The Annals of Statistics
1973, Vol. 1, No. 3, 454-471

ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD FUNCTION IN THE INDEPENDENT NOT IDENTICALLY DISTRIBUTED CASE[1]

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Let Θ be an open subset of R^k and for each $\theta \in \Theta$, let $X_1, \cdots, X_n$ be independent rv's defined on the probability space $(\mathscr{X}, \mathscr{A}, P_\theta)$, and let $p_{j,\theta}$ be the distribution of the rv X_j. Let $f_j(\cdot; \theta)$ be a specified version of the Radon–Nikodym derivative of $p_{j,\theta}$ with respect to a σ-finite measure μ and set $f_j(\theta) = f_j(X_j; \theta)$. Furthermore, for $\theta, \theta^* \in \Theta$, set $\phi_j(\theta, \theta^*) = [f_j(\theta^*)/f_j(\theta)]^{\frac{1}{2}}$ and suppose that $\phi_j(\theta, \theta^*)$ is differentiable in quadratic mean (qm) with respect to θ^* at (θ, θ), when the probability measure P_θ is employed, with qm derivative $\dot{\phi}_j(\theta)$. Set $\Delta_n(\theta) = 2n^{-\frac{1}{2}} \sum_{j=1}^n \dot{\phi}_j(\theta)$, $\Gamma_j(\theta) = 4\mathscr{E}_\theta[\dot{\phi}_j(\theta)\dot{\phi}_j'(\theta)]$, $\bar{\Gamma}_n(\theta) = n^{-1} \sum_{j=1}^n \Gamma_j(\theta)$, and suppose that $\bar{\Gamma}_n(\theta) \to \bar{\Gamma}(\theta)$ and $\bar{\Gamma}(\theta)$ is positive definite on Θ. Finally, for $h_n \to h \in R^k$, set $\theta_n = \theta + h_n n^{-\frac{1}{2}}$ and $\Lambda_n(\theta) = \log[dP_{n,\theta_n}/dP_{n,\theta}]$, where $P_{n,\theta}$ stands for the restriction of P_θ to $\mathscr{A}_n = \sigma(X_1, \cdots, X_n)$. Then, under suitable—and not too hard to verify—conditions, we obtain, the following results. The limits are taken as $n \to \infty$.

Theorem 1. $\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\frac{1}{2}h'\bar{\Gamma}(\theta)h$ *in* P_θ*-probability*, $\theta \in \Theta$.

Theorem 2. $\mathscr{L}[\Delta_n(\theta) \mid P_\theta] \Rightarrow N(0, \bar{\Gamma}(\theta))$, $\theta \in \Theta$.

Theorem 3. $\mathscr{L}[\Lambda_n(\theta) \mid P_\theta] \Rightarrow N(-\frac{1}{2}h'\bar{\Gamma}(\theta)h, h'\bar{\Gamma}(\theta)h)$, $\theta \in \Theta$.

Theorem 4. $\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\frac{1}{2}h'\bar{\Gamma}(\theta)h$ *in* P_{θ_n}*-probability*, $\theta \in \Theta$.

Theorem 5. $\mathscr{L}[\Lambda_n(\theta) \mid P_{\theta_n}] \Rightarrow N(\frac{1}{2}h'\bar{\Gamma}(\theta)h, h'\bar{\Gamma}(\theta)h)$, $\theta \in \Theta$.

Theorem 6. $\mathscr{L}[\Delta_n(\theta) \mid P_{\theta_n}] \Rightarrow N(\bar{\Gamma}(\theta)h, \bar{\Gamma}(\theta))$, $\theta \in \Theta$.

0. Summary. In the present paper, we consider a sequence of independent, but not necessarily identically distributed random variables, and give a set of non-standard (that is, not Cramér-type) conditions under which the asymptotic distributions of certain random functions of statistical importance are obtained. The approach is also non-standard in that the derivations rely heavily on results based on the concept of contiguity. The applicability of the assumptions made is illustrated by an example in which the conclusions of this manuscript are also specialized.

1. Introduction. In this paper, we consider the problem of deriving the asymptotic distributions of certain random functions of statistical importance in the case that the underlying process consists of independent not necessarily identically distributed (i.n.n.i.d.) random variables (rv's). Regularity conditions are given under which we obtain the asymptotic expansion, in the probability sense,

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Received February 1972; revised September 1972.

[1] This research was supported by the National Science Foundation, Grant GP-20036, and the University of Wisconsin Research Committee.

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of the log-likelihood function $\Lambda_n(\theta)$ (see (3.2) for its definition). Loosely speaking, we show that, for each fixed $\theta \in \Theta$, the likelihood function $e^{\Lambda_n(\theta)}$ may be approximated by an exponential family. On the exponent of this family, there appears a random vector $\Delta_n(\theta)$ (see (2.3) for its definition), which plays an important role in testing hypotheses problems. The asymptotic normality of $\Delta_n(\theta)$ is established along with that of $\Lambda_n(\theta)$. The above mentioned results are obtained under both the P_θ and P_{θ_n}-probability measures, where θ_n approaches θ at a specified rate (see (3.1) for the definition of θ_n).

Results analogous to the ones obtained herein, and under assumptions similar to the ones used in the present paper have been established by Roussas (1965) and Johnson and Roussas (1969), (1970) for stationary Markov processes. They are included in Roussas (1972), Chapter 2. The two sets of regularity conditions, the one employed here and the one utilized by the above mentioned authors, coincide in the special case that the rv's involved are i.i.d. Both sets include the basic assumption of differentiability in quadratic mean of a certain random function $\varphi_j(\theta, \theta^*)$ (see (2.1) for its definition), which replaces the usual assumptions about the existence of the second order derivatives of the densities involved.

The problem of deriving some of the asymptotic distributions mentioned above in the i.n.n.i.d. case has also been considered by LeCam (1960) as an application of his more general results in the framework of DAN families of distributions. He obtained an asympotic expansion for $\Lambda_n(\theta)$, but under Cramér-type assumptions. In a subsequent paper, LeCam (1966) proved that the asymptotic normality of $\Lambda_n(\theta)$ implies and is implied by the asymptotic normality of $\sum_{j=1}^n \{[\varphi_j(\theta, \theta_n)]^{2\delta} - 1\}$, provided that the i.n.n.i.d. rv's satisfy a certain uniformity condition. This result was first stated in LeCam (1960) for the case that $\delta = 1$. One direction of it and for the special case of $\delta = \frac{1}{2}$ was also independently established by Hájek and Šidák (1967), and was employed by them for some testing hypotheses problems in a non-parametric context.

Finally, asymptotic results of the same nature as the ones presented here, and in connection with linear rank statistics and also by utilizing contiguity arguments, have been obtained by Beran in a recent paper, Beran (1970).

The assumptions used in the present paper are non-standard (that is, not Cramér-type) and are designed to cover cases, where pointwise derivatives fail to exist. Also, they are of more probabilistic, rather than analytic, nature and not exceedingly difficult to verify. Finally, they enable us to directly derive the desired asymptotic distributions.

In Section 2, we introduce the necessary notation and summarize the assumptions under which the results in this paper are obtained. In Section 3, the main results are stated; their proof is deferred to Section 5, after some auxiliary results are established in Section 4. Finally, in Section 6, the applicability of the assumptions is illustrated by means of an example.

It is worth noting that the model considered here includes as a special case the regression model which need not be linear and where the residuals, although independent, need not be normally distributed. We shall return to this important case in a subsequent paper.

In order to avoid unnecessary repetitions in the sequel, all limits will be taken as $n \to \infty$ through positive integer values unless otherwise explicitly stated.

2. Notation and assumptions. Let Θ be an open subset of R^k and for each $\theta \in \Theta$, let $p_{j,\theta}$, $j = 1, \cdots, n$ $(n = 1, 2, \cdots)$ be probability measures on $(R_j, \mathscr{B}_j)$, where $(R_j, \mathscr{B}_j) = (R, \mathscr{B})$, the Borel real line. It is assumed that there is a σ-finite measure μ on $\mathscr{B}$ such that $p_{j,\theta} \ll \mu$, $\theta \in \Theta$, $j \geqq 1$, and set $f_j(\cdot; \theta) = dp_{j,\theta}/d\mu$ for a specified version of the Radon–Nikodym derivative involved. Set $(\mathscr{X}, \mathscr{A}) = \prod_{j=1}^{\infty} (R_j, \mathscr{B}_j)$ and let P_θ be the product measure of $p_{j,\theta}$, $j \geqq 1$, induced on $\mathscr{A}$. Then, if X_j, $j \geqq 1$, are the coordinate rv's, it follows that, for each $\theta \in \Theta$, these rv's are independent and the pdf of the jth rv is $f_j(\cdot; \theta)$. It is further assumed that, for each $j \geqq 1$, the set $\{x \in R; f_j(x; \theta) > 0\}$ is independent of $\theta \in \Theta$. In the following, we set $f_j(\theta)$ for the rv $f_j(X_j; \theta)$. Also the notation $\mathscr{A}_n = \sigma(X_1, \cdots, X_n)$ and $P_{n,\theta}$ for the σ-field induced by the rv's $X_1, \cdots, X_n$ and the restriction of the probability measure P_θ to $\mathscr{A}_n$, respectively, will occasionally prove useful.

For $\theta, \theta^* \in \Theta$, set

$$\varphi_j(\theta, \theta^*) = \varphi_j(\theta, \theta^*; X_j) = \left(\frac{f_j(X_j; \theta^*)}{f_j(X_j; \theta)}\right)^{\frac{1}{2}} = \left(\frac{f_j(\theta^*)}{f_j(\theta)}\right)^{\frac{1}{2}}, \tag{2.1}$$

so that $\mathscr{E}_\theta \varphi_j^2(\theta, \theta^*) = 1$, and

$$\Lambda(\theta, \theta^*) = \log \frac{dP_{n,\theta^*}}{dP_{n,\theta}} = \log \prod_{j=1}^{n} \frac{f_j(\theta^*)}{f_j(\theta)} = \log \prod_{j=1}^{n} \varphi_j^2(\theta, \theta^*) . \tag{2.2}$$

It will be assumed in the following that, for each $j \geqq 1$ and any arbitrary but fixed $\theta \in \Theta$, the random function $\varphi_j(\theta, \theta^*)$ is differentiable in quadratic mean (qm) with respect to θ^* at (θ, θ) when P_θ is used; $\dot{\varphi}_j(\theta)$ will denote its qm derivative at (θ, θ). Next, set

$$\Delta_n(\theta) = \frac{2}{n^{\frac{1}{2}}} \sum_{j=1}^{n} \dot{\varphi}_j(\theta) \tag{2.3}$$

and

$$\Gamma_j(\theta) = 4\mathscr{E}_\theta[\dot{\varphi}_j(\theta)\dot{\varphi}_j'(\theta)] , \qquad \bar{\Gamma}_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \Gamma_j(\theta) . \tag{2.4}$$

Now we gather together the various assumptions under which we will be able to derive several asymptotic results.

ASSUMPTIONS.

(A1) For each $j \geqq 1$ and every $\theta \in \Theta$, the rv X_j has a pdf $f_j(\cdot; \theta)$ with respect to a σ-finite measure μ on $\mathscr{B}$ and the set $\{x \in R; f_j(x; \theta) > 0\}$ is independent of $\theta \in \Theta$.

(A2) (i) For every $\theta \in \Theta$, the random functions $\varphi_j(\theta, \theta^*)$, $j \geqq 1$, are differentiable in qm with respect to θ^* at (θ, θ), when the probability measure P_θ is employed, uniformly in $j \geqq 1$. That is, there is a k-dimensional random vector

$\dot{\varphi}_j(\theta)$—the qm derivative of $\varphi_j(\theta, \theta^*)$ with respect to θ^* at (θ, θ)—such that

$$|\lambda^{-1}| \cdot |\varphi_j(\theta, \theta + \lambda h) - 1 - \lambda h'\dot{\varphi}_j(\theta)| \to 0 \qquad \text{in qm} \quad [P_\theta], \qquad \text{as } \lambda \to 0,$$

uniformly in j, uniformly on bounded sets of $h \in R^k$.

(ii) $\dot{\varphi}_j(\theta)$ is $X_j^{-1}(\mathscr{B}_j) \times \mathscr{C}$-measurable, where $\mathscr{C}$ is the σ-field of Borel subsets of Θ.

(A3) For every $\theta \in \Theta$ and every $h \in R^k$, $[h'\dot{\varphi}_j(\theta)]^2$, $j \geqq 1$, are uniformly integrable with respect to P_θ; i.e., uniformly in j

$$\int_{\{[h'\dot{\varphi}_j(\theta)]^2 > a\}} [h'\dot{\varphi}_j(\theta)]^2 \, dP_\theta \to 0, \qquad \text{as } a \to \infty.$$

(A4) For every $\theta \in \Theta$ and every $h \in R^k$, $(n^{(2+\delta)/2})^{-1} \sum_{j=1}^n \mathscr{E}_\theta |h'\dot{\varphi}_j(\theta)|^{2+\delta} \to 0$ for some $\delta > 0$, $0 < \delta \leqq 2$.

REMARK 2.1. A sufficient condition for (A3) and (A4) to be true, and which in many circumstances is easy to verify, is the following

$$\mathscr{E}_\theta |h'\dot{\varphi}_j(\theta)|^3 \leqq M(= M(h, \theta) < \infty), \qquad j \geqq 1.$$

(A5) Let $\Gamma_j(\theta)$, $j = 1, \cdots, n$ and $\bar{\Gamma}_n(\theta)$ be defined by (2.4). Then for every $\theta \in \Theta$, $\bar{\Gamma}_n(\theta) \to \bar{\Gamma}(\theta)$, where the convergence is convergence in norm (any one of the usual norms for matrices) and $\bar{\Gamma}(\theta)$ is positive definite.

3. Main results. Under Assumptions (A1)—(A5), we derive the results stated below. For their formulation, let

$$\theta_n = \theta + h_n/n^{\frac{1}{2}}, \quad h_n \to h \in R^k, \tag{3.1}$$

and with $\Lambda(\theta, \theta_n)$ given by (2.2), set

$$\Lambda_n(\theta) = \Lambda(\theta, \theta_n). \tag{3.2}$$

We may now proceed with the formulation of the main results in this paper.

THEOREM 3.1. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\tfrac{1}{2}h'\bar{\Gamma}(\theta)h \qquad \text{in } P_\theta\text{-probability},$$

where $\Delta_n(\theta)$ *and* $\Lambda_n(\theta)$ *are given by* (2.3) *and* (3.2), *respectively* (*and* θ_n *is defined by* (3.1)).

A loose interpretation of the result just stated is as follows. For large n, $\exp \Lambda_n(\theta)$ is approximately equal to $\exp[h'\Delta_n(\theta) - \frac{1}{2}h'\bar{\Gamma}(\theta)h]$. Thus, keeping θ fixed and letting h play the role of a parameter, one has that, for large n, the likelihood function $\exp \Lambda_n(\theta)$ behaves as if it were of an exponential form. This statement will be made precise in a subsequent paper, in which certain testing hypotheses problems will also be treated. The random vector $\Delta_n(\theta)$ plays the important role of the exponent in the approximating exponential family.

From the preceding comment, it also follows that the random vector $\Delta_n(\theta)$ is bound to be of fundamental importance for any statistical inferences in connection with the underlying process. The result stated below provides the asymptotic distribution of $\Delta_n(\theta)$. More precisely, one has

THEOREM 3.2. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$,

$$\mathscr{L}[\Delta_n(\theta) \mid P_\theta] \Rightarrow N[0, \bar{\Gamma}(\theta)],$$

where $\Delta_n(\theta)$ *is given by* (2.3).

Again from statistical inference point of view, one would like to know the asymptotic distribution of the log-likelihood function $\Lambda_n(\theta)$. This is provided by the following result.

THEOREM 3.3. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\mathscr{L}[\Lambda_n(\theta) \mid P_\theta] \Rightarrow N[-\tfrac{1}{2}h'\bar{\Gamma}(\theta)h,\ h'\bar{\Gamma}(\theta)h],$$

where $\Lambda_n(\theta)$ *is given by* (3.2) (*and* θ_n *is defined by* (3.1)).

The theorems stated so far provide asymptotic results when the parameter point θ—and hence the corresponding probability measure P_θ—is kept fixed. However, for statistical applications, one would have to allow the parameter point to vary with n in a certain way. When θ is replaced by θ_n, where θ_n is given by (3.1), then one can establish results similar to Theorems 3.1—3.3 with P_θ being replaced by P_{θ_n}. These are stated below as Theorems 3.4—3.6. Their statistical usefulness may be explained along the same lines as that of Theorems 3.1—3.3.

THEOREM 3.4. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\tfrac{1}{2}h'\bar{\Gamma}(\theta)h \qquad \text{in} \quad P_{\theta_n}\text{-probability},$$

where $\Delta_n(\theta)$, $\Lambda_n(\theta)$ *and* θ_n *are given by* (2.3), (3.2) *and* (3.1), *respectively.*

The appropriate version of Theorem 3.3 is as follows.

THEOREM 3.5. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\mathscr{L}[\Lambda_n(\theta) \mid P_{\theta_n}] \Rightarrow N[\tfrac{1}{2}h'\bar{\Gamma}(\theta)h,\ h'\bar{\Gamma}(\theta)h],$$

where $\Lambda_n(\theta)$ *and* θ_n *are given by* (3.2) *and* (3.1), *respectively.*

Finally, the analogue of Theorem 3.2 is stated below.

THEOREM 3.6. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\mathscr{L}[\Delta_n(\theta) \mid P_{\theta_n}] \Rightarrow N[\bar{\Gamma}(\theta)h,\ \bar{\Gamma}(\theta)],$$

where $\Delta_n(\theta)$ *and* θ_n *are given by* (2.3) *and* (3.1), *respectively.*

The proof of Theorems 3.1—3.6 is deferred to Section 5. In the next section, we establish some lemmas which will facilitate the proof of the theorems.

In closing this section, it should be mentioned that the results obtained herein under Assumptions (A1)—(A5), clearly, resemble results obtained by Roussas (1965) and Johnson and Roussas (1969), (1970) (see also Chapter 2 of Roussas (1972)) under related assumptions for stationary Markov processes. However, their results do not imply ours, since they consider rv's which are identically distributed but not (necessarily) independent, whereas we consider rv's which are independent but not (necessarily) identically distributed. For the i.i.d. case, our assumptions coincide with theirs and so do the results.

4. Some auxiliary results. The proof of the theorems in the previous section will follow after we have established a series of lemmas. In the sequel, it will be convenient to write $\varphi_{nj}(\theta)$ rather than $\varphi_j(\theta, \theta_n)$. Also θ will be often omitted from various expressions. Thus, for example, we shall write φ_{nj}, $\dot{\varphi}_j$, Γ_j, $\bar{\Gamma}_n$, $\bar{\Gamma}$, etc. rather than $\varphi_{nj}(\theta)$, $\dot{\varphi}_j(\theta)$, $\Gamma_j(\theta)$, $\bar{\Gamma}_n(\theta)$, $\bar{\Gamma}(\theta)$, etc., respectively.

LEMMA 4.1. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\sum_{j=1}^n [\varphi_{nj}(\theta) - 1]^2 - \frac{1}{n} \sum_{j=1}^n [h'\dot{\varphi}_j(\theta)]^2 \to 0 \quad \textit{in} \quad P_\theta\textit{-probability}.$$

PROOF. We start with the following remark which will be used below. For $j = 1, \cdots, n$, let Z_j and Z_{nj} be rv's defined on a probability space $(\Omega, \mathscr{A}, P)$ and let $\max(\mathscr{E}|Z_{nj} - Z_j|^2; 1 \leqq j \leqq n) \to 0$. Let also $\mathscr{E}Z_j^2 \leqq M\,(< \infty)$ for all j. Then

$$\mathscr{E}^{\frac{1}{2}}|Z_{nj}|^2 = \mathscr{E}^{\frac{1}{2}}|(Z_{nj} - Z_j) + Z_j|^2 \leqq \mathscr{E}^{\frac{1}{2}}|Z_{nj} - Z_j|^2 + \mathscr{E}^{\frac{1}{2}}|Z_j|^2$$

and

$$\mathscr{E}^{\frac{1}{2}}|Z_j|^2 = \mathscr{E}^{\frac{1}{2}}|(Z_{nj} - Z_j) - Z_{nj}|^2 \leqq \mathscr{E}^{\frac{1}{2}}|Z_{nj} - Z_j|^2 + \mathscr{E}^{\frac{1}{2}}|Z_{nj}|^2 ,$$

so that

$$|\mathscr{E}^{\frac{1}{2}}|Z_{nj}|^2 - \mathscr{E}^{\frac{1}{2}}|Z_j|^2| \leqq \mathscr{E}^{\frac{1}{2}}|Z_{nj} - Z_j|^2 .$$

Hence

$$\max(|\mathscr{E}^{\frac{1}{2}}Z_{nj}^2 - \mathscr{E}^{\frac{1}{2}}Z_j^2|; 1 \leqq j \leqq n) \to 0 .$$

This result, together with the assumption that $\mathscr{E}Z_j^2 \leqq M\,(< \infty)$ for all j, implies that $\mathscr{E}Z_{nj}^2 \leqq M\,(< \infty)$ for all j, where M is generic constant. Finally, this conclusion, along with the inequality

$$\mathscr{E}|Z_{nj}^2 - Z_j^2| \leqq (\mathscr{E}^{\frac{1}{2}}Z_{nj}^2 + \mathscr{E}^{\frac{1}{2}}Z_j^2)\mathscr{E}^{\frac{1}{2}}|Z_{nj} - Z_j|^2$$

and the assumption that $\max(\mathscr{E}|Z_{nj} - Z_j|^2; 1 \leqq j \leqq n) \to 0$, implies that

$$\max(\mathscr{E}|Z_{nj}^2 - Z_j^2|; 1 \leqq j \leqq n) \to 0 . \tag{4.1}$$

Next, by (A2) (i), we have

$$|\lambda^{-1}| \cdot |\varphi_j(\theta, \theta + \lambda h) - 1 - \lambda h'\dot{\varphi}_j(\theta)| \to 0 \qquad \text{in qm} \quad [P_\theta] ,$$
$$\text{as } \lambda \to 0 \text{ uniformly in } j,$$

uniformly on bounded sets of $h \in R^k$. Let $\lambda = 1/n^{\frac{1}{2}}$ and replace h by h_n, $h_n \to h \in R^k$. Then

$$\max[\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j|^2; 1 \leqq j \leqq n] \to 0 . \tag{4.2}$$

The non-uniform version of (4.2) was obtained by Roussas (1965) (see 3.1.2).

From (A3), one, clearly, has that $\mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \leqq M(h, \theta)\,(< \infty)$ for all j. By this fact and (4.2), relation (4.1) gives

$$\max\,[\mathscr{E}_\theta|n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2|;\ 1 \leqq j \leqq n] \to 0\,. \tag{4.3}$$

Next, for every $\varepsilon > 0$

$$\begin{aligned} P_\theta\left[\left| \sum_{j=1}^n (\varphi_{nj} - 1)^2 - \frac{1}{n} \sum_{j=1}^n (h'\dot{\varphi}_j)^2 \right| > \varepsilon\right] &= P_\theta\{|\sum_{j=1}^n [n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2]| > n\varepsilon\} \\ &\leqq P_\theta[\sum_{j=1}^n |n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2| > n\varepsilon] \\ &\leqq \frac{1}{n\varepsilon} \sum_{j=1}^n \mathscr{E}_\theta|n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2|\,. \end{aligned} \tag{4.4}$$

But (4.3) implies that, for $n \geqq N_1(\varepsilon, h, \theta)$, $\mathscr{E}_\theta|n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2| < \varepsilon^2$, $1 \leqq j \leqq n$. Thus the last expression on the right-hand side of (4.4) is $< (n\varepsilon)^{-1}n\varepsilon^2 = \varepsilon$ for all $n \geqq N_1(\varepsilon, h, \theta)$. The proof is completed.

LEMMA 4.2. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\frac{1}{n} \sum_{j=1}^n [h'\dot{\varphi}_j(\theta)]^2 \to \tfrac{1}{4}h'\bar{\Gamma}(\theta)h \qquad \textit{in} \quad P_\theta\textit{-probability}.$$

PROOF. By (2.4) and (A5), we have

$$\frac{1}{n} \sum_{j=1}^n \mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \to \tfrac{1}{4}h'\bar{\Gamma}h\,.$$

Therefore it suffices to show that

$$\frac{1}{n} \sum_{j=1}^n [(h'\dot{\varphi}_j)^2 - \mathscr{E}_\theta(h'\dot{\varphi}_j)^2] \to 0 \qquad \text{in} \quad P_\theta\text{-probability}. \tag{4.5}$$

Now (4.5) is the statement of the weak law of large numbers of independent but not necessarily identically distributed rv's, and a sufficient condition for it to hold true (see Loève (1955), page 275) is that for some δ', $0 < \delta' \leqq 1$,

$$\frac{1}{n^{1+\delta'}} \sum_{j=1}^n \mathscr{E}_\theta|(h'\dot{\varphi}_j)^2 - \mathscr{E}_\theta(h'\dot{\varphi}_j)^2|^{1+\delta'} \to 0\,. \tag{4.6}$$

But

$$\begin{aligned} &\frac{1}{n^{1+\delta'}} \sum_{j=1}^n \mathscr{E}_\theta|(h'\dot{\varphi}_j)^2 - \mathscr{E}_\theta(h'\dot{\varphi}_j)^2|^{1+\delta'} \\ &\qquad \leqq \frac{2^{\delta'}}{n^{1+\delta'}} \sum_{j=1}^n [\mathscr{E}_\theta|h'\dot{\varphi}_j|^{2(1+\delta')} + |\mathscr{E}_\theta(h'\dot{\varphi}_j)^2|^{1+\delta'}] \end{aligned} \tag{4.7}$$

by the c_r-inequality (see Loève (1955), page 155), and the expression on the right-hand side above is less than or equal to

$$\frac{2^{\delta'}}{n^{1+\delta'}} \sum_{j=1}^n [\mathscr{E}_\theta|h'\dot{\varphi}_j|^{2(1+\delta')} + \mathscr{E}_\theta|h'\dot{\varphi}_j|^{2(1+\delta')}]\,. \tag{4.8}$$

This is so because, if Z is a rv defined on the probability space $(\Omega, \mathscr{A}, P)$, then $\mathscr{E}|Z| \leqq \mathscr{E}^{1/r}|Z|^r$, so that $\mathscr{E}^r|Z| \leqq \mathscr{E}|Z|^r$, provided $r \geqq 1$. Finally, the expression in (4.8) is equal to

$$2^{1+\delta'} \cdot \frac{1}{n^{1+\delta'}} \sum_{j=1}^{n} \mathscr{E}_\theta |h'\dot{\varphi}_j|^{2(1+\delta')} .$$

Taking now $\delta' = \delta/2$, where δ is as in (A4), this last expression becomes

$$2^{(2+\delta)/2} \cdot \frac{1}{n^{(2+\delta)/2}} \sum_{j=1}^{n} \mathscr{E}_\theta |h'\dot{\varphi}_j|^{2+\delta}$$

which converges to zero, by Assumption (A4). This result implies (4.6), by means of (4.7) and (4.8), and hence (4.5). The desired result then follows.

To Lemmas 4.1 and 4.2 there is the following immediate corollary.

COROLLARY 4.1. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\sum_{j=1}^{n} [\varphi_{nj}(\theta) - 1]^2 \to \tfrac{1}{4} h'\bar{\Gamma}(\theta)h \qquad \text{in} \quad P_\theta\text{-probability}.$$

Next, we establish the following

LEMMA 4.3. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\max\,[|\varphi_{nj}(\theta) - 1|;\ 1 \leqq j \leqq n] \to 0 \qquad \text{in} \quad P_\theta\text{-probability}.$$

PROOF. Following Roussas (1972) (see Lemma 5.2), we set

$$R_{nj} = n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j .$$

Then

$$\begin{aligned} &P_\theta[\max\,(|\varphi_{nj} - 1|;\ 1 \leqq j \leqq n) > \varepsilon] \\ &\qquad \leqq P_\theta[\max\,(|h'\dot{\varphi}_j|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] \\ &\qquad\quad + P_\theta[\max\,(|R_{nj}|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] . \end{aligned} \tag{4.9}$$

But

$$\begin{aligned} P_\theta[\max\,(|R_{nj}|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] &\leqq \sum_{j=1}^{n} P_\theta(|R_{nj}| > \varepsilon n^{\frac{1}{2}}/2) \\ &\leqq \frac{4}{n\varepsilon^2} \sum_{j=1}^{n} \mathscr{E}_\theta |R_{nj}|^2 , \end{aligned}$$

and by the definition of R_{nj} and Assumption (A2) (i), $\mathscr{E}_\theta |R_{nj}|^2 < \varepsilon^3/8$ for all sufficiently large n, $n \geqq N_2(\varepsilon, h, \theta)$, say, and all j, $1 \leqq j \leqq n$. Therefore

$$P_\theta[\max\,(|R_{nj}|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] \leqq \varepsilon/2 \quad \text{for all} \quad n \geqq N_2(\varepsilon, h, \theta) . \tag{4.10}$$

Next, let $F_j = F_{j,h,\theta}$ be the distribution, under P_θ, of the rv $|h'\dot{\varphi}_j|$. Then by Assumption (A3), one has that for all sufficiently large n, $n \geqq N_3(\varepsilon, h, \theta)$, say, and all j, $1 \leqq j \leqq n$,

$$\int_{(\varepsilon n^{\frac{1}{2}}/2, \infty)} x^2 \, dF_j = \int_{(|h'\dot{\varphi}_j| > \varepsilon n^{\frac{1}{2}}/2)} |h'\dot{\varphi}_j|^2 \, dP_\theta < \varepsilon^3/8 .$$

Therefore, for all $n \geqq N_3(\varepsilon, h, \theta)$,

$$\begin{aligned} P_\theta[\max(|h'\dot{\varphi}_j|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] &\leqq \textstyle\sum_{j=1}^n P_\theta(|h'\dot{\varphi}_j| > \varepsilon n^{\frac{1}{2}}/2) \\ &= \textstyle\sum_{j=1}^n \int_{(\varepsilon n^{\frac{1}{2}}/2, \infty)} dF_j \\ &= \frac{4}{n\varepsilon^2} \textstyle\sum_{j=1}^n \int_{(\varepsilon n^{\frac{1}{2}}/2, \infty)} (\varepsilon n^{\frac{1}{2}}/2)^2 \, dF_j \\ &\leqq \frac{4}{n\varepsilon^2} \textstyle\sum_{j=1}^n \int_{(\varepsilon n^{\frac{1}{2}}/2, \infty)} x^2 \, dF_j \\ &\leqq \frac{4}{n\varepsilon^2} n \frac{\varepsilon^3}{8} = \frac{\varepsilon}{2}; \quad \text{i.e.,} \end{aligned}$$

(4.11) $$P_\theta[\max(|h'\dot{\varphi}_j|;\ 1 \leqq j \leqq n) > \varepsilon n^{\frac{1}{2}}/2] < \varepsilon/2 \quad \text{for all} \quad n \geqq N_3(\varepsilon, h, \theta).$$

The desired conclusion then follows from (4.9) by means of (4.10) and (4.11).

The following result shows that the log-likelihood function $\Lambda_n(\theta)$ may be expressed asymptotically and in P_θ-probability in terms of the $\varphi_{nj}(\theta)$ rv's. Namely,

LEMMA 4.4. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

$$\Lambda_n(\theta) - 2\{\textstyle\sum_{j=1}^n [\varphi_{nj}(\theta) - 1] - \frac{1}{2}\sum_{j=1}^n [\varphi_{nj}(\theta) - 1]^2\} \to 0 \quad \textit{in} \quad P_\theta\textit{-probability}.$$

PROOF. As has been mentioned in the course of the proof of Lemma 4.1, one has that $\mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \leqq M(h, \theta)$ $(< \infty)$ for all j. Therefore for any $M > 0$,

$$P_\theta\left[\left|\frac{1}{n}\textstyle\sum_{j=1}^n (h'\dot{\varphi}_j)^2\right| > M\right] \leqq \frac{1}{nM}\textstyle\sum_{j=1}^n \mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \leqq \frac{M(h, \theta)}{M},$$

so that $(1/n)\sum_{j=1}^n (h'\dot{\varphi}_j)^2$ is bounded in P_θ-probability. This result and Lemma 4.1 imply, in an obvious manner, that $\sum_{j=1}^n (\varphi_{nj} - 1)^2$ is also bounded in P_θ-probability. Therefore Lemma 4.3 gives

(4.12) $$[\max(|\varphi_{nj} - 1|;\ 1 \leqq j \leqq n)] \textstyle\sum_{j=1}^n (\varphi_{nj} - 1)^2 \to 0 \quad \text{in} \quad P_\theta\text{-probability}.$$

For a given $0 < \varepsilon$ $(\leqq \frac{1}{2})$, let

(4.13) $$A_n = A_n(\theta) = [\max(|\varphi_{nj} - 1|;\ 1 \leqq j \leqq n) > \varepsilon].$$

Then Lemma 4.3 implies that $P_\theta(A_n^c) > 1 - \varepsilon$, $n \geqq N(\varepsilon)$. Consider the expansion

(4.14) $$\begin{aligned} \log x &= \log[1 + (x - 1)] \\ &= (x - 1) - \tfrac{1}{2}(x - 1)^2 + c(x - 1)^3, \qquad \text{where } |c| \leqq 3. \end{aligned}$$

From (4.13) and (4.14), we have then that on the set A_n^c with $P_\theta(A_n^c) > 1 - \varepsilon$, $n \geqq N(\varepsilon)$,

$$\log\varphi_{nj} = \log[1 + (\varphi_{nj} - 1)] = (\varphi_{nj} - 1) - \tfrac{1}{2}(\varphi_{nj} - 1)^2 + c_{nj}(\varphi_{nj} - 1)^3, \quad |c_{nj}| \leqq 3, j = 1, \cdots, n.$$

Therefore on A_n^c with $n \geqq N(\varepsilon)$,

$$\sum_{j=1}^n \log \varphi_{nj} = \sum_{j=1}^n (\varphi_{nj} - 1) - \tfrac{1}{2} \sum_{j=1}^n (\varphi_{nj} - 1)^2 + \sum_{j=1}^n c_{nj}(\varphi_{nj} - 1)^3 .$$

But

$$|\sum_{j=1}^n c_{nj}(\varphi_{nj} - 1)^3| \leqq 3[\max (|\varphi_{nj} - 1|; 1 \leqq j \leqq n)] \sum_{j=1}^n (\varphi_{nj} - 1)^2 \to 0$$

in P_θ-probability by (4.12). Therefore

$$\sum_{j=1}^n \log \varphi_{nj}^2 - 2[\sum_{j=1}^n (\varphi_{nj} - 1) - \tfrac{1}{2} \sum_{j=1}^n (\varphi_{nj} - 1)^2] \to 0 \quad \text{in } P_\theta\text{-probability.} \tag{4.15}$$

But (2.2) and (3.2) give $\Lambda_n(\theta) = \sum_{j=1}^n \log \varphi_{nj}^2(\theta)$. Thus relation (4.15) implies the desired result.

From Lemma 4.4 and Corollary 4.1, one immediately has the following

COROLLARY 4.2. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$,

$$\Lambda_n(\theta) - 2 \sum_{j=1}^n [\varphi_{nj}(\theta) - 1] \to -\tfrac{1}{4} h'\bar{\Gamma}(\theta)h \quad \textit{in} \quad P_\theta\textit{-probability}.$$

The following lemma will also be needed in the sequel.

LEMMA 4.5. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

(i) $$\max \{\mathscr{E}_\theta|n^{\frac{1}{2}}[\varphi_{nj}^2(\theta) - 1] - 2h'\dot{\varphi}_j(\theta)|; 1 \leqq j \leqq n\} \to 0 ;$$

(ii) $$\mathscr{E}_\theta[h'\dot{\varphi}_j(\theta)] = 0 , \qquad j \geqq 1 .$$

PROOF. (i) Following Roussas (1965), we consider the identity

$$n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h'\dot{\varphi}_j = \{\varphi_{nj}[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j] + h'\dot{\varphi}_j(\varphi_{nj} - 1)\} + [n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j] ,$$

from which we get

$$\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h'\dot{\varphi}_j| \leqq 2\mathscr{E}_\theta^{\frac{1}{2}}[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j]^2 + \mathscr{E}_\theta^{\frac{1}{2}}(h'\dot{\varphi}_j)^2\mathscr{E}_\theta^{\frac{1}{2}}(\varphi_{nj} - 1)^2 , \tag{4.16}$$

by means of Hölder inequality, the inequality

$$\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j| \leqq \mathscr{E}_\theta^{\frac{1}{2}}[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j]^2$$

and the fact that $\mathscr{E}_\theta \varphi_{nj}^2 = 1$. By (A2) (i),

$$\max \{\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\varphi_j]^2; 1 \leqq j \leqq n\} \to 0 .$$

We have also seen, in the course of the proof of Lemma 4.1, that the quantities $\mathscr{E}_\theta(h'\dot{\varphi}_j)^2$, $j \geqq 1$, stay bounded, and

$$\max \{|\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1)]^2 - \mathscr{E}_\theta(h'\dot{\varphi}_j)^2|; 1 \leqq j \leqq n\} \to 0 .$$

This latter conclusion implies that $\max [\mathscr{E}_\theta(\varphi_{nj} - 1)^2; 1 \leqq j \leqq n] \to 0$. Therefore (i) follows from (4.16).

(ii) We have

$$|\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - \mathscr{E}_\theta(2h'\dot{\varphi}_j)| \leqq \mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h'\dot{\varphi}_j| \to 0 ,$$

by (i), so that $\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj}^2 - 1)] \to \mathscr{E}_\theta(2h'\dot{\varphi}_j)$. But $\mathscr{E}_\theta(\varphi_{nj}^2 - 1) = 0$. Thus $\mathscr{E}_\theta(h'\dot{\varphi}_j) = 0$, as was to be seen.

This section is closed with Lemma 4.6 below, which along with the previous lemmas, provides all we need for the proof of Theorems 3.1—3.3.

LEMMA 4.6. *Under Assumptions* (A1)—(A5) *and for each* $\theta \in \Theta$, *one has*

(i) $$2 \sum_{j=1}^n \mathscr{E}_\theta[\varphi_{nj}(\theta) - 1] + \frac{1}{n} \sum_{j=1}^n [h'\dot{\varphi}_j(\theta)]^2 \to 0 \quad \textit{in } P_\theta\textit{-probability},$$

and

(ii) $$\sum_{j=1}^n \{[\varphi_{nj}(\theta) - 1] - \mathscr{E}_\theta[\varphi_{nj}(\theta) - 1]\} - \frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^n h'\dot{\varphi}_j(\theta) \to 0 \quad \textit{in } P_\theta\textit{-probability}.$$

PROOF. (i) We have $\varphi_{nj}^2 - 1 = (\varphi_{nj} - 1)^2 + 2(\varphi_{nj} - 1)$, so that $0 = \mathscr{E}_\theta(\varphi_{nj}^2 - 1) = \mathscr{E}_\theta(\varphi_{nj} - 1)^2 + 2\mathscr{E}_\theta(\varphi_{nj} - 1)$. Hence

$$\sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)^2 = -2 \sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)\,. \tag{4.17}$$

Next,

$$\left|\sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)^2 - \frac{1}{n} \sum_{j=1}^n \mathscr{E}_\theta(h'\dot{\varphi}_j)^2\right| \leqq \frac{1}{n} \sum_{j=1}^n \mathscr{E}_\theta|n(\varphi_{nj} - 1)^2 - (h'\dot{\varphi}_j)^2| < \frac{1}{n} n\varepsilon = \varepsilon\,,$$

for all sufficiently large n, $n \geqq N_4(\varepsilon, h, \theta)$, say, by (4.3). Thus

$$\sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)^2 - \frac{1}{n} \sum_{j=1}^n \mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \to 0\,. \tag{4.18}$$

But $(1/n) \sum_{j=1}^n \mathscr{E}_\theta(h'\dot{\varphi}_j)^2 \to \frac{1}{4}h'\bar{\Gamma}h$, by (2.4) and (A5). Therefore (4.18) becomes

$$\sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)^2 \to \tfrac{1}{4}h'\bar{\Gamma}h\,. \tag{4.19}$$

Relation (4.19), together with Lemma 4.2, implies that

$$\sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1)^2 - \frac{1}{n} \sum_{j=1}^n (h'\dot{\varphi}_j)^2 \to 0 \quad \text{in } P_\theta\text{-probability.} \tag{4.20}$$

Therefore (4.17) and (4.20) give

$$2 \sum_{j=1}^n \mathscr{E}_\theta(\varphi_{nj} - 1) + \frac{1}{n} \sum_{j=1}^n (h'\dot{\varphi}_j)^2 \to 0 \quad \text{in } P_\theta\text{-probability},$$

as was to be shown.

(ii) For $j = 1, \cdots, n$, define the rv's Y_j by

$$Y_j = (\varphi_{nj} - 1) - \frac{1}{n^{\frac{1}{2}}} h'\dot{\varphi}_j - \mathscr{E}_\theta(\varphi_{nj} - 1)\,. \tag{4.21}$$

Then $\mathscr{E}_\theta Y_j = 0$, by Lemma 4.5 (ii). Therefore Kolmogorov's inequality and c_r-inequality, applied successively, give

$$P_\theta(|\sum_{j=1}^n Y_j| > \varepsilon) \leqq \frac{1}{\varepsilon^2} \sum_{j=1}^n \sigma_\theta^2(Y_j)$$

$$(4.22) \qquad = \frac{1}{n\varepsilon^2} \sum_{j=1}^n \mathscr{E}_\theta\{[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j] - n^{\frac{1}{2}}\mathscr{E}_\theta(\varphi_{nj} - 1)\}^2$$

$$\leqq \frac{2}{n\varepsilon^2} \sum_{j=1}^n \mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j]^2 + \frac{2}{n\varepsilon^2} \sum_{j=1}^n |\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1)]|^2 .$$

But $\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j]^2 < \varepsilon^3/2$ for all sufficiently large n, $n \geqq N_5(\varepsilon, h, \theta)$, say, and $1 \leqq j \leqq n$, by (A2) (i), so that

$$(4.23) \qquad \frac{2}{n\varepsilon^2} \sum_{j=1}^n \mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j]^2 < \varepsilon , \qquad n > N_5(\varepsilon, h, \theta) .$$

Next, since $\mathscr{E}_\theta(h'\dot{\varphi}_j) = 0$, by Lemma 4.5 (ii), we have

$$\begin{aligned} \max \{|\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1)]|;\ 1 \leqq j \leqq n\} &= \max \{|\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1)] - \mathscr{E}_\theta(h'\dot{\varphi}_j)|;\ 1 \leqq j \leqq n\} \\ &\leqq \max [\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j|;\ 1 \leqq j \leqq n] \\ &\leqq \max [\mathscr{E}_\theta^{\frac{1}{2}}|n^{\frac{1}{2}}(\varphi_{nj} - 1) - h'\dot{\varphi}_j|^2;\ 1 \leqq j \leqq n] \\ &\to 0 , \qquad \text{by (A2) (i).} \end{aligned}$$

Thus, for all sufficiently large n, $n \geqq N_6(\varepsilon, h, \theta)$, say, one has

$$(4.24) \qquad \frac{2}{n\varepsilon^2} \sum_{j=1}^n |\mathscr{E}_\theta[n^{\frac{1}{2}}(\varphi_{nj} - 1)]|^2 < \varepsilon .$$

Relations (4.22), (4.23) and (4.24) imply that $\sum_{j=1}^n Y_j \to 0$ in P_θ-probability. This result and the definition of Y_j by (4.21) imply (ii).

5. Proof of the main results. We may now proceed with the proof of the theorems stated in Section 3. The proof is based primarily on the lemmas established in Section 4.

PROOF OF THEOREM 3.1. In Lemma 4.6, we multiply the expression on the left-hand side of (ii) by 2 and then add up (i) and (ii). We get then

$$(5.1) \qquad 2 \sum_{j=1}^n (\varphi_{nj} - 1) + \frac{1}{n} \sum_{j=1}^n (h'\dot{\varphi}_j)^2 - \frac{2}{n^{\frac{1}{2}}} \sum_{j=1}^n h'\dot{\varphi}_j \to 0 \quad \text{in } P_\theta\text{-probability.}$$

In terms of the $\Delta_n(\theta)$ notation (see (2.3)), (5.1) becomes

$$2 \sum_{j=1}^n (\varphi_{nj} - 1) + \frac{1}{n} \sum_{j=1}^n (h'\dot{\varphi}_j)^2 - h'\Delta_n \to 0 \qquad \text{in } P_\theta\text{-probability.}$$

This result, along with Lemma 4.2, gives

(5.2) $$2 \sum_{j=1}^{n} (\varphi_{nj} - 1) - h'\Delta_n \to -\tfrac{1}{4}h'\bar{\Gamma}h \quad \text{in } P_\theta\text{-probability.}$$

Finally, combining (5.2) with Corollary 4.2, we obtain

$$\Lambda_n - h'\Delta_n \to -\tfrac{1}{2}h'\bar{\Gamma}h \quad \text{in } P_\theta\text{-probability,}$$

as was to be shown.

PROOF OF THEOREM 3.2. Set $Y_{nj} = (2/n^{\frac{1}{2}})h'\dot{\varphi}_j, j = 1, \cdots, n, 0 \neq h \in R^k$. Then $\mathscr{E}_\theta Y_{nj} = 0$, by Lemma 4.5 (ii), and $s_n^2 = \sum_{j=1}^{n} \sigma_\theta^2(Y_{nj}) = (4/n) \sum_{j=1}^{n} \mathscr{E}_\theta(h'\dot{\varphi}_j)^2 = h'\Gamma_n h \to h'\bar{\Gamma}h$, by (A5). Therefore

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \mathscr{E}_\theta |Y_{nj}|^{2+\delta} = \frac{2^{2+\delta}}{(s_n^2)^{1+\delta/2}} \frac{1}{n^{(2+\delta)/2}} \sum_{j=1}^{n} \mathscr{E}_\theta |h'\dot{\varphi}_j|^{2+\delta} \to 0 \,,$$

by (A4). Hence Liapounov's condition for the Central Limit Theorem to hold (see Loève (1955), page 275) is satisfied and therefore

$$\mathscr{L}\left(\frac{h'\Delta_n}{s_n} \,\Big|\, P_\theta\right) \Rightarrow N(0, 1) \,, \qquad \text{since} \quad h'\Delta_n = \sum_{j=1}^{n} Y_{nj} \,.$$

Now $h'\bar{\Gamma}h$ is positive, by (A5), and hence the last convergence implies

$$\mathscr{L}(h'\Delta_n \mid P_\theta) \Rightarrow N(0, h'\bar{\Gamma}h) \,.$$

Since this is true for every $h \in R^k$, it follows that

$$\mathscr{L}(\Delta_n \mid P_\theta) \Rightarrow N(0, \bar{\Gamma}) \,,$$

as was to be established.

PROOF OF THEOREM 3.3. It follows immediately from Theorems 3.1 and 3.2, and the standard Slutsky theorems.

We now proceed with the proof of Theorem 3.4.

PROOF OF THEOREM 3.4. From Theorem 3.3 herein and a corollary to LeCam's first lemma (see Hájek and Šidák (1967), page 204), it follows that $\{P_{n,\theta_n}\}$ is contiguous with respect to $\{P_{n,\theta}\}$. The proof of the theorem is then completed on account of Theorem 3.1.

PROOF OF THEOREM 3.5. It follows from Theorem 3.3, the contiguity of $\{P_{n,\theta_n}\}$ and $\{P_{n,\theta}\}$ mentioned in the proof of the previous theorem and Corollary 7.2, Chapter 1, in Roussas (1972). Alternatively,

$$P_{\theta_n}(\Lambda_n \leqq x) = \int_{(\Lambda_n \leqq x)} dP_{\theta_n} = \int_{(\Lambda_n \leqq x)} e^{\Lambda_n} dP_\theta = \int e^z I_{(-\infty,x]}(z)\, d\mathscr{L}(\Lambda_n \mid P_\theta) \,,$$

and $\mathscr{L}(\Lambda_n \mid P_\theta) \Rightarrow N(-\frac{1}{2}h'\bar{\Gamma}h, h'\bar{\Gamma}h) = Q_h$, by Theorem 3.3, whereas the set of discontinuities of $e^z I_{(-\infty,x]}(z)$, $\{x\}$, has Q_h-measure zero. Therefore

$$\begin{aligned} \int e^z I_{(-\infty,x]}(z)\, d\mathscr{L}(\Lambda_n \mid P_\theta) &\Rightarrow \int e^z I_{(-\infty,x]}(z)\, dQ_h \\ &= \int_{(-\infty,x]} e^z\, dN(-\tfrac{1}{2}h'\bar{\Gamma}h, h'\bar{\Gamma}h) \,. \end{aligned}$$

But $$\int_{(-\infty,x]} e^z \, dN(-\tfrac{1}{2}h'\bar{\Gamma}h,\ h'\bar{\Gamma}h) = \int_{(-\infty,x]} dN(\tfrac{1}{2}h'\bar{\Gamma}h,\ h'\bar{\Gamma}h)$$

as is easily seen, and this establishes the theorem.

For the proof of Theorem 3.6, we need an auxiliary result which is formulated and proved below.

PROPOSITION 5.1. *For each* $n = 1, 2, \cdots$, *let* P_n *be a probability measure defined on the measurable space* $(\Omega, \mathscr{F}_n)$, *and for each* $h \in R^k$, *let* $Z_n = Z_n(h)$ *and* T_n *be a* rv *and a* k*-dimensional random vector, respectively, defined on* $(\Omega, \mathscr{F}_n)$ *and such that*

$$\mathscr{L}(T_n \mid P_n) \Rightarrow N(0, \Gamma)\,, \tag{5.3}$$

where Γ *is a* $k \times k$ *nonsingular covariance matrix, and*

$$Z_n - h'T_n \to -\tfrac{1}{2}\sigma^2 \quad \textit{in } P_n\textit{-probability}, \tag{5.4}$$

where $\sigma^2 = \sigma^2(h) = h'\Gamma h$.

Then $\{\mathscr{L}[(Z_n, T_n')' \mid P_n]\}$ *converges (weakly) to a* $((k+1)$*-dimensional) normal law with mean and covariance given by*

$$(-\tfrac{1}{2}\sigma^2, 0, \cdots, 0)' \quad \textit{and} \quad \begin{bmatrix} \sigma^2 & h'\Gamma \\ \Gamma h & \Gamma \end{bmatrix}, \quad \textit{respectively.}$$

PROOF. For $t_0 \in R$ and $t \in R^k$, set $u' = (t_0, t')$. Then it suffices to show that, for every such $u \in R^{k+1}$, the rv $u'\binom{Z_n}{T_n}$ converges (weakly) to the normal law with mean and variance

$$u' \begin{pmatrix} -\frac{1}{2}\sigma^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad u' \begin{bmatrix} \sigma^2 & h'\Gamma \\ \Gamma h & \Gamma \end{bmatrix} u\,, \quad \text{respectively.}$$

Now (5.3) implies that

$$\mathscr{L}[(t_0 h + t)'T_n \mid P_n] \Rightarrow N(0, (t_0 h + t)'\Gamma(t_0 h + t))\,, \tag{5.5}$$

whereas (5.4) implies that

$$t_0(Z_n - h'T_n) \to -\tfrac{1}{2}t_0\sigma^2 \quad \text{in } P_n\text{-probability.} \tag{5.6}$$

Next,

$$\begin{aligned} \mathscr{L}[u'\tbinom{Z_n}{T_n} \mid P_n] &= \mathscr{L}[(t_0 Z_n + t'T_n) \mid P_n] \\ &= \mathscr{L}\{[t_0(Z_n - h'T_n) + (t_0 h + t)'T_n] \mid P_n\} \end{aligned}$$

which, by means of (5.5) and (5.6) and the standard Slutsky theorems, converges to the normal law with mean and variance $-\frac{1}{2}t_0\sigma^2$ and $(t_0 h + t)'\Gamma(t_0 h + t)$, respectively. But

$$-\tfrac{1}{2}t_0\sigma^2 = u' \begin{pmatrix} -\frac{1}{2}\sigma^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad (t_0 h + t)'\Gamma(t_0 h + t) = u' \begin{bmatrix} \sigma^2 & h'\Gamma \\ \Gamma h & \Gamma \end{bmatrix} u\,,$$

as is easily checked by taking into consideration the fact that $\sigma^2 = h'\Gamma h$. The proof is completed.

We are now in a position to present the proof of Theorem 3.6.

PROOF OF THEOREM 3.6. In Proposition 5.1, take $Z_n = \Lambda_n(\theta)$ and $T_n = \Delta_n(\theta)$. Then, by Theorems 3.2 and 3.1, conditions (5.3) and (5.4) of Proposition 5.1 are satisfied. Therefore $\{\mathscr{L}[(\Lambda_n, \Delta_n')' \mid P_\theta]\}$ converges (weakly) to a normal law with mean and covariance given by

$$(-\tfrac{1}{2}\sigma^2, 0, \cdots, 0)' \qquad \text{and} \qquad \begin{bmatrix} \sigma^2 & h'\bar{\Gamma} \\ \bar{\Gamma}h & \bar{\Gamma} \end{bmatrix}, \qquad \text{respectively, where } \sigma^2 = h'\bar{\Gamma}h\,.$$

Next, $\{P_{n,\theta}\}$ and $\{P_{n,\theta_n}\}$ are contiguous, as was pointed out in the proof of Theorem 3.4. Therefore Theorem 2.1 (6) in LeCam (1960) applies and gives that

$$\mathscr{L}(\Delta_n \mid P_{\theta_n}) \Rightarrow N(\bar{\Gamma}h, \bar{\Gamma})\,,$$

as was to be seen. The same conclusion may be arrived at by an obvious modification of LeCam's third lemma, as is formulated and proved in Hájek and Šidák (1967), page 208.

6. An example. In this section, the applicability of Assumptions (A1)—(A5) and the results obtained in this paper are illustrated by means of an example.

EXAMPLE. Let X_j, $j = 1, \cdots, n$ be independent rv's, such that the probability density of the rv X_j is given by

$$f_j(x_j; \theta) = \tfrac{1}{2}\exp[-|x_j - \lambda_j\theta|]\,, \qquad x_j \in R, \theta \in R\,, \tag{6.1}$$

where the λ_j's are assumed to satisfy the relations: $0 < \lambda_j < M$ for some $M < \infty$, and $(1/n)\sum_{j=1}^n \lambda_j^2 \to \lambda > 0$.

Assumption (A1) is, clearly, satisfied.

In order to establish (A2), we proceed as follows. We have

$$\varphi_j(\theta, \theta^*) = \exp\{-\tfrac{1}{2}[|X_j - \lambda_j\theta^*| - |X_j - \lambda_j\theta|]\}\,. \tag{6.2}$$

We note that the pointwise derivative of $\varphi_j(\theta, \theta^*)$ with respect to θ^* does not exist at (θ, θ) when $\lambda_j\theta = X_j$. The derivative in quadratic mean, however, exists and is given by $Z_j(\theta)$, where

$$\begin{aligned} Z_j(\theta) &= -\frac{\lambda_j}{2}\,, && \text{if } X_j < \lambda_j\theta \\ &= 0\,, && \text{if } X_j = \lambda_j\theta \\ &= \frac{\lambda_j}{2}\,, && \text{if } X_j > \lambda_j\theta\,. \end{aligned} \tag{6.3}$$

Then

$$\begin{aligned} \mathscr{E}_\theta Z_j(\theta) &= -\frac{\lambda_j}{2}P_\theta(X_j < \lambda_j\theta) + \frac{\lambda_j}{2}P_\theta(X_j > \lambda_j\theta) \\ &= -\frac{\lambda_j}{2}\cdot\frac{1}{2} + \frac{\lambda_j}{2}\cdot\frac{1}{2} \\ &= 0\,, \end{aligned} \tag{6.4}$$

and

$$(6.5)\qquad \mathscr{E}_\theta[Z_j(\theta)]^2 = \frac{\lambda_j^2}{4}\cdot\frac{1}{2} + \frac{\lambda_j^2}{4}\cdot\frac{1}{2} = \frac{\lambda_j^2}{4}\,.$$

Next,

$$\mathscr{E}_\theta\varphi_j(\theta,\theta+h) = \mathscr{E}_\theta\exp\{-\tfrac{1}{2}[|X_j-\lambda_j(\theta+h)| - |X_j-\lambda_j\theta|]\} = \begin{cases}\tfrac{1}{2}(2-\lambda_j h)e^{\lambda_j\cdot h/2}\,, & \text{if } h<0\\ \tfrac{1}{2}(2+\lambda_j h)e^{-\lambda_j\cdot h/2}\,, & \text{if } h>0\,,\end{cases}$$

as is easily seen. Thus

$$(6.6)\qquad \frac{2}{h^2}[1-\mathscr{E}_\theta\varphi_j(\theta,\theta+h)] = \begin{cases}\dfrac{1}{2t^2}(1-e^{\lambda_j t}+\lambda_j te^{\lambda_j t})\,, & \text{if } t=\dfrac{h}{2}<0\\ \dfrac{1}{2t^2}(1-e^{\lambda_j t}-\lambda_j te^{\lambda_j t})\,, & \text{if } t=\dfrac{h}{2}>0\end{cases} \to \frac{\lambda_j^2}{4}\,,\qquad \text{as } h\to 0,\ \text{uniformly in } j.$$

Finally,

$$\mathscr{E}_\theta[Z_j(\theta)\varphi_j(\theta,\theta+h)] = -\frac{\lambda_j}{4}\int_{-\infty}^{\lambda_j\theta}\exp\{-\tfrac{1}{2}[|x_j-\lambda_j(\theta+h)| + |x_j-\lambda_j\theta|]\}\,dx_j + \frac{\lambda_j}{4}\int_{\lambda_j\theta}^{\infty}\exp\{-\tfrac{1}{2}[|x_j-\lambda_j(\theta+h)| + |x_j-\lambda_j\theta|]\}\,dx_j$$

$$= \begin{cases}\dfrac{\lambda_j^2}{4}he^{\lambda_j\cdot h/2}\,, & \text{if } h<0\\ \dfrac{\lambda_j^2}{4}he^{-\lambda_j\cdot h/2}\,, & \text{if } h>0\end{cases}$$

which implies

$$(6.7)\qquad \frac{1}{h}\mathscr{E}_\theta[Z_j(\theta)\varphi_j(\theta,\theta+h)] = \begin{cases}\dfrac{\lambda_j^2}{4}e^{\lambda_j\cdot h/2}\,, & \text{if } h<0\\ \dfrac{\lambda_j^2}{4}e^{-\lambda_j\cdot h/2}\,, & \text{if } h>0\end{cases} \to \frac{\lambda_j^2}{4}\,,\qquad \text{as } h\to 0,\ \text{uniformly in } j.$$

Relations (6.3)—(6.7) show that (A2)(i) is satisfied with $\dot\varphi_j(\theta) = Z_j(\theta)$ given by (6.3). Since $\dot\varphi_j(\theta)$ is, clearly, $X_j^{-1}(\mathscr{B}_j)\times\mathscr{C}$-measurable, (A2)(ii) also holds.

We now observe that

$$(6.8)\qquad \mathscr{E}_\theta|Z_j(\theta)|^3 = \left|-\frac{\lambda_j}{2}\right|^3\cdot\frac{1}{2} + \left|\frac{\lambda_j}{2}\right|^3\cdot\frac{1}{2} = \frac{\lambda_j^3}{8} < \frac{M^3}{8}\,,$$

so that (A3) and (A4) are satisfied, by means of Remark 2.1.

Lastly, (A5) is true, since

$$\bar{\Gamma}_n(\theta) = \frac{4}{n} \sum_{j=1}^n \mathscr{E}_\theta[\dot{\varphi}_j(\theta)]^2 = \frac{1}{n} \sum_{j=1}^n \lambda_j^2 \to \lambda = \bar{\Gamma}(\theta) > 0 . \tag{6.9}$$

It is then easily seen, by means of (6.1)—(6.3) and (6.9), that one has here

$$\Lambda_n(\theta) = \sum_{j=1}^n (|X_j - \lambda_j\theta| - |X_j - \lambda_j\theta_n|) ,$$

$$\Delta_n(\theta) = \frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^n \lambda_j[I_{(\lambda_j\theta,\infty)}(X_j) - I_{(-\infty,\lambda_j\theta)}(X_j)]$$

and $\bar{\Gamma}(\theta) = \lambda$.

Therefore the main results of this paper, in connection with the present example, become as follows:

$$\sum_{j=1}^n \left\{(|X_j - \lambda_j\theta| - |X_j - \lambda_j\theta_n|) - \frac{1}{n^{\frac{1}{2}}} h\lambda_j[I_{(\lambda_j\theta,\infty)}(X_j) - I_{(-\infty,\lambda_j\theta)}(X_j)]\right\}$$

$$\to -\tfrac{1}{2}h^2\lambda \quad \text{both in} \quad P_\theta \quad \text{and} \quad P_{\theta_n}\text{-probability},$$

$$\mathscr{L}\left\{\frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^n \lambda_j[I_{(\lambda_j\theta,\infty)}(X_j) - I_{(-\infty,\lambda_j\theta)}(X_j)] \mid P_\theta\right\} \Rightarrow N(0, \lambda) ,$$

$$\mathscr{L}[\sum_{j=1}^n (|X_j - \lambda_j\theta| - |X_j - \lambda_j\theta_n|) \mid P_\theta] \Rightarrow N(-\tfrac{1}{2}h^2\lambda, h^2\lambda) ,$$

$$\mathscr{L}[\sum_{j=1}^n (|X_j - \lambda_j\theta| - |X_j - \lambda_j\theta_n|) \mid P_{\theta_n}] \Rightarrow N(\tfrac{1}{2}h^2\lambda, h^2\lambda)$$

and

$$\mathscr{L}\left\{\frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^n \lambda_j[I_{(\lambda_j\theta,\infty)}(X_j) - I_{(-\infty,\lambda_j\theta)}(X_j)] \mid P_{\theta_n}\right\} \Rightarrow N(h\lambda, \lambda) .$$

In a forthcoming paper, an example involving a multidimensional parameter will be presented.

Acknowledgment. The authors wish to express their thanks to the referee for his thoughtful and constructive comments which helped to improve a previous version of this paper.

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