

UNIFORMLY MORE POWERFUL, ONE-SIDED TESTS FOR HYPOTHESES ABOUT LINEAR INEQUALITIES

BY HUIMEI LIU AND ROGER L. BERGER

*National Chung Hsing University and North Carolina State
University*

Let \mathbf{X} have a multivariate, p -dimensional normal distribution ($p \geq 2$) with unknown mean $\boldsymbol{\mu}$ and known, nonsingular covariance $\boldsymbol{\Sigma}$. Consider testing $H_0: \mathbf{b}'_i \boldsymbol{\mu} \leq 0$, for some $i = 1, \dots, k$, versus $H_1: \mathbf{b}'_i \boldsymbol{\mu} > 0$, for all $i = 1, \dots, k$, where $\mathbf{b}_1, \dots, \mathbf{b}_k$, $k \geq 2$, are known vectors that define the hypotheses. For any $0 < \alpha < 1/2$, we construct a size- α test that is uniformly more powerful than the size- α likelihood ratio test (LRT). The proposed test is an intersection-union test. Other authors have presented uniformly more powerful tests under restrictions on the covariance matrix and on the hypothesis being tested. Our new test is uniformly more powerful than the LRT for all known nonsingular covariance matrices and all hypotheses. So our results show that, in a very general class of problems, the LRT can be uniformly dominated.

1. Introduction. Let $\mathbf{X} = (X_1, \dots, X_p)'$, $p \geq 2$, be a p -variate normal random variable with mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)'$ and known, nonsingular covariance matrix $\boldsymbol{\Sigma}$. We consider the problem of testing

$$(1.1) \quad \begin{aligned} &H_0: \mathbf{b}'_i \boldsymbol{\mu} \leq 0 \quad \text{for some } i = 1, \dots, k \\ &\text{versus} \\ &H_1: \mathbf{b}'_i \boldsymbol{\mu} > 0 \quad \text{for all } i = 1, \dots, k. \end{aligned}$$

Here $\mathbf{b}_1, \dots, \mathbf{b}_k$, $k \geq 2$, are specified p -dimensional vectors that define the hypotheses. Berger (1989) gives several examples of hypotheses that can be expressed in this way. We assume H_1 is nonempty, so the testing problem is meaningful. (We use the symbol H_1 to denote the set of $\boldsymbol{\mu}$ vectors specified by the hypothesis, as well as the statement of the hypothesis.) We also assume that the set $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ has no redundant vectors in it, that is, there is no \mathbf{b}_j such that $\{\boldsymbol{\mu}: \mathbf{b}'_i \boldsymbol{\mu} > 0, i = 1, \dots, k\} = \{\boldsymbol{\mu}: \mathbf{b}'_i \boldsymbol{\mu} > 0, i = 1, \dots, k, i \neq j\}$. Sasabuchi (1980) discusses conditions that are equivalent to our two assumptions.

In this paper, for any testing problem of the form (1.1) and any $0 < \alpha < 1/2$, we propose a new size- α test that is uniformly more powerful than the size- α likelihood ratio test (LRT). First we consider hypotheses that have only two linear restrictions ($k = 2$). Two new tests, ϕ_o and ϕ_a , are proposed for the cases $\mathbf{b}'_1 \boldsymbol{\Sigma} \mathbf{b}_2 \geq 0$ and $\mathbf{b}'_1 \boldsymbol{\Sigma} \mathbf{b}_2 < 0$, respectively. In both cases, the rejection

Received August 1993; revised May 1994.

AMS 1991 subject classifications. Primary 62F03; secondary 62F04, 62F30, 62H15.

Key words and phrases. Uniformly more powerful test, linear inequalities hypotheses, likelihood ratio test, acute cone, obtuse cone, intersection-union test.

region of the new test is like Berger's (1989) in that it contains the rejection region of the LRT and an additional set, but the size of the new test is still α . So the new test is uniformly more powerful than the LRT. Berger (1989) proposed a more powerful test for the $\mathbf{b}'_1 \boldsymbol{\Sigma} \mathbf{b}_2 \leq 0$ case. The test ϕ_a we propose is different than Berger's test and, in some cases, appears to be more powerful. Then, recognizing that, for $k > 2$, H_1 can be written as the intersection of sets each defined by two inequalities, we use the intersection-union method to combine tests of the form ϕ_o and ϕ_a to obtain a test ϕ_g that is uniformly more powerful for the general problem (1.1).

The initial work on testing problems where both null and alternative hypotheses are determined by k linear inequalities was by Sasabuchi (1980). Sasabuchi (1980) treats the problem where the null hypothesis corresponds to the boundary of a convex polyhedral cone determined by linear inequalities and the alternative corresponds to its interior. His problem is to test

$$(1.2) \quad \begin{array}{l} H_{0S}: \mathbf{b}'_i \boldsymbol{\mu} \geq 0 \quad \text{for all } i = 1, \dots, k, \text{ where equality holds for} \\ \qquad \qquad \qquad \text{at least one value of } i \\ \text{versus} \\ H_{1S}: \mathbf{b}'_i \boldsymbol{\mu} > 0 \quad \text{for all } i = 1, 2, \dots, k. \end{array}$$

Sasabuchi (1980) showed that the size- α likelihood ratio test (LRT) of problem (1.2) is the test that rejects H_{0S} if

$$Z_i = \frac{\mathbf{b}'_i \mathbf{X}}{(\mathbf{b}'_i \boldsymbol{\Sigma} \mathbf{b}_i)^{1/2}} \geq z_\alpha \quad \text{for all } i = 1, \dots, k,$$

where z_α is the upper 100α percentile of the standard normal distribution. Berger (1989) shows that, although $H_{0S} \subset H_0$ and H_0 is a much bigger set than H_{0S} , the size- α LRT in problem (1.1) is the same as in Sasabuchi (1980). The LRT has some optimal properties. Lehmann (1952), Cohen, Gatsonis and Marden (1983) and Sierra-Cavazos (1992) prove under various conditions that the LRT is uniformly most powerful among all monotone, size- α tests. Cohen, Gatsonis and Marden (1983) also show that, in a bivariate problem, the LRT is admissible. However, the LRT is a biased test. Berger (1989) points out that the power will be approximately α^p when $\boldsymbol{\mu}$ is close to $\mathbf{0}$ in the sign testing problem. Lehmann (1952) showed that in some problems of this type, no unbiased, nonrandomized test exists. Iwasa (1991) also points out the LRT is d -admissible but not α -admissible in a bivariate problem. The α -admissibility would guarantee the nonexistence of a uniformly more powerful test of size α , but the d -admissibility does not. So it is possible that we can find a nonmonotone test which is uniformly more powerful than the LRT, and several researchers have worked on finding such tests. We believe the increased power of these tests outweighs their lack of monotonicity, even in practical situations. However, if monotonicity is the primary criterion, then the LRT is the most powerful test.

Gutmann (1987) constructs two tests, when X_1, \dots, X_k are independent, that are uniformly more powerful than the uniformly most powerful mono-

tone test in the sign testing problem. Nomakuchi and Sakata (1987) also give a uniformly more powerful test in the bivariate normal case, which is a special case of Sasabuchi's (1980) problem. Berger (1989) gives a class of tests which are more powerful than the LRT if $\mathbf{b}'_1 \boldsymbol{\Sigma} \mathbf{b}_2 \leq 0$. If \mathbf{X} is a normal random vector, then Gutmann's (1987) problem is a special case of Berger's (1989), and Berger's test is more powerful than Gutmann's test. Iwasa (1991) generalized the Nomakuchi–Sakata test to an exponential family. In the same paper, he also generalized Berger's test to an exponential family in the sign testing problem when $k = 2$. Shirley (1992) proposes a test that is more powerful than Gutmann's when $k = 3$.

To simplify computation we consider the transformed version of the original problem that is similar to the one used by Sasabuchi (1980) and Berger (1989). Let Γ be a $p \times p$ nonsingular matrix such that $\Gamma \boldsymbol{\Sigma} \Gamma' = \mathbf{I}_p$, the $p \times p$ identity matrix. So $\Gamma^{-1}(\Gamma^{-1})' = \boldsymbol{\Sigma}$. Make the transformation $\mathbf{Y} = \Gamma \mathbf{X}$. Then $\mathbf{Y} \sim N_p(\boldsymbol{\theta}, \mathbf{I}_p)$, where $\boldsymbol{\theta} = \Gamma \boldsymbol{\mu}$. Let $\|\mathbf{a}\| = (\mathbf{a}'\mathbf{a})^{1/2}$ denote the norm of a vector. Define $\mathbf{h}'_i = \mathbf{b}'_i \Gamma^{-1} / \|\mathbf{b}'_i \Gamma^{-1}\|$. Then $\mathbf{b}'_i \boldsymbol{\mu} = \mathbf{h}'_i \boldsymbol{\theta} \|\mathbf{b}'_i \Gamma^{-1}\|$. Therefore, problem (1.1) is equivalent to observing \mathbf{Y} and testing

$$(1.3) \quad \begin{aligned} H_0: \mathbf{h}'_i \boldsymbol{\theta} &\leq 0 \quad \text{for some } i = 1, \dots, k \\ \text{versus} \\ H_1: \mathbf{h}'_i \boldsymbol{\theta} &> 0 \quad \text{for all } i = 1, \dots, k. \end{aligned}$$

We will use \mathbf{Y} , \mathbf{h}_i and $\boldsymbol{\theta}$ throughout the rest of the paper. Note that $\|\mathbf{h}'_i\| = 1$. This will simplify some expressions. For example, in terms of these variables, the size- α LRT of (1.1) or (1.3) is the test that rejects H_0 if $\mathbf{h}'_i \mathbf{Y} \geq z_\alpha$, for all $i = 1, \dots, k$.

In Section 2 we propose a new test ϕ_o for the case $k = 2$ and $\mathbf{h}'_1 \mathbf{h}_2 \geq 0$ ($\mathbf{b}'_1 \boldsymbol{\Sigma} \mathbf{b}_2 \geq 0$). To our knowledge, this is the first more powerful test described for these problems except that Berger considered the $\mathbf{h}'_1 \mathbf{h}_2 = 0$ case. We compare the power of ϕ_o and the LRT in an example. We also discuss a restriction on size- α tests that shows why some types of construction will not give uniformly more powerful, size- α tests in this case. In Section 3 we consider the $k = 2$ and $\mathbf{h}'_1 \mathbf{h}_2 < 0$ case, which was also considered by Berger (1989). We imitate the strategy used in Section 2 to propose another test ϕ_a , which is more powerful than the LRT. We compare the power functions of ϕ_a , Berger's test and the LRT in an example. In Section 4, we construct a uniformly more powerful, intersection–union test based on ϕ_o and ϕ_a , for the general, $k \geq 2$, problem (1.1). Section 5 contains some general comments on intersection–union tests.

2. Uniformly more powerful test when the cone is obtuse. In this section, we will consider the testing problem (1.3) when $k = 2$ and $\mathbf{h}'_1 \mathbf{h}_2 \geq 0$. Let τ be the angle between the vectors \mathbf{h}_1 and \mathbf{h}_2 . Since $\cos(\tau) = \mathbf{h}'_1 \mathbf{h}_2 \geq 0$, τ is acute. However, the angle in the cone $\mathcal{C} = \{\boldsymbol{\theta}: \mathbf{h}'_1 \boldsymbol{\theta} \geq 0, \mathbf{h}'_2 \boldsymbol{\theta} \geq 0\}$ is $\xi = \pi - \tau$, which is obtuse. So we say H_1 is an obtuse cone when $\mathbf{h}'_1 \mathbf{h}_2 \geq 0$. Figure 1 illustrates this. Berger (1989) describes a test in the opposite case, $\mathbf{h}'_1 \mathbf{h}_2 \leq 0$.

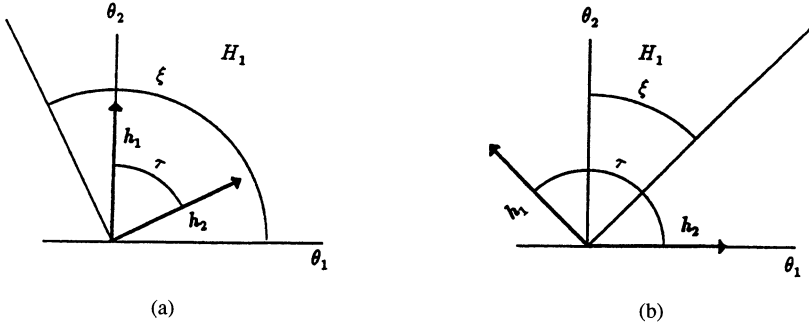


FIG. 1. (a) obtuse cone, $\mathbf{h}'_1 \mathbf{h}_2 \geq 0$; (b) acute cone, $\mathbf{h}'_1 \mathbf{h}_2 < 0$.

His method of test construction does not yield a size- α test when $\mathbf{h}'_1 \mathbf{h}_2 > 0$. We discuss this in Section 2.2.

2.1. *A test that is uniformly more powerful than the LRT.* In this section we will describe a new test that is uniformly more powerful than the LRT when the alternative hypothesis is an obtuse cone. We start by defining the test ϕ_o . Then we show that ϕ_o is a size- α test and is uniformly more powerful than the LRT.

Before describing the test ϕ_o , we will define the functions and set which will be used to construct the rejection region for the test ϕ_o .

DEFINITION 2.1. For any s , $-\infty < s < \infty$, let $c_s = (\sqrt{1+s^2} - s)z_\alpha$ and let L_s be the two-dimensional set defined by

$$L_s = \left\{ (u, v) : \frac{u + sv}{\sqrt{1+s^2}} \geq z_\alpha, v \geq z_\alpha \right\}.$$

The set L_s is an obtuse cone if $s \geq 0$, and L_s is acute cone if $s < 0$. Examples of each are shown in Figure 2 and in Figure 6. The vertex of the cone is (c_s, z_α) . We will eventually express the LRT in terms of L_s . Throughout the rest of the paper, $\varphi(v)$ and $\Phi(v)$ denote the standard normal pdf and cdf, respectively.

DEFINITION 2.2. For any u , $-\infty < u < \infty$, define

$$P_s(u) = \alpha - \int_{L_s(u)} \varphi(v) dv,$$

where $L_s(u) = \{v : (u, v) \in L_s\}$. Specifically, for $s > 0$,

$$P_s(u) = \begin{cases} \alpha - \left\{ 1 - \Phi\left(\frac{\sqrt{1+s^2}z_\alpha - u}{s}\right) \right\}, & u < c_s, \\ 0, & u \geq c_s. \end{cases}$$

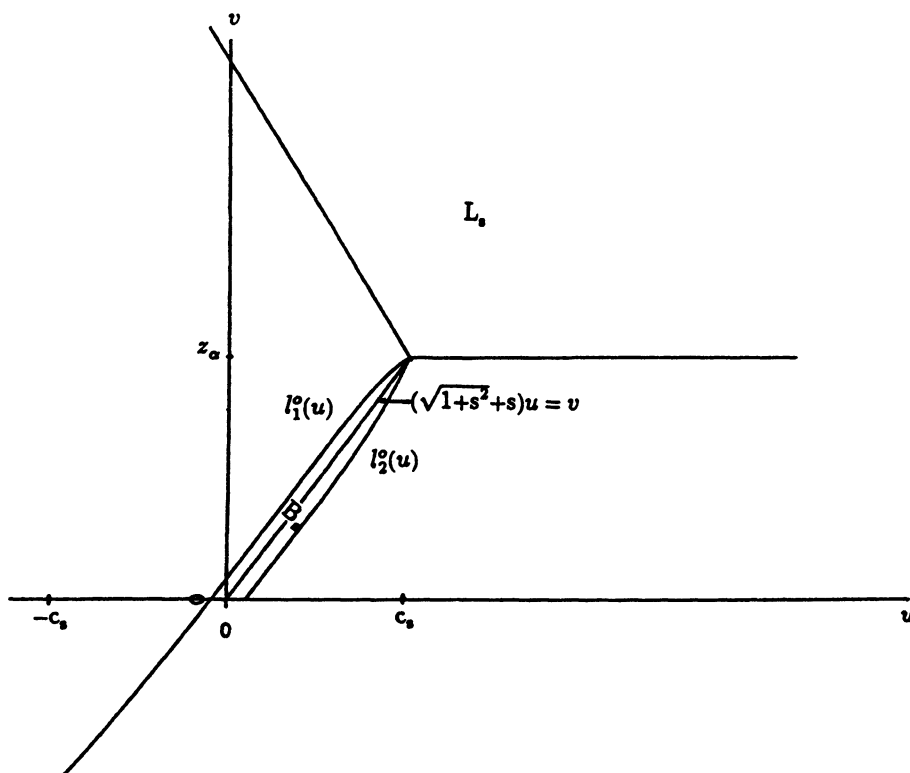


FIG. 2. The set $A_s = L_s \cup B_s$ and functions $l_1^o(u)$ and $l_2^o(u)$.

For $s = 0$,

$$P_s(u) = \begin{cases} \alpha, & u < c_s = z_\alpha, \\ 0, & u \geq c_s = z_\alpha. \end{cases}$$

For $s < 0$,

$$P_s(u) = \begin{cases} \alpha, & u < c_s, \\ 1 - \Phi\left(\frac{\sqrt{1+s^2}z_\alpha - u}{s}\right), & u \geq c_s. \end{cases}$$

The specific formulas for $P_s(u)$ are easily verified by using the definition of L_s ; $0 \leq P_s(u) \leq \alpha$ for all u ; $P_0(u)$ is the limit of $P_s(u)$ as $s \rightarrow 0$, and, if $(U, V) \sim N_2((\mu, 0), I_2)$, $P((U, V) \in L_s) = \int (\alpha - P_s(u))\varphi(u - \mu) du$. The line between the origin and (c_s, z_α) , the vertex of L_s , has the equation $v = z_\alpha u / c_s = (\sqrt{1+s^2} + s)u$. We now define a set that contains this line, for $s \geq 0$.

DEFINITION 2.3. For $s \geq 0$, $0 < d < 1$ and $0 < \alpha < 1/2$, let B_s be the set defined by

$$B_s = \{(u, v): -c_s \leq u \leq c_s, l_2^o(u) \leq v \leq l_1^o(u)\},$$

where

$$l_1^o(u) = \text{Min}\left\{\Phi^{-1}\left(\Phi\left(\sqrt{1+s^2}+s\right)u\right) + dP_s(u)\right\}, su - sc_s + z_\alpha\right\},$$

$$l_2^o(u) = \text{Max}\left\{\Phi^{-1}\left(\Phi(l_1^o(u)) - P_s(u)\right), 0\right\}.$$

An example of the set B_s is shown in Figure 2; B_s is a set that touches L_s at the vertex of L_s and extends down toward the origin. Note, the restriction $\alpha < 1/2$ ensures that the vertex (c_s, z_α) is in the first orthant and that B_s is a nonempty set above the $v = 0$ axis. Ignoring the Max and Min, the constant d is the proportion of the probability $P_s(u)$ that is placed above the line $v = (\sqrt{1+s^2}+s)u$. Increasing d moves the lines $l_1^o(u)$ and $l_2^o(u)$ upward. If $l_1^o(u) \geq 0$, then $l_2^o(u) \leq l_1^o(u)$, but if $l_1^o(u) < 0$, then $l_2^o(u) > l_1^o(u)$. The set B_s does not contain any (u, v) for which $l_2^o(u) > l_1^o(u)$. In Figure 2, $l_2^o(u) = 0 > l_1^o(u)$ for most negative values of u . The following lemma is the key fact that will ensure that the size of ϕ_o is α .

LEMMA 2.1. *Let $(U, V) \sim N_2((\mu, \nu), \mathbf{I}_2)$ where $\nu \leq 0$. Let $s \geq 0$, $0 < \alpha < 1/2$ and $A_s = L_s \cup B_s$. Then $P_{(\mu, \nu)}((U, V) \in A_s) \leq \alpha$.*

PROOF. For every $(u, v) \in A_s$, $v \geq 0$. Since $\nu \leq 0$, by Theorem 2.2 of Berger (1989),

$$\begin{aligned} P_{(\mu, \nu)}((U, V) \in A_s) &\leq P_{(\mu, 0)}((U, V) \in A_s) \\ &= \int_{-\infty}^{+\infty} \left(\int_{L_s(u)} \varphi(v) dv + \int_{B_s(u)} \varphi(v) dv \right) \varphi(u - \mu) du \\ (2.1) \quad &= \int_{-\infty}^{+\infty} \left(\alpha - P_s(u) + \int_{B_s(u)} \varphi(v) dv \right) \varphi(u - \mu) du, \end{aligned}$$

where $L_s(u)$ is defined in Definition 2.2 and

$$B_s(u) = \{v: (u, v) \in B_s\} = \begin{cases} \emptyset, & u < -c_s \text{ or } u > c_s, \\ \{v: l_2^o(u) \leq v \leq l_1^o(u)\}, & -c_s \leq u \leq c_s. \end{cases}$$

The expression in parentheses in (2.1) is clearly bounded above by α if $u < -c_s$ or $u > c_s$. For $-c_s \leq u \leq c_s$, $B_s(u) = \emptyset$ and the integral over $B_s(u)$ is zero, if $l_1^o(u) < l_2^o(u)$. Otherwise,

$$\begin{aligned} \int_{B_s(u)} \varphi(v) dv &= \Phi(l_1^o(u)) - \Phi(l_2^o(u)) \\ &\leq \Phi(l_1^o(u)) - \Phi(\Phi^{-1}(\Phi(l_1^o(u)) - P_s(u))) = P_s(u). \end{aligned}$$

So again the expression in parentheses is bounded above by α , and, hence, $P_{(\mu, \nu)}((U, V) \in A_s) \leq \alpha$. \square

Our new tests will be defined in terms of variables U_1, V_1, U_2 and V_2 , which we now define.

DEFINITION 2.4. Let \mathbf{h}_1 and \mathbf{h}_2 be noncollinear vectors ($|\mathbf{h}'_1\mathbf{h}_2| < \|\mathbf{h}_1\| \cdot \|\mathbf{h}_2\| = 1 \cdot 1 = 1$). Let $\mathbf{g}_1 = \mathbf{h}_2 - (\mathbf{h}'_1\mathbf{h}_2)\mathbf{h}_1$ and $\mathbf{g}_2 = \mathbf{h}_1 - (\mathbf{h}'_1\mathbf{h}_2)\mathbf{h}_2$ (\mathbf{g}_1 and \mathbf{g}_2 are vectors spanned by \mathbf{h}_1 and \mathbf{h}_2 that are orthogonal to \mathbf{h}_1 and \mathbf{h}_2 ; $\mathbf{g}'_1\mathbf{h}_1 = 0$, $\mathbf{g}'_2\mathbf{h}_2 = 0$). Define $\mathbf{h}'_i\mathbf{y} = v_i$ and $\mathbf{g}'_i\mathbf{y}/\|\mathbf{g}_i\| = u_i$, $i = 1, 2$. Also define the corresponding random vectors $\mathbf{h}'_i\mathbf{Y} = V_i$ and $\mathbf{g}'_i\mathbf{Y}/\|\mathbf{g}_i\| = U_i$, $i = 1, 2$.

Note that $\|\mathbf{g}_1\| = \|\mathbf{g}_2\| = \sqrt{1 - (\mathbf{h}'_1\mathbf{h}_2)^2}$. Since $\mathbf{g}'_i\mathbf{h}_i = 0$, we know that U_i and V_i are independent.

Now we define the test ϕ_o . In fact, we define a whole family of tests, indexed by the constant d , $0 < d < 1$, that appears in Definition 2.3.

DEFINITION 2.5. Consider the testing problem (1.3) for vectors \mathbf{h}_1 and \mathbf{h}_2 that satisfy $\mathbf{h}'_1\mathbf{h}_2 \geq 0$. Fix d , $0 < d < 1$. Let $s = \mathbf{h}'_1\mathbf{h}_2(1 - (\mathbf{h}'_1\mathbf{h}_2)^2)^{-1/2}$. For any α that satisfies $0 < \alpha < 1/2$, define ϕ_o as the test that rejects H_0 if $\mathbf{Y} \in S_1^* \cap S_2^*$, where $S_1^* = \{\mathbf{y}: (u_1, v_1) \in A_s\}$, $S_2^* = \{\mathbf{y}: (u_2, v_2) \in A_s\}$ and $A_s = L_s \cup B_s$.

The following lemma will show that the rejection region for the LRT is a subset of that for ϕ_o .

LEMMA 2.2. Consider the testing problem (1.3) when $k = 2$ and $0 < \alpha < 1/2$. The rejection region for the size- α LRT is $R_L = \{\mathbf{y}: \mathbf{h}'_1\mathbf{y} \geq z_\alpha \text{ and } \mathbf{h}'_2\mathbf{y} \geq z_\alpha\}$. Let $L_s^i = \{\mathbf{y}: (u_i, v_i) \in L_s\} \subset S_i^*$, $i = 1, 2$, where $s = \mathbf{h}'_1\mathbf{h}_2(1 - (\mathbf{h}'_1\mathbf{h}_2)^2)^{-1/2}$. Then $L_s^i = R_L$ for $i = 1, 2$. Hence, the rejection region for ϕ_o , namely, $S_1^* \cap S_2^*$, contains R_L .

PROOF. For $i = 1$,

$$u_1 = \frac{\mathbf{g}'_1\mathbf{y}}{\|\mathbf{g}_1\|} = \frac{\mathbf{h}'_2\mathbf{y} - (\mathbf{h}'_1\mathbf{h}_2)\mathbf{h}'_1\mathbf{y}}{\sqrt{1 - (\mathbf{h}'_1\mathbf{h}_2)^2}} \quad \text{and} \quad v_1 = \mathbf{h}'_1\mathbf{y}.$$

Since $s/\sqrt{1 + s^2} = \mathbf{h}'_1\mathbf{h}_2$, then $\|\mathbf{g}_1\| = \sqrt{1 - (\mathbf{h}'_1\mathbf{h}_2)^2} = 1/\sqrt{1 + s^2}$. Hence,

$$\frac{u_1 + sv_1}{\sqrt{1 + s^2}} = \mathbf{a}'_1\mathbf{y},$$

where

$$\mathbf{a}'_1 = \left\{ \frac{\mathbf{h}'_2 - (\mathbf{h}'_1\mathbf{h}_2)\mathbf{h}'_1}{\|\mathbf{g}_1\|} \|\mathbf{g}_1\| + (\mathbf{h}'_1\mathbf{h}_2)\mathbf{h}'_1 \right\} = \mathbf{h}'_2.$$

Therefore, $L_s^1 = R_L$. For $i = 2$, similar algebra yields $(u_2 + sv_2)/\sqrt{1 + s^2} = \mathbf{h}'_1\mathbf{y}$ and $v_2 = \mathbf{h}'_2\mathbf{y}$. So $L_s^2 = R_L$, also. \square

Note that Lemma 2.2 is true for any \mathbf{h}_1 and \mathbf{h}_2 , not just obtuse cones with $\mathbf{h}'_1\mathbf{h}_2 \geq 0$. Another way to state Lemma 2.2 is to say that the three events, $\{\mathbf{Y} \in R_L\}$, $\{(U_1, V_1) \in L_s\}$ and $\{(U_2, V_2) \in L_s\}$, are all the same event. The following theorem shows ϕ_o is a size- α test and uniformly more powerful than the LRT.

THEOREM 2.1. *For the testing problem (1.3) when $k = 2$, suppose that $\mathbf{h}'_1 \mathbf{h}_2 \geq 0$. If $0 < \alpha < 1/2$, then ϕ_o has size exactly α , and ϕ_o is uniformly more powerful than the size- α LRT.*

PROOF. From Lemma 2.2 we know the rejection region of the size- α LRT, R_L , is a subset of the rejection region of ϕ_o . Hence, ϕ_o is uniformly more powerful than the size- α LRT. Also,

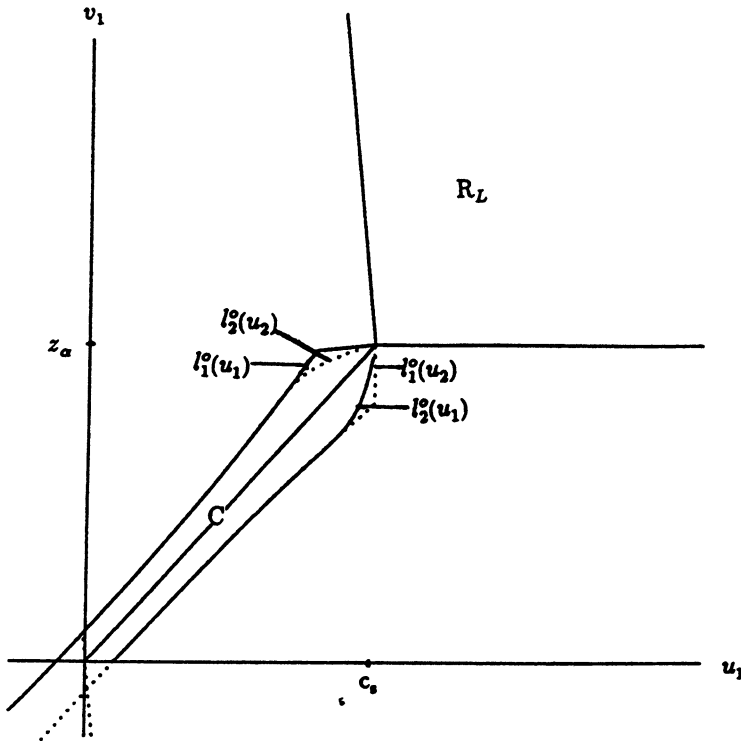
$$(2.2) \quad \text{the size of } \phi_o \geq \text{size of LRT} = \alpha.$$

For any $\theta \in H_0$, $\mathbf{h}'_i \theta \leq 0$ for either $i = 1$ or $i = 2$. For this i ,

$$(2.3) \quad P_\theta(\mathbf{Y} \in S_1^* \cap S_2^*) \leq P_\theta(\mathbf{Y} \in S_i^*) = P_\theta((U_i, V_i) \in A_s) \leq \alpha.$$

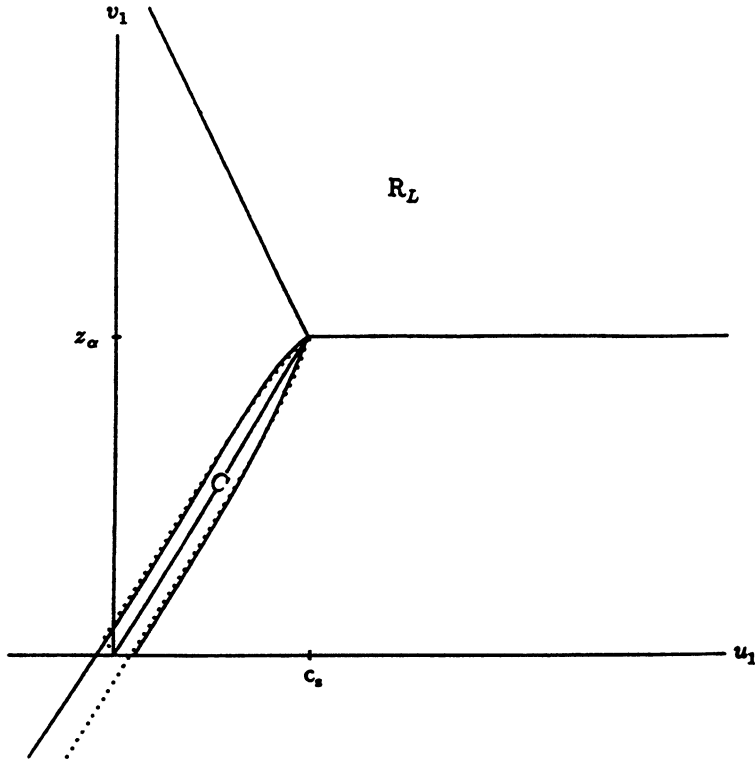
The last inequality is from Lemma 2.1 since U_i and V_i are independent normal random variables, each with variance 1, and $EV_i = \mathbf{h}'_i \theta \leq 0$. Since, (2.3) is true for any $\theta \in H_0$, the size of ϕ_o is less than or equal to α . With (2.2) this implies ϕ_o has size exactly α . \square

Figure 3 shows two examples (different s and d , $\alpha = 0.1$) of the rejection region of ϕ_o . Consider $p = 2$, $\mathbf{h}'_1 = (0, 1)$ and $\mathbf{h}'_2 = (1/\sqrt{1+s^2}, s/\sqrt{1+s^2})$



(a)

FIG. 3(a). Rejection region of ϕ_o when $s = 0.1$, $d = 1/2$ and $\alpha = 0.1$.



(b)

FIG. 3(b). Rejection region of ϕ_o when $s = 0.5$, $d = 4/9$ and $\alpha = 0.1$.

so that $(y_1, y_2) = (u_1, v_1)$. In Figure 3a and 3b, the solid line above the line from $(0, 0)$ to the vertex of R_L is $l_1^o(u_1)$ and that below the line is $l_2^o(u_1)$. The lower dotted line is $l_1^o(u_2)$ and the upper dotted line is $l_2^o(u_2)$. These are the same functions, l_1^o and l_2^o , but these are graphed in the (u_2, v_2) axes. The intersection of the region between the solid lines and the region between the dotted lines is the additional set which is added to the rejection region of the LRT. Specifically, this is $C = \{y: (u_1, v_1) \in B_s\} \cap \{y: (u_2, v_2) \in B_s\}$. The region $R_L \cup C$ is the rejection region of ϕ_o . When s increases as in Figure 3b, the added area decreases. The constant d that will produce the biggest intersection, and hence the highest power, depends on s . We discuss this more in Section 3.

EXAMPLE 2.1. Suppose Y_1 and Y_2 are independent and $Y_i \sim N_1(\theta_i, 1)$. Consider $\mathbf{h}'_1 = (0, 1)$, $\mathbf{h}'_2 = (1/\sqrt{1+s^2}, s/\sqrt{1+s^2})$, $s > 0$, so that we are testing $H_0: \theta_2 \leq 0$ or $\theta_1 + s\theta_2 \leq 0$ against $H_1: \theta_2 > 0$ and $\theta_1 + s\theta_2 > 0$. Here we selected $\alpha = 0.1$, $s = 0.1$ and $d = 1/2$ (as in Figure 3a) to compute the power of ϕ_o and LRT. Let $\beta_L(\boldsymbol{\theta})$ and $\beta_{\phi_o}(\boldsymbol{\theta})$ be the power functions of the LRT

TABLE 1
Power of LRT and ϕ_o for $s = 0.1$, $d = 1/2$ and $\alpha = 0.1$

	θ								
	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$\beta_L(\theta, 0)$	0.013	0.027	0.046	0.065	0.081	0.092	0.097	0.099	0.100
$\beta_{\phi_o}(\theta, 0)$	0.048	0.066	0.081	0.091	0.096	0.099	0.100	0.100	0.100
$\beta_L(0.905\theta, \theta)$	0.013	0.059	0.181	0.390	0.633	0.827	0.936	0.981	0.996
$\beta_{\phi_o}(0.905\theta, \theta)$	0.048	0.106	0.224	0.416	0.643	0.829	0.936	0.982	0.996
$\beta_L(0.4505\theta, \theta)$	0.013	0.043	0.108	0.219	0.367	0.526	0.668	0.781	0.863
$\beta_{\phi_o}(0.4505\theta, \theta)$	0.048	0.087	0.153	0.255	0.389	0.536	0.672	0.782	0.863

and ϕ_o , respectively. Values of these two functions for certain θ values are in Table 1. These values are calculated by two steps. First, we calculate the cross-sectional probability $\int_{A(u)} \varphi(v - \theta_2) dv = f(u, \theta_2)$, which is a function of u and θ_2 . Second, we calculate $\int_{-\infty}^{\infty} f(u, \theta_2) \varphi(u - \theta_1) du$ using the trapezoidal rule with 300 points. The first part of the tables are for values of $\theta' = (\theta, 0)$, $\theta \geq 0$. These values are on the boundary of H_0 , so the powers are less than $\alpha = 0.1$. If a test is unbiased, then the power is equal to α for the values of θ which are on the boundary of H_0 . Here we can see that the LRT and ϕ_o are biased, but the difference between α and the power of ϕ_o is considerably smaller than that between α and the power of the LRT. The second part of the table is for values of $\theta' = ((\sqrt{1 + s^2} - s)\theta, \theta)$ which are on the line from the origin to the vertex (c_s, z_α) . For example, $\beta_{\phi_o}(0.4525, 0.5) / \beta_L(0.4525, 0.5) \approx 1.80$ and $\beta_{\phi_o}(0.4525, 0.5) > \alpha > \beta_L(0.4525, 0.5)$. $\beta_{\phi_o}(\theta)$ is clearly bigger than $\beta_L(\theta)$ for $\theta \leq 1.5$. The largest difference is 0.047. The bottom of the table is for values of $\theta' = (0.5(\sqrt{1 + s^2} - s)\theta, \theta)$; $\beta_{\phi_o}(\theta)$ is clearly larger than $\beta_L(\theta)$ for $\theta \leq 2.5$. As s increases, there is less space to add to the rejection region of the LRT. Figure 3b shows this fact. So we cannot improve the power as much when s is large.

2.2 Restriction on the construction of a size- α test. The set that is added to the rejection region of the LRT to construct ϕ_o touches the LRT rejection region R_L at only a single point (see Figure 3). One might ask, if we can add a set along the boundary of R_L to obtain a more powerful test. Gutmann (1987) constructed such a set for a nonnormal problem. The following theorem will show that a test with a rejection region like this will have size greater than α in a normal problem. It will also show that a construction like Berger (1989) used for acute cone problems will not work for obtuse cone problems. Liu (1992) also showed this.

THEOREM 2.2. *Let $(U, V) \sim N_2((\theta, 0), \mathbf{I}_2)$. For a constant c , let*

$$R = \{(u, v) : u \geq c, v \geq z_\alpha\} \quad \text{and} \quad Q_1 = \{(u, v) : u \geq c, v < z_\alpha\}.$$

If $Q \subset Q_1$ and $P_\theta((U, V) \in Q) > 0$, then $P_\theta((U, V) \in R \cup Q) > \alpha$ for all large θ .

PROOF. Suppose $U \sim N(\theta, 1)$ and $\psi(u)$ is a nonnegative function such that $P_\theta(U > c \text{ and } \psi(U) > 0) > 0$. Then, as in an argument in Stein (1956),

$$(2.4) \quad \frac{E_\theta \psi(U)}{P_\theta(U < c)} = \frac{\int_{-\infty}^{\infty} \psi(u) \exp(\theta(u - c)) \varphi(u) du}{\int_{-\infty}^c \exp(\theta(u - c)) \varphi(u) du} \rightarrow \infty \text{ as } \theta \rightarrow \infty,$$

because the numerator goes to ∞ and the denominator goes to 0. Now define $\psi(u) = P(Q|U = u)$ and $R_1 = \{(u, v): u < c, v \geq z_\alpha\}$. Then $R_1 \subset \{(u, v): u < c\}$ and

$$\begin{aligned} \alpha &= P_{(\theta, 0)}(R_1 \cup R) < P_\theta(U < c) + P_{(\theta, 0)}(R) \\ &< E_\theta \psi(U) + P_{(\theta, 0)}(R) = P_{(\theta, 0)}(Q \cup R), \end{aligned}$$

for large enough θ , where the second inequality is from (2.4). \square

Now consider a test ϕ^* whose rejection region R^* contains R_L . In Section 2.1, we saw that $R_L = \{(u_1, v_1): (u_1, v_1) \in L_s\}$ and this set contains $\{(u_1, v_1): u_1 \geq c_s, v_1 \geq z_\alpha\}$. So, by Theorem 2.2, if ϕ^* has size α , then R^* cannot contain any part of $Q_1 = \{(u_1, v_1): u_1 \geq c_s, v_1 < z_\alpha\}$ (except for a set with probability zero). Similarly, we can also write $R_L \supset \{(u_2, v_2): u_2 \geq c_s, v_2 \geq z_\alpha\}$ and R^* cannot contain any part of $Q_2 = \{(u_2, v_2): u_2 \geq c_s, v_2 < z_\alpha\}$. These sets are shown in Figure 4. The set added to R_L to form a more powerful, size- α test cannot be in the shaded region $Q_1 \cup Q_2$. It must lie in the triangular region labeled Q_3 , which is where the set that defines ϕ_o lies.

Berger (1989) constructed tests that were uniformly more powerful than the LRT for problems in which $\mathbf{h}'_1 \mathbf{h}_2 \leq 0$. Theorem 2.2 can be used to show that Berger's method of construction will not give a size- α test if $\mathbf{h}'_1 \mathbf{h}_2 > 0$. Figure 4 illustrates this. The region R_L is the rejection region of the LRT. The diamond shapes $R_L \cup R_2 \cup \dots \cup R_5$ would be the rejection region of Berger's test. This rejection region contains some area in the shaded region which causes the size of the test to be greater than α . Berger's method cannot be applied to obtuse cone alternative hypotheses when $p = 2$.

3. Uniformly more powerful test when the cone is acute. In this section, we describe a size- α test that is uniformly more powerful than the size- α LRT for problems in which $\mathbf{h}'_1 \mathbf{h}_2 < 0$. In these problems, the cone defined by the alternative hypothesis is acute. Berger (1989) described a size- α test ϕ_b that is more powerful than the LRT for these problems. Here we describe a new test ϕ_a that has smoother boundaries and sometimes appears more powerful than ϕ_b . The method we use to construct ϕ_a is very similar to the method we used to construct ϕ_o . So we will omit the formal proofs that ϕ_a has the described properties. One difference in this acute case

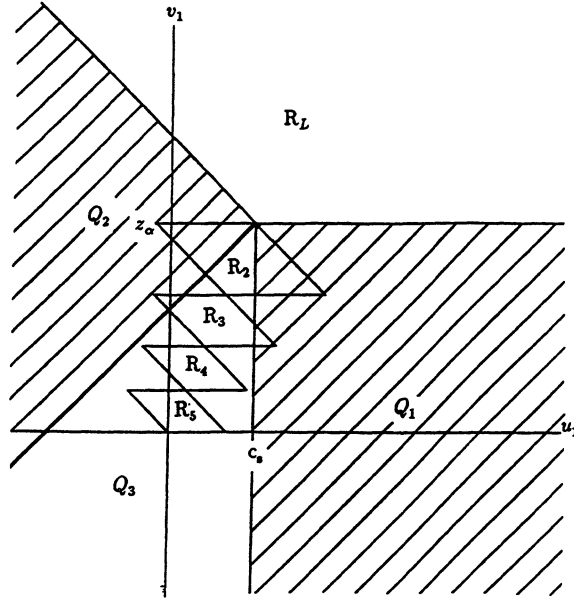


FIG. 4. Constraints on a size- α rejection region and rejection region of ϕ_b for an obtuse cone hypothesis.

is that, unlike in Section 2.2, the rejection region for ϕ_a completely surrounds and enlarges upon the rejection region of the LRT.

Our description of ϕ_a will be similar to our description of ϕ_o in Section 2.1. The set L_s , constant c_s , function $P_s(u)$ and variables (u_1, v_1) and (u_2, v_2) are as defined in Section 2.1. Lemma 2.2 remains valid, and the LRT's rejection region can be expressed in any of the following ways:

$$R_L = \{\mathbf{y}: \mathbf{h}'_1 \mathbf{y} \geq z_\alpha \text{ and } \mathbf{h}'_2 \mathbf{y} \geq z_\alpha\} = \{\mathbf{y}: (u_1, v_1) \in L_s\} = \{\mathbf{y}: (u_2, v_2) \in L_s\},$$

where $s = \mathbf{h}'_1 \mathbf{h}_2 (\mathbf{h}'_1 \mathbf{h}_2)^{-1/2}$. The test ϕ_a is defined in terms of a set A_s that we now define.

DEFINITION 3.1. For $s < 0$, $0 < \alpha < 1/2$ and $0 < d < 1$, let A_s be the set defined by

$$A_s = \{(u, v): u \geq 0, l_2^a(u) \leq v \leq l_1^a(u)\},$$

where

$$l_1^a(u) = \begin{cases} \Phi^{-1}\{\Phi((\sqrt{1+s^2} + s)u) + dP_s(u)\}, & 0 \leq u < c_s, \\ \Phi^{-1}\{1 - (1-d)P_s(u)\}, & u \geq c_s, \end{cases}$$

and

$$l_2^a(u) = \text{Max}\{\Phi^{-1}(\Phi(l_1^a(u)) - \alpha), 0\}, \quad u \geq 0.$$

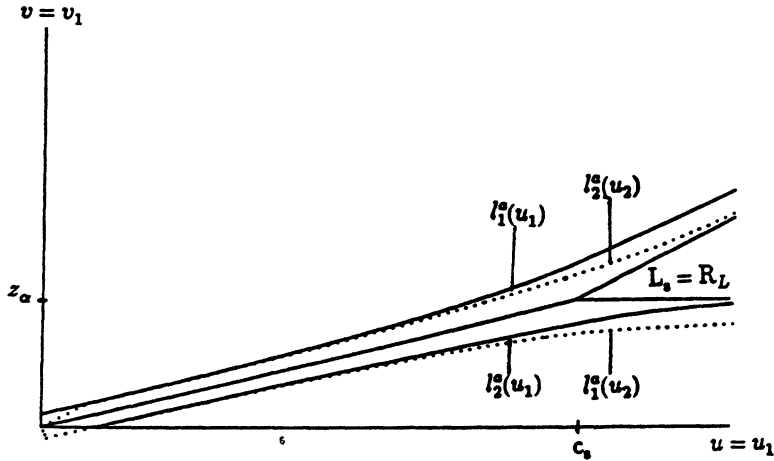


FIG. 5. Rejection region of ϕ_α .

Examples of the sets L_s and A_s and the lines $l_1^\alpha(u)$ and $l_2^\alpha(u)$ are shown in Figure 5. In this figure, $s = -2$, $d = 1/2$ and $\alpha = 0.1$. The solid lines are $l_1^\alpha(u)$ and $l_2^\alpha(u)$. For $u \geq c_s$ the line $l_1^\alpha(u)$ lies above the upper boundary of L_s , which is given by the line $v = (\sqrt{1 + s^2} z_\alpha - u)/s$. This is true since $1 - (1 - d)P_s(u) = \Phi((\sqrt{1 + s^2} z_\alpha - u)/s) + dP_s(u) > \Phi((\sqrt{1 + s^2} z_\alpha - u)/s)$. For $u \geq c_s$, $l_2^\alpha(u)$ is below the lower boundary of L_s because the lower boundary is $z_\alpha = \Phi^{-1}(1 - \alpha) > l_2^\alpha(u)$. Therefore, $L_s \subset A_s$ and, for $i = 1$ or 2 , we have

$$(3.1) \quad R_L = \{\mathbf{y}: (u_i, v_i) \in L_s\} \subset \{\mathbf{y}: (u_i, v_i) \in A_s\}.$$

If $(U, V) \sim N_2((\mu, \nu), \mathbf{I}_2)$, with $\nu \leq 0$, then $P_{(\mu, \nu)}((U, V) \in A_s) < \alpha$. This follows as in the proof of Lemma 2.1. In this case we have

$$(3.2) \quad \begin{aligned} P_{(\mu, \nu)}((U, V) \in A_s) &\leq P_{(\mu, 0)}((U, V) \in A_s) \\ &= \int_0^\infty \left\{ \int_{l_2^\alpha(u)}^{l_1^\alpha(u)} \varphi(v) dv \right\} \varphi(u - \mu) du \\ &= \int_0^\infty \{ \Phi(l_1^\alpha(u)) - \Phi(l_2^\alpha(u)) \} \varphi(u - \mu) du, \\ &\leq \int_0^\infty \{ \Phi(l_1^\alpha(u)) - \Phi(\Phi^{-1}(\Phi(l_1^\alpha(u)) - \alpha)) \} \varphi(u - \mu) du \\ &= \int_0^\infty \alpha \varphi(u - \mu) du < \alpha. \end{aligned}$$

Thus, we can define a size- α test, just as we did in Section 2.1.

DEFINITION 3.2. Consider the testing problem (1.3) for vectors \mathbf{h}_1 and \mathbf{h}_2 that satisfy $\mathbf{h}'_1 \mathbf{h}_2 < 0$. Fix d , $0 < d < 1$. Let $s = \mathbf{h}'_1 \mathbf{h}_2 (1 - (\mathbf{h}'_1 \mathbf{h}_2)^2)^{-1/2}$. For

any α that satisfies $0 < \alpha < 1/2$, define ϕ_α as the test that rejects H_0 if $\mathbf{Y} \in S_1^* \cap S_2^*$, where $S_i^* = \{y: (u_i, v_i) \in A_s\}$ (A_s is defined in Definition 3.1).

Because (3.1) and (3.2) are true, as in Theorem 2.1, we can show that ϕ_α is a size- α test that is uniformly more powerful than the size- α LRT.

Consider the testing problem with $\mathbf{h}'_1 = (0, 1)$ and $\mathbf{h}'_2 = (1/\sqrt{5}, -2/\sqrt{5})$, so that $(y_1, y_2) = (u_1, v_1)$ and $s = -2$. Let $d = 1/2$ and $\alpha = 0.1$. Then in Figure 5, the solid lines are $l_1^\alpha(u_1)$ and $l_2^\alpha(u_1)$ and the region between them is S_1^* . The dotted lines are $l_1^\alpha(u_2)$ (lower line) and $l_2^\alpha(u_2)$ (upper line), and the region between them is S_2^* . The rejection region is $S_1^* \cap S_2^*$, and it contains $L_s = R_L$, the LRT's rejection region. In Figure 6, the union of the diamond shaped regions, $R_L \cup R_2 \cup \dots \cup R_5$, is the rejection region for Berger's (1989) test ϕ_b for this problem. Note that the rejection region for ϕ_b is almost completely contained in the rejection region for ϕ_α . In fact, ϕ_α may be uniformly more powerful than ϕ_b . In general, as s decreases, the containment of ϕ_b in ϕ_α comes closer and closer to reality.

For this same problem, the power functions of the LRT, ϕ_b and ϕ_α are compared in Table 2. Denote these power function by $\beta_L(\boldsymbol{\theta})$, $\beta_b(\boldsymbol{\theta})$ and $\beta_{\phi_\alpha}(\boldsymbol{\theta})$, respectively. These values are calculated in the same way as in Example 2.2. The first part of the table is for values of $\boldsymbol{\theta}' = (\theta, 0)$, $\theta \geq 0$. These values are on the boundary of H_0 , so the powers are less than $\alpha = 0.1$. Again here we can see that the LRT, ϕ_b and ϕ_α are biased, but the bias of ϕ_α is considerably smaller than the bias of the LRT. The power of the LRT is much smaller than those of ϕ_b and ϕ_α when $\boldsymbol{\theta}$ is close to $\mathbf{0}$; $\beta_{\phi_\alpha}(0, 0) > \beta_b(0, 0)$. Both tests improve greatly on the LRT; $\beta_b(2, 0) = 0.052$ and $\beta_{\phi_\alpha}(2, 0) = 0.091$, but $\beta_L(2, 0) \approx 0.0$. The largest difference between $\beta_{\phi_\alpha}(\theta, 0)$ and $\beta_L(\theta, 0)$ is 0.095. The largest difference between $\beta_{\phi_\alpha}(\theta, 0)$ and $\beta_b(\theta, 0)$ is 0.042. The second part of the table is for values of $\boldsymbol{\theta}' = ((\sqrt{1 + s^2} - s)\theta, \theta)$, values on the line from $(0, 0)$ to (c_s, z_α) ; $\beta_{\phi_\alpha}(4.236, 1)/\beta_L(4.236, 1) \approx 15.3$. The largest difference is 0.143; $\beta_{\phi_\alpha}(\boldsymbol{\theta})$ is significantly bigger than $\beta_L(\boldsymbol{\theta})$ for $\theta \leq 3$; $\beta_{\phi_\alpha}(4.236, 1)/\beta_b(4.236, 1) \approx 1.75$. The largest difference is 0.066; $\beta_{\phi_\alpha}(\boldsymbol{\theta})$ is

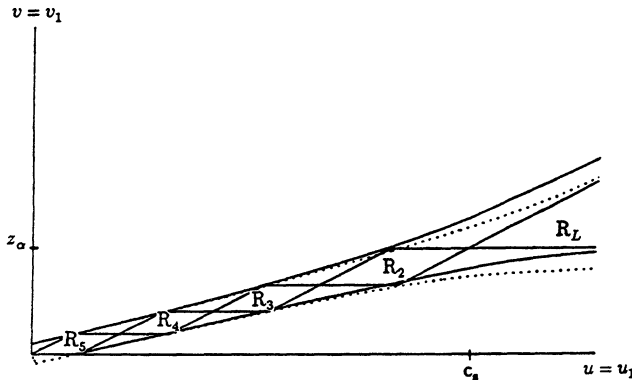


FIG. 6. Rejection regions of ϕ_α and ϕ_b .

TABLE 2
Power of LRT, ϕ_b and ϕ_a for $s = -2.0$, $d = 1/2$ and $\alpha = 0.1$

	θ								
	0	1	2	3	4	5	6	7	8
$\beta_L(\theta, 0)$	0.000	0.000	0.000	0.000	0.002	0.014	0.040	0.071	0.089
$\beta_b(\theta, 0)$	0.026	0.045	0.052	0.053	0.053	0.050	0.054	0.073	0.090
$\beta_{\phi_a}(\theta, 0)$	0.029	0.069	0.091	0.095	0.092	0.090	0.090	0.092	0.096
$\beta_L(4.236\theta, \theta)$	0.000	0.010	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_b(4.236\theta, \theta)$	0.026	0.087	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_{\phi_a}(4.236\theta, \theta)$	0.029	0.153	0.536	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_L(2.118\theta, \theta)$	0.000	0.000	0.135	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_b(2.118\theta, \theta)$	0.026	0.067	0.151	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_{\phi_a}(2.118\theta, \theta)$	0.029	0.119	0.222	0.484	0.691	0.830	0.920	0.968	0.989

clearly bigger than $\beta_b(\theta)$ for $\theta \leq 2$. The bottom of the table is for $\theta' = (0.5(\sqrt{1 + s^2} - s)\theta, \theta)$. Again ϕ_a improves on ϕ_b for these parameter values; $\beta_{\phi_a}(2.118, 1)/\beta_L(2.118, 1) > 100$. The largest difference is 0.119; $\beta_{\phi_a}(\theta)$ is significantly bigger than $\beta_L(\theta)$ and $\beta_b(\theta)$ for $\theta \leq 2$; $\beta_{\phi_a}(2.118, 1)/\beta_b(2.118, 1) \approx 1.8$. The largest different is 0.071. The power of ϕ_a may be greater than that of ϕ_b for all θ , at $s = -2$.

A size- α test that is a good deal more powerful than ϕ_a (or ϕ_o in Section 2) could be constructed by allowing the set A_s (or B_s) to extend into the third orthant, in a construction similar to the one in Berger [(1989), Section 4]. Such tests reject H_0 for sample points at which the estimate of θ does not satisfy the inequalities in H_1 . Because of this, such tests do not have much practical value. We have not considered them in this article, but from a power standpoint they do exhibit a further improvement that is possible.

For given values of s and θ , we can numerically find the constant d that gives the test with highest power at θ . This we have done and the results are given in Table 3, for several values of s and two values of θ . For $\theta = (c_s/2, z_\alpha/2)$, the optimal d is slightly less than $1/2$, but, for $\theta = (c_s, z_\alpha)$ and very acute cones (small s), the optimal d is somewhat smaller than $1/2$. The reason for this can be seen in Figure 5, where $d = 1/2$. Near $u = c_s$, $l_1^\alpha(u)$ and $l_2^\alpha(u)$ do not coincide. When d is reduced, these boundaries are closer

TABLE 3
Optimal d for ϕ_o and ϕ_a and $\alpha = 0.1$

θ	s^1						
	-2.41	-1.00	-0.41	0.00	0.41	1.00	2.41
$(c_s/2, z_\alpha/2)$	0.477	0.480	0.488	0.500	0.491	0.479	0.493
(c_s, z_α)	0.430	0.455	0.480	0.500	0.484	0.477	0.492

¹These s -values correspond to $\xi = i\pi/8$, $i = 1, \dots, 7$, in Figure 1.

together, and the rejection region $S_1^* \cap S_2^*$ is larger in the neighborhood of (c_g, z_α) . Hence, the power is increased.

4. A more powerful test in the general problem. We will now describe a size- α test that is uniformly more powerful than the size- α LRT for the general problem (1.3) when $k \geq 2$ and $0 < \alpha < 1/2$. We will denote this test by ϕ_g . The intersection-union method will be used to construct ϕ_g . A summary of this method may be found in Casella and Berger [(1990), Sections 8.2.4 and 8.3.5] or in Berger (1982).

To use the intersection-union method, $H_0: \mathbf{h}'\boldsymbol{\theta} \leq 0$, for some $i = 1, \dots, k$, must be written as a union. Let D denote any division of the indices $\{1, \dots, k\}$ into the minimal number of subsets of size two such that each value $1, \dots, k$ appears at least once. Elements of D are just pairs of indices (i, j) . If k is even, D has $k^* = k/2$ elements and each index appears once. If k is odd, D has $k^* = (k + 1)/2$ elements. All indices appears once except one appears twice. To construct a more powerful test, any such division of $\{1, \dots, k\}$ will work, but different divisions will lead to different tests.

For each $(i, j) \in D$, consider testing $H_{0ij}: \mathbf{h}'_i\boldsymbol{\theta} \leq 0$ or $\mathbf{h}'_j\boldsymbol{\theta} \leq 0$ versus $H_{1ij}: \mathbf{h}'_i\boldsymbol{\theta} > 0$ and $\mathbf{h}'_j\boldsymbol{\theta} > 0$. If $\mathbf{h}'_i\mathbf{h}_j \geq 0$, let C_{ij} denote the size- α rejection region of ϕ_o (for some d) from Section 2. If $\mathbf{h}'_i\mathbf{h}_j < 0$, let C_{ij} denote the size- α rejection region of ϕ_a (for some d) from Section 3. Since $H_0 = \bigcup_{(i,j) \in D} H_{0ij}$, we can define an intersection-union test based on the C_{ij} .

DEFINITION 4.1. For the testing problem (1.3) with $k \geq 2$ and $0 < \alpha < 1/2$, let ϕ_g be the test that rejects H_0 if $\mathbf{Y} \in \bigcap_{(i,j) \in D} C_{ij}$.

THEOREM 4.1. For $0 < \alpha < 1/2$, the test ϕ_g is a size- α test of H_0 versus H_1 , and ϕ_g is uniformly more powerful than the size- α LRT.

PROOF. Since each of C_{ij} is a size- α rejection region for testing H_{0ij} , by Berger [(1982), Theorem 1], ϕ_g has size less than or equal to α . However, the size- α LRT's rejection region is

$$R_L = \{\mathbf{y}: \mathbf{h}'_i\mathbf{y} \geq z_\alpha, i = 1, \dots, k\} \subset \{\mathbf{y}: \mathbf{h}'_i\mathbf{y} \geq z_\alpha \text{ and } \mathbf{h}'_j\mathbf{y} \geq z_\alpha\} \subset C_{ij},$$

for every $(i, j) \in D$. Hence R_L is contained in the rejection region of ϕ_g , the size of ϕ_g is greater than or equal to the size of the LRT = α , and ϕ_g is uniformly more powerful than the LRT. \square

The test ϕ_g is, in fact, strictly more powerful than the LRT because ϕ_g 's rejection region contains an open set that is not in R_L . Let \mathbf{y} denote a point satisfying $\mathbf{h}'_i\mathbf{y} = z_\alpha, i = 1, \dots, k$. (If $k \geq p$, there is only one such \mathbf{y} . If $k < p$, there are many such \mathbf{y} 's.) Every C_{ij} contains an open set that contains the line from \mathbf{y} to the origin. So the intersection of the C_{ij} 's, ϕ_g 's rejection region, contains an open set containing this line, and this open set is not in R_L .

As mentioned earlier, different choices of D will lead to different tests. More work needs to be done to determine which D 's yield generally more

powerful tests, but one principle seems reasonable. In Tables 1 and 2, we see that the improvement in power over the LRT is much greater for small s (acute cones) than for large s (obtuse cones). So it seems that we will get more improvement from ϕ_g if D is chosen so that the values $s_{ij} = \mathbf{h}'_i \mathbf{h}_j (1 - (\mathbf{h}'_i \mathbf{h}_j)^2)^{-1/2}$, $(i, j) \in D$, are small rather than large.

EXAMPLE 4.1. Consider the hypothesis testing problem defined by the three vectors $\mathbf{h}'_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\mathbf{h}'_2 = (1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$ and $\mathbf{h}'_3 = (1/\sqrt{3}, -\sqrt{2}/\sqrt{3}, 0)$. Then $s_{12} \approx 5.83$, $s_{13} \approx -0.17$ and $s_{23} \approx -0.35$. So we conjecture that $D = \{(1, 3), (2, 3)\}$ will give a generally more powerful ϕ_g than will $D = \{(1, 2), (1, 3)\}$. However, we would not expect the first test to be uniformly more powerful than the second.

5. Further comments on intersection-union tests. In Section 4, ϕ_g was explicitly constructed as an intersection-union test (IUT). In fact, most of the tests considered in this paper are naturally thought of as IUT's.

For $i = 1, \dots, k$, $R_{L_i} = \{\mathbf{y}: \mathbf{h}'_i \mathbf{y} \geq z_\alpha\}$ is the size- α LRT of $H_{0i}: \mathbf{h}'_i \boldsymbol{\theta} \leq 0$ versus $H_{1i}: \mathbf{h}'_i \boldsymbol{\theta} > 0$. Since $H_0 = \bigcup_{i=1}^k H_{0i}$, the test with rejection region $R_L = \bigcap_{i=1}^k R_{L_i}$ is a level- α IUT of H_0 versus H_1 . This test is just the size- α LRT. Berger [(1982), Theorem 1] shows that this test is level- α . A more specific analysis, such as in Berger (1989), is required to show the test is size- α .

The tests ϕ_o and ϕ_a are also constructed as IUT's for their $k = 2$ problems. For example, consider an obtuse cone problem. By Lemma 2.1 and Definition 2.5, for $i = 1$ or 2 the test with rejection region $S_i^* = \{\mathbf{y}: (u_i, v_i) \in A_s\}$ is a size- α test of $H_{0i}: \mathbf{h}'_i \boldsymbol{\theta} \leq 0$ versus $H_{1i}: \mathbf{h}'_i \boldsymbol{\theta} > 0$. So the test with rejection region $S_1^* \cap S_2^*$, that is, ϕ_o , is a level- α IUT of H_0 versus H_1 . Since $R_L \subset S_1^* \cap S_2^*$ and we know R_L is size- α , ϕ_o must in fact have size equal to α .

For the $k = 2$ case, both the LRT and ϕ_o (or ϕ_a) are IUT's constructed starting from the same individual hypotheses, H_{01} and H_{02} . This illustrates that some foresight in choosing the rejection region for the individual hypotheses, foresight concerning how the rejection regions will intersect, might result in increased power in the resulting IUT. Starting with the more complicated regions S_1^* and S_2^* , rather than the simpler R_{L_1} and R_{L_2} , yields a more powerful test. Also, although the R_{L_i} 's have certain optimality properties, for example, R_{L_i} is an unbiased test of H_{0i} , whereas S_i^* is not, this optimality does not carry over to the IUT's.

The test ϕ_b could also be described as an IUT in terms of the variables (u_i, v_i) , $i = 1, \dots, k$, but it was not described in this way in Berger (1989).

Acknowledgments. Portions of this paper are from the Ph.D. dissertation of Huimei Liu, written under the direction of Roger L. Berger at North Carolina State University. We thank an Associate Editor for the simplified proof of Theorem 2.2.

REFERENCES

- BERGER, R. L. (1982). Multiparameter hypothesis testing and acceptance sampling. *Technometrics* **24** 295–300.
- BERGER, R. L. (1989). Uniformly more powerful tests for hypotheses concerning linear inequalities and normal means. *J. Amer. Statist. Assoc.* **84** 192–199.
- CASELLA, G. AND BERGER, R. L. (1990). *Statistical Inference*. Wadsworth & Brooks/Cole, Pacific Grove, CA.
- COHEN, A., GATSONIS, C. AND MARDEN, J. I. (1983). Hypothesis tests and optimality properties in discrete multivariate analysis. In *Studies in Econometrics, Time Series and Multivariate Statistics* (S. Karlin, T. Amemiya and L. A. Goodman, eds.) 379–405. Academic Press, New York.
- GUTMANN, S. (1987). Tests uniformly more powerful than uniformly most powerful monotone tests. *J. Statist. Plann. Inference* **17** 279–292.
- IWASA, M. (1991). Admissibility of unbiased tests for a composite hypothesis with a restricted alternative. *Ann. Inst. Statist. Math.* **43** 657–665.
- LEHMANN, E. L. (1952). Testing multiparameter hypotheses. *Ann. Math. Statist.* **23** 541–552.
- LIU, W. (1992). On the existence of the uniformly more powerful tests. Technical report, Univ. Southampton.
- NOMAKUCHI, K. AND SAKATA, T. (1987). A note on testing two-dimensional normal mean. *Ann. Inst. Statist. Math.* **39** 489–495.
- SASABUCHI, S. (1980). A test of a multivariate normal mean with composite hypotheses determined by linear inequalities. *Biometrika* **67** 429–439.
- SHIRLEY, A. G. (1992). Is the minimum of several location parameters positive? *J. Statist. Plann. Inference* **31** 67–79.
- SIERRA-CAVAZOS, J. H. (1992). Tests of hypotheses defined by linear inequalities for elliptically contoured families. Ph.D. dissertation, North Carolina State Univ.
- STEIN, C. (1956). The admissibility of Hotelling's T^2 -test. *Ann. Math. Statist.* **27** 616–623.

DEPARTMENT OF STATISTICS
NATIONAL CHUNG HSING UNIVERSITY
TAIPEI, TAIWAN 10433
REPUBLIC OF CHINA

DEPARTMENT OF STATISTICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA 27695-8203