr{D}} \phi^{-1})(x \otimes_{\mathscr{D}} x)\} n(\Sigma \otimes_{\mathscr{D}} \phi_1) \\ &\qquad\qquad \times \exp\{-\tfrac{1}{2}(\Sigma^{-1} \otimes_{\mathscr{D}} \phi_1{}^{-1})(y - \xi) \otimes_{\mathscr{R}} (y - \xi)\} \end{aligned}$$

with respect to a Lebesgue measure $\lambda \otimes_{\mathscr{R}} \lambda_1$, where λ and λ_1 are Lebesgue measures on $(F \otimes_{\mathscr{D}} L)$ and $(F \otimes_{\mathscr{D}} L_1)$ respectively.

Let $B \in \mathscr{S}_{\mathscr{D}}(\bar{L}^*)$ be defined by $\phi = \mathscr{R}(B)$. The mapping

$$\begin{aligned} \delta : (F \times L) \times (F \times L) &\to F \otimes_{\mathscr{D}} \bar{F} \\ (f, l, f', l') &\to f \otimes_{\mathscr{D}} B^{-1}(l, l') f' \end{aligned}$$

determined one and only one $\mathscr{R}$-linear mapping ([3], Section 3) $\Theta_\phi : (F \otimes_{\mathscr{D}} L) \otimes_{\mathscr{R}} (F \otimes_{\mathscr{D}} L) \to F \otimes_{\mathscr{D}} \bar{F}$, with the property $\Theta_\phi((f \otimes_{\mathscr{D}} l) \otimes_{\mathscr{R}} (f' \otimes_{\mathscr{D}} l')) = \delta(f, l, f', l')$. For $x \in F \otimes_{\mathscr{D}} L$ define $s(x) = \Theta_\phi(x \otimes_{\mathscr{R}} x)$. Since $(\Sigma^{-1} \otimes_{\mathscr{D}} \phi^{-1})(x \otimes_{\mathscr{R}} x) = \Sigma^{-1}(s(x))$ it follows that $(s(x), y)$ is a sufficient statistic for (Σ, ξ). $s(x)$ is usually called the unnormed empirical variance. $s(x) \in P_{\mathscr{D}}(F) \subset F \otimes_{\mathscr{D}} \bar{F}$ since $F \otimes_{\mathscr{D}} L$ is $\mathscr{R}$-isomorphic with $L_{\mathscr{D}}(L^*, F)$ ([3], Chapter 2, Section 4, No. 2) and a direct calculation shows that $s(x) = P_{\mathscr{D}}(x)(\phi^{-1})$. If $\dim_{\mathscr{D}}(F) \leqq \dim_{\mathscr{D}}(L)$ then the elements of $F \otimes_{\mathscr{D}} L$ ($\cong L_{\mathscr{D}}(L^*, F)$) of full rank constitute an open dense subset $(F \otimes_{\mathscr{D}} L)_r$. Hence $s(x) \in P_{\mathscr{D}}(F)_r$ with probability 1 (with respect to all the distributions in the model) and s is a surjection from $(F \otimes_{\mathscr{D}} L)_r$ to $P_{\mathscr{D}}(F)_r$. A direct calculation shows that s commutes with the natural left actions of $GL_{\mathscr{D}}(F)$ on $(F \otimes_{\mathscr{D}} L)_r$ and $P_{\mathscr{D}}(F)_r$. The fact that s commutes with the actions of $GL_{\mathscr{D}}(F)$ gives that the measure ν on $P_{\mathscr{D}}(F)_r$ defined by $d\nu(\theta) = n(\theta \otimes_{\mathscr{D}} \phi)^{-1} d(s(\lambda))(\theta)$ is the $GL_{\mathscr{D}}(F)$-invariant measure on $P_{\mathscr{D}}(F)_r$. The following calculation gives the distribution of $s(x)$.

$$\begin{aligned} d(sN(\Sigma \otimes_{\mathscr{D}} \phi, 0))(\theta) &= n(\Sigma \otimes_{\mathscr{D}} \phi)^{-1} \exp\{\tfrac{1}{2}\Sigma^{-1}(\theta)\} d(s(\lambda))(\theta) \\ &= \frac{n(\theta \otimes_{\mathscr{D}} \phi)}{n(\Sigma \otimes_{\mathscr{D}} \phi)} \exp\{-\tfrac{1}{2}\Sigma^{-1}(\theta)\}\, d\nu(\theta)\,. \end{aligned}$$

DEFINITION. The measure

$$dW(\Sigma, \phi)(\theta) = \frac{n(\theta \otimes_{\mathscr{D}} \phi)}{n(\Sigma \otimes_{\mathscr{D}} \phi)} \exp\{-\tfrac{1}{2}\Sigma^{-1}(\theta)\}\, d\nu(\theta)$$

is called the *$\mathscr{D}$-Wishart distribution* on $P_{\mathscr{D}}(F)_r$ with parameters (Σ, ϕ).

5.7. PROPOSITION. *The maximum likelihood estimator* $(\hat{\Sigma}, \hat{\xi})$ *for* (Σ, ξ) *in a $\mathscr{D}$-normal model is*

$$(\dim_{\mathscr{R}} (L \oplus L_1)^{-1} \cdot s(x), y)\,.$$

PROOF. This proof is copied from [7] with $\mathscr{R}$ replaced by $\mathscr{D}$. The density of the sufficient statistic $(s(x), y)$ is

$$\begin{aligned} &\frac{n(s(x) \otimes_{\mathscr{D}} \phi)}{n(\Sigma \otimes_{\mathscr{D}} \phi)} \exp\{-\tfrac{1}{2}\Sigma^{-1}(s(x))\} \\ &\qquad \times \frac{1}{n(\Sigma \otimes_{\mathscr{D}} \phi_1)} \exp\{-\tfrac{1}{2}(\Sigma^{-1} \otimes_{\mathscr{D}} \phi_1^{-1})(y - \xi) \otimes_{\mathscr{R}} (y - \xi)\}\,. \end{aligned}$$

It is evident that $\hat{\xi} = y$, and it only remains to maximize

$$\begin{aligned} &\frac{n(s(x) \otimes_{\mathscr{D}} \phi)}{n(\Sigma \otimes_{\mathscr{D}} \phi) n(\Sigma \otimes_{\mathscr{D}} \phi_1)} \exp\{-\tfrac{1}{2}\Sigma^{-1}(s(x))\} \\ &\qquad = \frac{n(s(x) \otimes_{\mathscr{D}} (\phi \oplus \phi_1))}{n(\Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1))} \exp\{-\tfrac{1}{2}\Sigma^{-1}(s(x))\} \cdot \frac{1}{n(s(x) \otimes_{\mathscr{D}} \phi_1)}\,. \end{aligned}$$

$[n(s(x) \otimes_{\mathscr{D}} \phi_1)]^{-1}$ does not depend on Σ, so the problem is reduced to the problem of maximizing the density of the Wishart distribution with parameters $(\Sigma, \phi \oplus \phi_1)$.

We consider therefore the mean-value zero $\mathscr{D}$-normal model given by $E = F \otimes_{\mathscr{D}} (L \oplus L_1)$ and

$$\begin{aligned} \iota : P_{\mathscr{D}}(F)_r &\rightarrow P_{\mathscr{R}}(F \otimes_{\mathscr{D}} (L \oplus L_1))_r \\ \Sigma &\rightarrow \Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1)\,. \end{aligned}$$

The sufficient statistic $t = s(z)$ will have a Wishart distribution with parameters $(\Sigma, \phi \oplus \phi_1)$. This is an exponential family and the maximum likelihood estimator is defined by the equation ($W = W(\Sigma, \phi \oplus \phi_1)$)

$$\int_{P_{\mathscr{D}}(F)_r} \theta \, dW(\theta) = t \, .$$

Now

($N = N(\Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1), 0)$)

$$\int_{P_{\mathscr{D}}(F)_r} \theta \, dW(\theta) = \int_E s(z) \, dN(z) = \int_E \theta_{\phi \oplus \phi_1}(z \otimes_{\mathscr{R}} z) \, dN(z) \, .$$

For $f, h \in L_{\mathscr{R}}(F \otimes_{\mathscr{D}} (L \oplus L_1), F)$, $\int_E (f \otimes_{\mathscr{R}} h)(z \otimes_{\mathscr{R}} z) \, dN(z) = (f \otimes_{\mathscr{R}} h) \int_E (z \otimes_{\mathscr{R}} z) \, dN(z) = (f \otimes_{\mathscr{R}} h)(\Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1))$.
This relation can be extended by linearity, and

$$L_{\mathscr{R}}(F \otimes_{\mathscr{D}} (L \oplus L_1), F) \otimes_{\mathscr{R}} L_{\mathscr{R}}(F \otimes_{\mathscr{D}} (L \oplus L_1), F) \rightarrow L_{\mathscr{R}}((F \otimes_{\mathscr{D}} (L \oplus L_1) \otimes_{\mathscr{R}} (F \otimes_{\mathscr{D}} (L \oplus L_1)), F \otimes_{\mathscr{D}} \bar{F})$$

being a surjection, we get

$$\int_{P_{\mathscr{D}}(F)_r} \theta \, dW(\theta) = \Theta_{\phi \oplus \phi_1}(\Sigma \otimes_{\mathscr{D}} (\phi \oplus \phi_1)) = \dim_{\mathscr{R}} (L \oplus L_1) \cdot \Sigma \, .$$

Hence $\hat{\Sigma} = t/[\dim_{\mathscr{R}} (L \oplus L_1)]$ in this problem.

From this the result follows.

5.8. DEFINITION. An *invariant normal model* on an $\mathscr{R}$-space E is a decomposition $E = \bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} (L_q \oplus L_{1q}))$ of E together with a parametrized family of regular normal distributions on E given by the injections

$$m: \bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} L_{1q}) \rightarrow \bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} (L_q \oplus L_{1q}))$$
$$(f_q \otimes_{\mathscr{D}_q} l_{1q})_{q \in Q} \rightarrow (f_q \otimes_{\mathscr{D}_q} (0, l_{1q}))_{q \in Q}$$

and

$$v: \bigoplus_{q \in Q} P_{\mathscr{D}_q}(F_q)_r \rightarrow P_{\mathscr{R}}(\bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} (L_q \oplus L_{1q})))_r$$
$$(\Sigma_q)_{q \in Q} \rightarrow \bigoplus_{q \in Q} (\Sigma_q \otimes_{\mathscr{D}_q} (\phi_q \oplus \phi_{1q})) \, ,$$

where for every $q \in Q$, $\phi_q \in P_{\mathscr{D}_q}(L_q)_r$ and $\phi_{1q} \in P_{\mathscr{D}_q}(L_{1q})_r$ are given.

5.9. THEOREM. *Let an invariant normal model on the $\mathscr{R}$-space E be given as in 5.8. Define*

$$\Theta: \bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} L_q) \oplus (F_q \otimes_{\mathscr{D}_q} L_{1q}) \rightarrow \bigoplus_{q \in Q} P_{\mathscr{D}_q}(F_q) \times [\bigoplus_{q \in Q} (F_q \otimes_{\mathscr{D}_q} L_{1q})]$$
$$(x_q, y_q)_{q \in Q} \rightarrow ((\dim_{\mathscr{R}} (L_q \oplus L_{1q})^{-1} s_q(x_q))_{q \in Q}, (y_q)_{q \in Q}) \, .$$

If $\forall q \in Q: \dim L_q \geqq \dim F_q$ *then* Θ *is the maximum likelihood estimator in the invariant normal model and the distribution of* $\Theta(x)$, $x \in E$ *is given by*

$$(\bigotimes_{q \in Q} W(\Sigma_q, \phi_q)) \otimes (\bigotimes_{q \in Q} N(\xi_q, \Sigma_q \otimes_{\mathscr{D}_q} \phi_{1q})) \, .$$

PROOF. Since the model splits into a product of independent $\mathscr{D}_q$-normal models the result follows from 5.6 and 5.7.

The following theorem shows why the name invariant normal model is chosen.

5.10. THEOREM. *Let π be a representation of a group G on the $\mathscr{R}$-space E. The mean-zero normal model arising from the family of regular normal distributions with mean value zero invariant under π is a mean-zero invariant normal model* (see 5.8).

PROOF. Let λ be a Lebesgue measure on E and let $\phi_\Sigma \lambda$ be a regular mean-zero normal distribution with variance Σ on E. It follows from 5.2 that $\phi_\Sigma\lambda$ is invariant under π if and only if $P_{\mathscr{R}}(\pi(g))\Sigma = \Sigma$ for every $g \in G$. But $P_{\mathscr{R}}(\pi(g))\Sigma = \Sigma$ if and only if Σ is π^*-invariant (4.9 and the isomorphism in 5.1). If the model is not empty there exists a π^*-invariant positive definite form on E. 4.20 gives that π^* and therefore also π is reducible. The decomposition of E with respect to π and the parametrization in 4.20 gives that the model is a mean-zero invariant normal model.

6. Matrix formulation and examples.

6.1. The case $\mathscr{D} \cong \mathscr{H}$.

Let $1, i, j, k$ be a basis for the $\mathscr{R}$-space $\mathscr{D}$ with the properties $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. Every element $d \in \mathscr{D}$ can be written uniquely as $d = p + qi + rj + sk$, $r, s, p, q \in \mathscr{R}$. $\bar{d} = p - qi - rj - sk$. We have $\operatorname{Re}(d) = p$, $I(d) = q$, $J(d) = r$, and $K(d) = s$.

Let F be a right and L a left $\mathscr{D}$-space. Let $\Sigma \in \mathscr{T}_{\mathscr{D}}(\bar{F}^*)_r$ and $\phi \in \mathscr{T}_{\mathscr{D}}(\bar{L}^*)_r$. Let $A \in \mathscr{S}_{\mathscr{D}}(\bar{F}^*)_r$ and $B \in \mathscr{S}_{\mathscr{D}}(\bar{L}^*)_r$ be defined by $\Sigma = R(A)$ and $\phi = R(B)$. The formulas in 3.2 give ($d = i$ and $e = j$) that

$$B(x, u) = \phi(x, u) + \phi(x, iu)i + \phi(x, ju)j + \phi(x, ku)k\,, \qquad x, u \in \bar{L}^*$$

and

$$A(y, v) = \Sigma(y, v) + \Sigma(yi, v)i + \Sigma(yj, v)j + \Sigma(yk, v)k\,, \qquad y, v \in \bar{F}^*\,.$$

$I(A)$, $J(A)$, $K(A)$, $I(B)$, $J(B)$, and $K(B)$ are all antisymmetric $\mathscr{R}$-bilineary forms.

$$\begin{aligned}(\Sigma \otimes_{\mathscr{D}} \phi)(y \otimes_{\mathscr{D}} x, v \otimes_{\mathscr{D}} u) = {} & \Sigma(y, v)\phi(x, u) + \Sigma(yi, v)\phi(x, iu) \\ & + \Sigma(yj, v)\phi(x, ju) + \Sigma(yk, v)\phi(x, ku)\,, \\ & \qquad y \otimes_{\mathscr{D}} x, v \otimes_{\mathscr{D}} u \in (\bar{F}^* \otimes_{\mathscr{D}} \bar{L}^*)\,.\end{aligned}$$

Let $f_1, \cdots, f_m$ resp. $l_1, \cdots, l_n$ be a basis for F resp. L, such that $B = I_n$ (identity matrix of dimension n) with respect to $l_1^*, \cdots, l_n^*$ (the dual basis to $l_1, \cdots, l_n$). Since the $\mathscr{R}$-dual basis to $\mathscr{R}$-basis $f_1, \cdots, f_m, f_1 i, \cdots, f_m i, f_1 j, \cdots, f_m j, f_1 k, \cdots, f_m k$ for F is $f_1^*, \cdots, f_m^*, f_1^* i, \cdots, f_m^* i, f_1^* j, \cdots, f_m^* j, f_1^* k, \cdots, f_m^* k$ in $\bar{F}^*$, and analogous for L only, we have $\phi = I_{4n}$ and

$$\Sigma = \begin{Bmatrix} \operatorname{Re}(A) & -I(A) & -J(A) & -K(A) \\ +I(A) & \operatorname{Re}(A) & -K(A) & +J(A) \\ +J(A) & +K(A) & \operatorname{Re}(A) & -I(A) \\ +K(A) & -J(A) & +I(A) & \operatorname{Re}(A) \end{Bmatrix} \quad \begin{array}{l}(A \text{ is the matrix of } A \text{ with} \\ \text{respect to } f_1^*, \cdots, f_m^*)\end{array}$$

with respect to the $\mathscr{R}$-basis above.

$$\Sigma \otimes_{\mathscr{D}} \phi = \begin{Bmatrix} \Sigma & & & 0 \\ & \Sigma & & n\text{-times} \\ & & \ddots & \\ 0 & & & \Sigma \end{Bmatrix} \quad 4\,nm \text{ columns and rows}$$

with respect to the $\mathscr{R}$-basis

$$\begin{array}{llll} f_1^* \otimes_{\mathscr{D}} l_1^*, \cdots, & f_1^* \otimes_{\mathscr{D}} l_n^*, & f_1^* \otimes_{\mathscr{D}} il_1^*, \cdots, & f_1^* \otimes_{\mathscr{D}} il_n^*, \\ \vdots & \vdots & \vdots & \vdots \\ f_m^* \otimes_{\mathscr{D}} l_1^*, \cdots, & f_m^* \otimes_{\mathscr{D}} l_n^*, & f_m^* \otimes_{\mathscr{D}} il_1^*, \cdots, & f_m^* \otimes_{\mathscr{D}} il_n^*, \end{array}$$

$$\begin{array}{llll} f_1^* \otimes_{\mathscr{D}} jl_1^*, \cdots, & f_1^* \otimes_{\mathscr{D}} jl_n^*, & f_1^* \otimes_{\mathscr{D}} kl_1^*, \cdots, & f_1^* \otimes_{\mathscr{D}} kl_n^* \\ \vdots & \vdots & \vdots & \vdots \\ f_m^* \otimes_{\mathscr{D}} jl_1^*, \cdots, & f_m^* \otimes_{\mathscr{D}} jl_n^*, & f_m^* \otimes_{\mathscr{D}} kl_1^*, \cdots, & f_m^* \otimes_{\mathscr{D}} kl_n^* . \end{array}$$

In the case $\mathscr{D} \cong \mathscr{C}$ we get with an analogous notation $\phi = I_{2n}$ and

$$\Sigma = \begin{bmatrix} \mathrm{Re}\,(A) & -I(A) \\ I(A) & \mathrm{Re}\,(A) \end{bmatrix} \qquad \Sigma \otimes_{\mathscr{D}} \phi = \begin{bmatrix} \Sigma & & & & 0 \\ & \Sigma & & n\text{-times} & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \Sigma \end{bmatrix} \quad \begin{array}{l} 2\,nm \text{ columns} \\ \text{and rows} \end{array}$$

and for $\mathscr{D} = \mathscr{R}$

$$\phi = B = I_n \qquad \Sigma \otimes_{\mathscr{D}} \phi = \begin{bmatrix} A & & & & 0 \\ & A & & n\text{-times} & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & A \end{bmatrix} \quad nm \text{ columns and rows}$$

$\Sigma = A$.

6.2. We shall now give the estimation in the case of a fixed basis. We take the case with mean value 0 ($n \geqq m$).

$\mathscr{D} \cong \mathscr{H}$.

Let

$$\begin{array}{llllllll} X_{11}^r, \cdots, & X_{m1}^r, & X_{11}^i, \cdots, & X_{m1}^i, & X_{11}^j, \cdots, & X_{m1}^j, & X_{11}^k, \cdots, & X_{m1}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{1n}^r, \cdots, & X_{mn}^r, & X_{1n}^i, \cdots, & X_{mn}^i, & X_{1n}^j, \cdots, & X_{mn}^j, & X_{1n}^k, \cdots, & X_{mn}^k \end{array}$$

be the coordinates with respect to the $\mathscr{R}$-basis

$$\begin{array}{llll} f_1 \otimes_{\mathscr{D}} l_1, \cdots, & f_m \otimes_{\mathscr{D}} l_1, & f_1 i \otimes_{\mathscr{D}} l_1, \cdots, & f_m i \otimes_{\mathscr{D}} l_1, \\ \vdots & \vdots & \vdots & \vdots \\ f_1 \otimes_{\mathscr{D}} l_n, \cdots, & f_m \otimes_{\mathscr{D}} l_n, & f_1 i \otimes_{\mathscr{D}} l_n, \cdots, & f_m i \otimes_{\mathscr{D}} l_n, \end{array}$$

$$\begin{array}{llll} f_1 j \otimes_{\mathscr{D}} l_1, \cdots, & f_m j \otimes_{\mathscr{D}} l_1, & f_1 k \otimes_{\mathscr{D}} l_1, \cdots, & f_m k \otimes_{\mathscr{D}} l_1 \\ \vdots & \vdots & \vdots & \vdots \\ f_1 j \otimes_{\mathscr{D}} l_n, \cdots, & f_m j \otimes_{\mathscr{D}} l_n, & f_1 k \otimes_{\mathscr{D}} l_n, \cdots, & f_m k \otimes_{\mathscr{D}} l_n \end{array}$$

in $F \otimes_{\mathscr{D}} L$.

We set

$$X^r = (X^r_{\mu\nu})_{\substack{\nu=1,\cdots,n;\\ \mu=1,\cdots,m.}}, \qquad X^i = (X^i_{\mu\nu})_{\substack{\nu=1,\cdots,n;\\ \mu=1,\cdots,m.}},$$
$$X^j = (X^j_{\mu\nu})_{\substack{\nu=1,\cdots,n;\\ \mu=1,\cdots,m.}}, \qquad X^k = (X^k_{\mu\nu})_{\substack{\nu=1,\cdots,n;\\ \mu=1,\cdots,m.}}$$

and the maximum likelihood estimator $\hat{\Sigma}$ for Σ is given by

$$\hat{\Sigma}=\frac{1}{4n}\left\{\begin{array}{c|c|c|c}
\begin{matrix}X^r(X^r)'+X^i(X^i)'+\\ X^j(X^j)'+X^k(X^k)'\end{matrix} & \begin{matrix}X^k(X^j)'-X^j(X^k)'+\\ X^i(X^r)'-X^r(X^i)'\end{matrix} & \begin{matrix}X^i(X^k)'-X^k(X^i)'+\\ X^j(X^r)'-X^r(X^j)'\end{matrix} & \begin{matrix}X^j(X^i)'-X^i(X^j)'+\\ X^k(X^r)'-X^r(X^k)'\end{matrix} \\ \hline
\begin{matrix}X^j(X^k)'-X^k(X^j)'+\\ X^r(X^i)'-X^i(X^r)'\end{matrix} & \begin{matrix}X^r(X^r)'+X^i(X^i)'+\\ X^j(X^j)'+X^k(X^k)'\end{matrix} & \begin{matrix}X^j(X^i)'-X^i(X^j)'+\\ X^k(X^r)'-X^r(X^k)'\end{matrix} & \begin{matrix}X^k(X^i)'-X^i(X^k)'+\\ X^r(X^j)'-X^j(X^r)'\end{matrix} \\ \hline
\begin{matrix}X^k(X^i)'-X^i(X^k)'+\\ X^r(X^j)'-X^j(X^r)'\end{matrix} & \begin{matrix}X^i(X^j)'-X^j(X^i)'+\\ X^r(X^k)'-X^k(X^r)'\end{matrix} & \begin{matrix}X^r(X^r)'+X^i(X^i)'+\\ X^j(X^j)'+X^k(X^k)'\end{matrix} & \begin{matrix}X^k(X^j)'-X^j(X^k)'+\\ X^i(X^r)'-X^r(X^i)'\end{matrix} \\ \hline
\begin{matrix}X^i(X^j)'-X^j(X^i)'+\\ X^r(X^k)'-X^k(X^r)'\end{matrix} & \begin{matrix}X^i(X^k)'-X^k(X^i)'+\\ X^j(X^r)'-X^r(X^j)'\end{matrix} & \begin{matrix}X^j(X^k)'-X^k(X^j)'+\\ X^r(X^i)'-X^i(X^r)'\end{matrix} & \begin{matrix}X^r(X^r)'+X^i(X^i)'+\\ X^j(X^j)'+X^k(X^k)'\end{matrix}
\end{array}\right\}.$$

In the case $\mathscr{D} \cong \mathscr{C}$ we get with analogous notation

$$\hat{\Sigma} = \frac{1}{2n}\left\{\begin{array}{c|c} X^r(X^r)' + X^i(X^i)' & X^i(X^r)' - X^r(X^i)' \\ \hline X^r(X^i)' - X^i(X^r)' & X^r(X^r)' + X^i(X^i)' \end{array}\right\}$$

and for $\mathscr{D} \cong \mathscr{R}$

$$\hat{\Sigma} = \frac{1}{n} X(X)' .$$

The second and the last estimation problem is well known (see [1], [9], [10] and [7]).

6.3. In the Introduction we suggested that our invariant normal model contains examples from the literature of normal models given by symmetry and (or) independent observations from the same normal distribution. From the matrix formulation above it follows that the real complex and quaternion normal models (see the Introduction) are invariant normal models. Mean-value zero normal models given by symmetry are also invariant normal models. This follows from 5.10. For mean value different from zero all the examples given in the Introduction are still invariant normal models. To illustrate this we must translate the formulation in the papers to the invariant form in 5.7. We will give two examples of this.

6.4. Example. (Wilks [15]). Let $(X_1, \cdots, X_k) \in \mathscr{R}^k$ be an observation from a p-dimensional normal distribution with mean-value zero and regular covariance matrix given by

$$\Gamma = \begin{bmatrix} \sigma^2 & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \lambda \\ \lambda & \cdot & \cdot & \cdot & \lambda & \sigma^2 \end{bmatrix}.$$

This situation arises from the hypothesis that the distribution of $(X_1, \cdots, X_k)$ is independent of permutations of the elements $X_1, \cdots, X_k$. In terms of representation theory the hypothesis is that the normal distribution on $\mathscr{R}^k$ is invariant under the representation

$$\pi: \mathscr{S}(k) \to GL_{\mathscr{R}}(\mathscr{R}^k)$$
$$\sigma \to ((x_1, \cdots, x_k) \to (x_{\sigma(1)}, \cdots, x_{\sigma(k)}))$$

where $S(k)$ is the symmetric group of order k. This representation splits into a direct sum of two irreducible representations in the following way

$$\mathscr{R}^k = L_e \oplus L_0 \quad (=(\mathscr{R} \otimes_{\mathscr{R}} L_e) \oplus (\mathscr{R} \otimes_{\mathscr{R}} L_0)) ,$$
$$\pi = \pi_e \oplus \pi_0$$

where $L_e = \{(x_1, \cdots, x_k) \in \mathscr{R}^k \mid x_1 = \cdots = x_k\}$, $L_0 = \{(x_1, \cdots, x_k) \in \mathscr{R}^k \mid \sum_{i=1}^k x_i = 0\}$, $\pi_e = \pi \mid L_e$ and $\pi_0 = \pi \mid L_0$. Since there is a natural basis in $\mathscr{R}^k$ we can identify $(\mathscr{R}^k)^*$ and $\mathscr{R}^k$. Hence Γ has the form $\xi_1^2\phi_e \oplus \xi_2^2\phi_0$, where ϕ_e and ϕ_0 are given by $\phi_e(\underline{x}, \underline{x}) = x^2$, $\underline{x} = (x, \cdots, x)$ and $\phi_0(\underline{y}, \underline{y}) = \sum_{i=1}^k y_i^2$, $y = (y_1, \cdots, y_k)$ with $\sum_{i=1}^k y_i = 0$. The transformation of the parameter is given by $\xi_1^2 = k\sigma^2 + k(k-1)\lambda$, $\xi_2^2 = \sigma^2 - \lambda$. Hence the model is transformed into a product of two well-known trivial models.

If we now have n independent observations $\underline{X}_1, \cdots, \underline{X}_n$ from the normal distribution above we get a family of normal distributions on the space $\mathscr{R}^{kn} = \mathscr{R}^k \otimes_{\mathscr{R}} \mathscr{R}^n$; the above decomposition of $\mathscr{R}^k$ gives the decomposition

$$\mathscr{R}^{kn} = [\mathscr{R} \otimes_{\mathscr{R}} (L_e \otimes_{\mathscr{R}} \mathscr{R}^n)] \oplus [\mathscr{R} \otimes_{\mathscr{R}} (L_0 \otimes_{\mathscr{R}} \mathscr{R}^n)] ,$$
$$\Gamma \otimes_{\mathscr{R}} I_n = [\xi_1^2 \otimes_{\mathscr{R}} (\phi_e \otimes_{\mathscr{R}} I_n)] \oplus [\xi_2^2 \otimes_{\mathscr{R}} (\phi_0 \otimes_{\mathscr{R}} I_n)] .$$

Again we get a product of two well-known trivial models.

Let now the normal distribution for $\underline{X}_i$ have the mean value $\underline{\mu} \in \mathscr{R}^k$, $i = 1, \cdots, n$. Then we have the decomposition:

$$\mathscr{R}^{kn} = \mathscr{R} \otimes_{\mathscr{R}} ((L_e \otimes_{\mathscr{R}} V_0) \oplus (L_e \otimes_{\mathscr{R}} V_e))$$
$$\oplus (\mathscr{R} \otimes_{\mathscr{R}} ((L_0 \otimes_{\mathscr{R}} V_0) \oplus (L_0 \otimes_{\mathscr{R}} V_e))) ,$$
$$\Gamma \otimes I_n = \xi_1^2 \cdot ((\phi_e \otimes_{\mathscr{R}} \phi_0') \oplus (\phi_e \otimes_{\mathscr{R}} \phi_e')) \oplus \xi_2^2 \cdot ((\phi_0 \otimes_{\mathscr{R}} \phi_0') \oplus (\phi_0 \otimes_{\mathscr{R}} \phi_e'))$$

where V_0, V_e, ϕ_0', ϕ_e' is defined in an analogous manner as subspaces in $\mathscr{R}^n$ and positive definite forms on these. The mean value is then parametrized by the subspace $(L_e \otimes_{\mathscr{R}} V_e) \oplus (L_0 \otimes_{\mathscr{R}} V_e) = \mathscr{R}^k \otimes_{\mathscr{R}} V_e$ $(\dim_{\mathscr{R}} V_e = 1)$. This again is a product of trivial well-known models. If $\underline{\mu} = (\mu_1, \cdots, \mu_k)$ and the mean-value structure is given by $\mu_1 = \cdots = \mu_k$ we have the decomposition:

$$\mathscr{R}^{pn} = \mathscr{R} \otimes_{\mathscr{R}} ((L_e \otimes_{\mathscr{R}} V_0) \oplus (L_e \otimes_{\mathscr{R}} V_e)) \oplus (\mathscr{R} \otimes_{\mathscr{R}} (L_0 \otimes_{\mathscr{R}} \mathscr{R}^n)) ,$$
$$\Gamma \otimes I_n = \xi_1^2((\phi_e \otimes_{\mathscr{R}} \phi_0') \oplus (\phi_e \otimes_{\mathscr{R}} \phi_e')) \oplus \xi_2^2(\phi_0 \otimes_{\mathscr{R}} I_n) .$$

The mean value is then parametrized by the subspace $L_e \otimes V_e$ in $\mathscr{R}^k \otimes \mathscr{R}^n = \mathscr{R}^{kn}$.

6.5. EXAMPLE. Arnold [2]: For the sake of convenience we will only treat the case where the mean value is zero. The case of mean value different from zero can be treated as above (6.4).

Let $(X_1, \cdots, X_k)$ be interchangeable p-dimensional observations from a p-dimensional normal distribution. The covariance-matrix for the pk-dimensional observation is given by

$$\Gamma = \begin{bmatrix} \Sigma & \Lambda & \cdot & \cdot & \cdot & \Lambda \\ \Lambda & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & & \Lambda \\ \Lambda & \cdot & \cdot & \cdot & \Lambda & \Sigma \end{bmatrix}$$

where Σ is positive definite and Λ is symmetric such that Γ is positive definite. In Arnold [2] this is called pattern A_k. Since the covariance is given by invariance under the representation

$$\begin{aligned} \pi_p : \mathscr{S}(k) &\to GL_{\mathscr{R}}(\mathscr{R}^{pk}) \\ \sigma &\to ((x_{11}, \cdots, x_{1p}, \cdots, x_{k1}, \cdots, x_{kp}) \\ &\to (x_{\sigma(1)1}, \cdots, x_{\sigma(1)p}, \cdots, x_{\sigma(k)1}, \cdots, x_{\sigma(k)p})) \end{aligned}$$

the model is an invariant normal model and therefore splits into a product of well-known trivial models. The decomposition is given by

$$\begin{aligned} \mathscr{R}^{pk} &= \mathscr{R}^p \otimes \mathscr{R}^k = (\mathscr{R}^p \otimes_{\mathscr{R}} L_e) \oplus (\mathscr{R}^p \otimes_{\mathscr{R}} L_0) \\ \pi_p &= I_p \otimes \pi = (I_p \otimes_{\mathscr{R}} \pi_1) \oplus (I_p \otimes_{\mathscr{R}} \pi_2) \,. \end{aligned}$$

The family of covariance is therefore

$$\{(\Xi_1 \otimes_{\mathscr{R}} \phi_e) \oplus (\Xi_2 \otimes_{\mathscr{R}} \phi_0) \mid (\Xi_1, \Xi_2 \in \mathscr{S}_{\mathscr{R}}(\mathscr{R}^p)_r\} \,.$$

The transformation of the parameter is trivial given by

$$\Xi_1 = k\Sigma + k(k-1)\Lambda \,, \qquad \Xi_2 = \Sigma - \Lambda \,.$$

So the model splits into a product of two real normal models.

Now this generalization by Arnold [2] of the model of Wilks [15] was already treated by Votaw [16] in 1948. If one writes the observation $(X_1, \cdots, X_k) = (X_{11}, \cdots, X_{1p}, \cdots, X_{k1}, \cdots, X_{kp})$ as $(X_{11}, \cdots, X_{k1}, \cdots, X_{1p}, \cdots, X_{kp})$ we get that the model of Votaw (see the symmetry model (4) in the Introduction. In short, we can say that Arnold uses the isomorphism $\mathscr{R}^p \otimes \mathscr{R}^k \cong \mathscr{R}^p \oplus \cdots \oplus \mathscr{R}^p$ (k-times) and Votaw uses the isomorphism $\mathscr{R}^p \otimes \mathscr{R}^k \cong \mathscr{R}^k \oplus \cdots \oplus \mathscr{R}^k$ (p-times).

6.6. Other examples can be found in McLaren [13]. In the paper by Olkin and Press [14] irreducible representation of a complex type occurs, but only with multiplicity one.

REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] ARNOLD, S. F. (1973). Application of the theory of products of problems to certain patterned covariance matrices. *Ann. Statist.* **1** 682–699.

[3] BOURBAKI, N. (1970). *Elements de Mathematique, Algebra.* Chapitre 1 à 3. Hermann, Paris.

[4] BOURBAKI, N. (1958). *Elements de Mathematique, Algebra.* Chapitre 8. Hermann, Paris.

[5] BOURBAKI, N. (1959). *Elements de Mathematique, Algebra.* Chapitre 9, Hermann, Paris.

[6] BOURBAKI, N. (1963). *Elements de Mathematique, Integration.* Chapitre 7. Hermann, Paris.

[7] BRØNS, H., HENNINGSEN, I., and JENSEN, S. T. (1971). A canonical statistical hypothesis in the multidimensional normal distribution. Preprint No. 7, Institute of Mathematical Statistics, University of Copenhagen. Unpublished.

[8] CONSUL, P. C. (1969). On the exact distribution of Votaw's criteria for testing compound symmetry of a covariance matrix. *Ann. Math. Statist.* **40** 836–843.

[9] GOODMANN, N. R. (1963). Statistical analysis based on a certain multivariate complex Gausian distribution. *Ann. Math. Statist.* **34** 152–177.

[10] JAMES, A. T. (1957). Relations algebra and experimental design. *Ann. Math. Statist.* **28** 993–1002.

[11] KHATRI, C. G. (1965). A test for reality of covariance matrix in a certain complex Gausian distribution. *Ann. Math. Statist.* **36** 115–119.

[12] KHATRI, C. G. (1965). Classical statistical analysis based on certain multivariate complex distributions. *Ann. Math. Statist.* **36** 98–114.

[13] MACLAREN, A. S. (1963). On group representations and invariant stochastic processes. *Proc. Cambridge Philos. Soc.* **59** 431–450.

[14] OLKIN, I. and PRESS, S. J. (1969). Testing and Estimation for a circular stationary model. *Ann. Math. Statist.* **40** 1358–1373.

[15] WILKS, S. S. (1946). Sample criteria for testing equality of means, equality of variances and equality of covariances in a normal multivariate distribution. *Ann. Math. Statist.* **17** 257–281.

[16] VOTAW, D. F. (1948). Testing compound symmetry in a normal multivariate distribution. *Ann. Math. Statist.* **19** 447–473.

INSTITUTE OF MATHEMATICAL STATISTICS
5 UNIVERSITETSPARKEN
2100 COPENHAGEN Ø, DENMARK