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A GENERAL METHOD FOR THE APPROXIMATION OF TAIL AREAS

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For a density function $f(x)$, the tail area

$$\alpha(x) = \int_x^\infty f(x)\,dx\,,$$

may be approximated by

$$\hat{\alpha}(x) = \frac{f(x)}{g(x)} \cdot (K-1)^{-1} \cdot \left\{1 + \frac{1}{2}\left(\frac{g'(x)}{g^2(x)} - (K)\right)\right\}\,,$$

where $g(x) = f(x)/f'(x)$, and $K = \lim_{x\to\infty} \{g'(x)/g^2(x)\}$. The formula requires only one constant and three function evaluations; g and g' are typically elementary functions. Such approximations are useful for programmed calculators or very small computers where only a few constants can be stored. The accuracy of the approximation is calculated for some common distributions. The approximation is very accurate for a large class of distributions.

1. Introduction. Tail areas, or significance levels, are used widely in statistical problems. The nature of their use frequently does not require that these levels be known to any great precision. A proportional error of several hundred percent typically makes very little difference. In these cases an approximation accurate to within 5 or 10 percent is quite acceptable. Here a fairly general method of approximating such areas is given. The error of the approximation has been evaluated for some common distributions.

2. The basis for the approximation. Many distributions have tails that "look" exponential. See Fig. 1. The following discussion applies only to such distributions, not for example to the uniform. It is assumed that the density function and its first and second derivatives exist, are continuous, and are non-zero in the region studied. For an exponential distribution the tangent at x bisects the tail area. For exponential-tailed distributions the tangent at x partitions the tail area in a fairly constant proportion. See Fig. 2. Let

$$\alpha(x) = \int_x^\infty f(x)\,dx\,,$$

the tail area to be approximated.

For the exponential distribution $\alpha(x)$ is twice the area of the triangle formed by the tangent to f at x and the line $y = x$. Thus

$$\alpha(x) = -f^2(x)/f'(x)\,.$$

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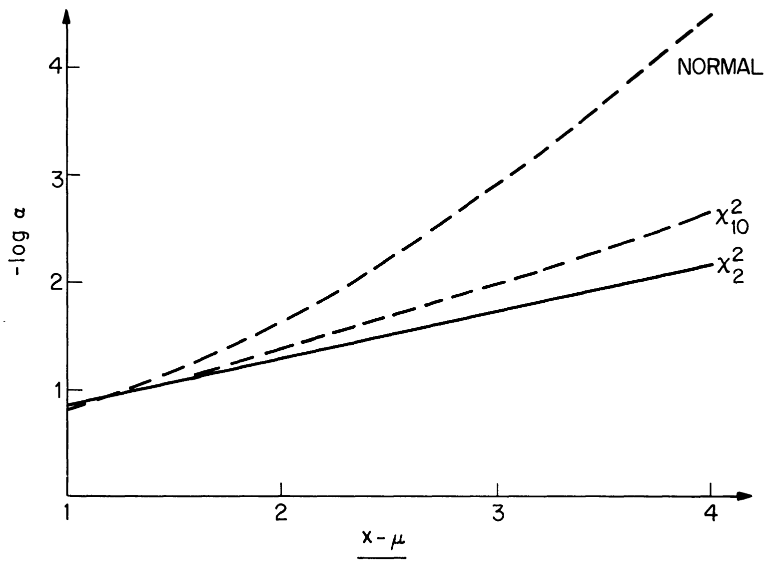

FIG. 1. The logarithm of the tail area, α, is, for the exponential distribution, linear in x—for some other distributions—almost linear.

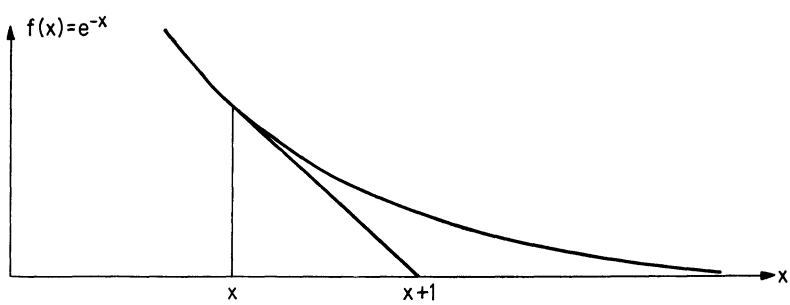

FIG. 2. The tangent at x bisects the tail area of the exponential distribution.

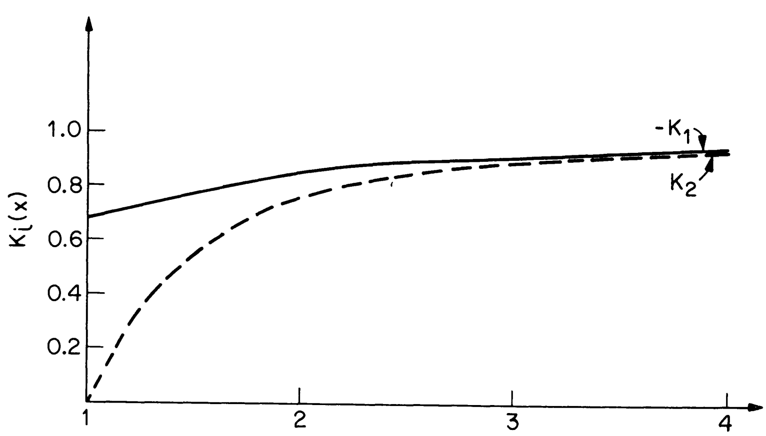

FIG. 3. The functions $K_1(x)$, $K_2(x)$ plotted for the normal distribution.

In general

$$\alpha(x) = \frac{f^2(x)}{f'(x)} \cdot K_1(x)$$

where $K_1(x) = \alpha(x)f'(x)/f^2(x)$. Typically $K_1(x)$ is a slowly varying function of x and in the region of interest, for small $\alpha(x)$, is not far from its limiting value

$$K_1 = \lim_{x\to\infty} K_1(x) .$$

See Fig. 3.

So as a first approximation let

$$\hat{\alpha}_1(x) = \frac{f^2(x)}{f'(x)} \cdot K_1 .$$

The exponential shape of the density function in the tail suggests that the ratio of two successive derivatives of the distribution function is approximately a constant. It suggests, then, that the behavior of $-K_1(x)$ will be not unlike that of

$$K_2(x) = f(x)f''(x)/[f'(x)f'(x)]$$

obtained by taking the derivative of each term in $K_1(x)$. See Fig 3. An approximation that makes use of this is

$$\hat{\alpha}(x) = \frac{f^2(x)}{f'(x)} \hat{K}_1(x) ,$$

where

$$\hat{K}_1(x) = K_1\left(1 + \frac{K_2(x) - K_2}{2}\right) ,$$

and $K_2 = \lim_{x\to\infty} K_2(x)$.

3. Algebraic considerations. The approximation has a more rigorous algebraic basis. Let $g(x)$ be defined by

$$f'(x) = g(x)f(x) .$$

For many distributions $g(x)$ is a surprisingly simple function. See, for example, Table 1. Note also that $g(x)$ is Fisher's score for location.

It follows from L'Hopitals rule and the definition

$$\begin{aligned} K_2(x) &= f(x)f''(x)/[f'(x)]^2 \\ &= 1 + [g'(x)/g^2(x)] \end{aligned}$$

that

$$\begin{aligned} -K_1 &= \lim_{x\to\infty} \frac{-\alpha(x)}{f^2(x)/f'(x)} \\ &= \lim_{x\to\infty} \frac{f(x)}{2f(x) - f^2(x)f''(x)/[f'(x)]^2} \\ &= \lim_{x\to\infty} \frac{1}{2 - K_2(x)} . \end{aligned}$$

Let

$$K_2 = \lim_{x\to\infty} K_2(x) .$$

The tail area is given by

$$\begin{aligned} a(x) &= \textstyle\int_x^\infty \{f'(x)/g(x)\}\, dx \\ &= \frac{-f(x)}{g(x)} + \textstyle\int_x^\infty f'(x) \frac{g'(x)}{g^3(x)}\, dx \\ &= \frac{-f(x)}{g(x)} - f(x)\frac{g'(\xi)}{g^3(\xi)} , \qquad x < \xi < \infty , \end{aligned}$$

since $g'(x)/g^3(x)$ is a continuous function of x. Thus $K_1(x)$ is given by

$$\begin{aligned} \frac{\alpha(x)f'(x)}{f(x)f(x)} &= -1 - \frac{g(x)}{g(\xi)} \cdot \frac{g'(\xi)}{g^2(\xi)} \\ &= K_1\left(1 - \frac{g(x)}{g(\xi)}\frac{1}{K_1}\frac{g'(\xi)}{g^2(\xi)} - (K_2 - 1)\right) \\ &= K_1\left(1 - \frac{g(x)}{g(\xi)}\frac{1}{K_1}(K_2(\xi) - 1) - (K_2 - 1)\right), \end{aligned}$$

since $-K_1(2 - K_2) = 1$.

Now typically ξ is close to x and the ratio $g(x)/g(\xi)$ is not far from 1. In many cases K_1 also is not far from -1. Therefore it may not be unreasonable to approximate the term $g(x)/g(\xi)K_1$ with -1. Furthermore, if $K_2(\xi)$ is monotonic in the region $\xi > x$ it follows that

$$0 < \frac{K_2(\xi) - K_2}{K_2(x) - K_2} < 1 .$$

This ratio may be approximated with $\frac{1}{2}$. In many cases the errors introduced by these two approximations are compensating to some degree.

The resulting approximation for the tail area,

$$\hat{\alpha}(x) = \frac{f^2(x)}{f'(x)} K_1\left(1 + \frac{K_2(x) - K_2}{2}\right)$$

is very good for a wide range of distributions. Note especially Table 2D.

The final expression may be simplified by letting

$$K = \lim_{x\to\infty} \frac{g'(x)}{g^2(x)} .$$

Then

$$\hat{\alpha} = \frac{f(x)}{g(x)}(K - 1)^{-1}\left\{1 + \frac{1}{2}\left(\frac{g'(x)}{g^2(x)} - K\right)\right\} .$$

The functions required for the approximation of some common distributions are given in Table 1. The proportional error is given in Table 2.

If the limiting value, K, is difficult to obtain it may be approximated by

$$\hat{K} = g'(X)/g^2(X)$$

where $X = 10x$ say.

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TABLE 1
Approximating functions for common distributions

Density	$g(x)$	$g'(x)$	K
f	f'/f	$g'(x)$	$\lim_{x\to\infty} \dfrac{g'(x)}{g^2(x)}$
Standard normal	$-x$	-1	0
t, n degrees freedom	$-\left(1+\dfrac{1}{n}\right)\dfrac{t}{\left(1+\dfrac{t^2}{n}\right)}$	$-\left(1+\dfrac{1}{n}\right)\dfrac{\left(1-\dfrac{t^2}{n}\right)}{\left(1+\dfrac{t^2}{n}\right)^2}$	$\dfrac{1}{n+1}$
χ^2, n degrees freedom	$\dfrac{n-2}{2x}-\dfrac{1}{2}$	$-\dfrac{n-2}{2x^2}$	0
F_{n_1,n_2}	$\dfrac{1}{2}\left[\dfrac{n_1-2}{x}-\dfrac{(n_1+n_2)n_1}{(n_2+n_1x)}\right]$	$\dfrac{1}{2}\left[\dfrac{2-n_1}{x^2}+\dfrac{(n_1+n_2)n_1^2}{(n_2+n_1x)^2}\right]$	$\dfrac{2}{2+n_2}$

TABLE 2A
t distribution
Relative error of approximation, $(\hat{\alpha}-\alpha)/\alpha$

	α				
n	0.05	0.025	0.01	0.005	5×10^{-6}
2	0.01	0.008	0.003	0.002	4×10^{-6}
4	0.03	0.02	0.02	0.01	4×10^{-4}
10	0.03	0.03	0.03	0.03	0.006
20	0.03	0.04	0.04	0.03	0.01
∞	0.02	0.04	0.04	0.04	0.02

TABLE 2B
χ^2 *distribution*
Relative error of approximation, $(\hat{\alpha}-\alpha)/\alpha$
Degrees of freedom $= n$

	α			
n	0.05	0.025	0.01	0.001
4	0.009	0.009	0.007	0.004
6	0.01	0.01	0.01	0.007
10	0.02	0.02	0.02	0.01
20	0.02	0.02	0.02	0.01
40	0.02	0.03	0.03	0.02
80	0.02	0.03	0.03	0.02

TABLE 2C

F distribution
n_1, n_2 degrees of freedom
Relative error of approximation, $(\hat{\alpha} - \alpha)/\alpha$,
For $\alpha = 0.05$ and $\alpha = 0.01$

		n_1			
α	n_2	2	4	10	20
.05	2	~0	0.001	0.002	0.002
.01		~0	~0	−0.0002	−0.0007
.05	4	−0.0001	0.0002	0.002	0.002
.01		~0	−0.0002	0.004	0.004
.05	10	0.0003	0.002	0.01	0.007
.01		~0	0.0009	0.002	0.004
.05	20	−0.002	0.002	0.01	0.02
.01		−0.00001	0.0001	0.001	0.003

~0 is $< 10^{-5}$

TABLE 2D

Relative error of approximation for some other distributions

Distribution	$f(x)$	x	$\alpha(x)$	$\hat{\alpha}(x)$	$\frac{\hat{\alpha}(x) - \alpha(x)}{\alpha(x)}$
Log-normal	$(2\pi)^{-\frac{1}{2}} \frac{1}{\lvert x \rvert} \exp\{-\frac{1}{2} \ln^2 \lvert x \rvert\}$	7.39	0.0227	0.0200	0.12
Logistic	$\frac{e^x}{(1 + e^x)^2}$	ln 19	0.0500	0.0497	0.006
Extreme value	$e^{-x} \exp\{-e^{-x}\}$	ln 20	0.0488	0.0487	0.002
Stable ($\alpha = \frac{1}{2}$)	$(2\pi)^{-\frac{1}{2}} \frac{1}{x^{\frac{3}{2}}} \exp\left\{-\frac{1}{2x}\right\} \quad x > 0$	625	0.0319	0.0319	$\sim 2 \times 10^{-8}$
Log-Cauchy	$\frac{1}{\pi \lvert x \rvert (1 + \ln^2 \lvert x \rvert)}$	$K = 1$, approximation fails			
Cauchy	$\frac{1}{\pi} \frac{1}{1 + x^2}$	10	0.0317	0.0317	0.002

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