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EXACT ASYMPTOTICS FOR SOME PROBABILITY DISTRIBUTIONS ON COMPACT MANIFOLDS[1]

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Let M be a compact, smooth, orientable manifold without boundary, and let $f: M \to \mathbb{R}$ be a smooth function. Let dm be a volume form on M with total volume 1, and denote by X the corresponding random variable. Using a theorem of Kirwan, we obtain necessary conditions under which the method of stationary phase returns an exact evaluation of the characteristic function of $f(X)$. As an application to the Langevin distribution on the sphere S^{d-1}, we deduce that the method of stationary phase provides an exact evaluation of the normalizing constant for that distribution when, and only when, d is odd.

1. Introduction. Let M be a compact, smooth, orientable manifold without boundary, and let dm be a given volume form on M. We assume that dm has a total volume 1 so that we may view dm as a probability distribution on M; we then denote by X the corresponding random variable. Given a smooth (i.e., C^∞) function $f: M \to \mathbb{R}$, we consider the problem of deriving asymptotic expansions for the characteristic function of $f(X)$,

$$\hat{f}(t) := \int_M \exp(itf(s))\, dm(s), \tag{1}$$

as $t \to \infty$.

An example of this problem arises when $M = S^{d-1}$, the unit sphere in $\mathbb{R}^d$; $dm(s)$ is the normalized surface measure on S^{d-1}; and $f(s) = \langle \nu, s \rangle$, $s \in S^{d-1}$, for some fixed $\nu \in S^{d-1}$. In this case it is well known that (1) may be written in terms of the classical Bessel functions. Another example stems from the choice $M = O(n)$, the group of orthogonal $n \times n$ matrices; $dm(s)$ is the normalized Haar measure on $O(n)$; and $f(s) = \operatorname{tr} UsVs^{-1}$, $s \in O(n)$, where U and V are $n \times n$, real symmetric matrices. In this case $\hat{f}(t) = {}_0F_0(itU, V)$, where ${}_0F_0$ is a hypergeometric function of two matrix arguments [Muirhead (1982), page 262].

One procedure used to derive asymptotic expansions of (1) is the *method of stationary phase* [Barndorff-Nielsen and Cox (1989); Hörmander (1983)] (cf. Section 3 below). This method requires that we locate the critical points of the

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function f, and then we approximate (1) by certain sums depending on the values of f and its higher derivatives at the critical points. (This should be contrasted with *Laplace's method* [Muirhead (1982), page 391], which provides asymptotic expansions for integrals of certain real-valued integrands.)

Here, we consider the problem of finding random variables X for which the method of stationary phase produces the *exact value* of the characteristic function (1). Our interest in this problem began on reading the result of Daniels (1980) [cf. Blæsild and Jensen (1985)] that the saddlepoint approximation for the mean of a random sample is exact if and only if the data follow either a normal, gamma or inverse Gaussian distribution. On reading the results of Daniels and of Blæsild and Jensen, we were motivated to search for exactness results for other approximation methods used in statistical theory.

2. Preliminaries. Consider the integral on the sphere S^2 coordinatized by standard Euler angles (θ, ϕ),

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\exp(it\cos\theta)\sin\theta\,d\theta\,d\phi = \frac{(\exp(it) - \exp(-it))}{2it}, \tag{2}$$

where $t \neq 0$. This may be viewed as calculating the Fourier transform of the random variable $\cos\Theta$, where the pair (Θ, Φ) parametrizes the uniform distribution on S^2. For later purposes we note also that (2), with t replaced by t/i, represents an evaluation of the normalizing constant for the Fisher distribution [Watson (1983)] on S^2.

The right-hand side of (2) is a sum of two terms, each term arising from a critical point of the function $\cos\theta$ (at the two critical points $\theta = 0$ and π). It may be checked that the method of stationary phase, applied to the left-hand side of (2), produces the right-hand side as the final value. Therefore, (2) provides an example where the method of stationary phase obtains an exact asymptotic expansion. This example is a simple instance of a phenomenon treated by Duistermaat and Heckman (1982); those authors characterized, in a more general context than ours, a class of integrals such that, when the method of stationary phase is applied to those integrals, the remainder terms are identically zero. As an application of the results of Duistermaat and Heckman we also find that, in the second example above, where $f(s) = \operatorname{tr} UsVs^{-1}$, $s \in O(n)$, the method of stationary phase does not return the exact result.

To describe explicitly the phenomenon of exact stationary phase, we need some preliminaries on manifolds and differential forms, all of which are provided by Muirhead (1982) or Wijsman (1990), and some topological results which we have abstracted from Milnor (1963).

As before, let M be an n-dimensional, smooth, orientable, compact manifold without boundary, and with a given volume form dm. For $0 \le p \le n$, let $\wedge^p(M)$ denote the space of *smooth p-forms on M*. Thus, in a system of local coordinates $s = (s_1, \ldots, s_n)$ on M, each $\omega \in \wedge^p(M)$ is a differential form of the type

$$\omega = \sum_{i_1 < \cdots < i_p} h_{i_1, i_2, \ldots, i_p}(s_1, \ldots, s_n)\, ds_{i_1} \wedge \cdots \wedge ds_{i_p},$$

where the $h_{i_1, i_2, \ldots, i_p}(s_1, \ldots, s_n)$ are smooth functions of $(s_1, \ldots, s_n)$.

Let $d\colon \wedge^p(M) \to \wedge^{p+1}(M)$ denote exterior differentiation, with range Ran_p and kernel Ker_p. Since $d \circ d = 0$, then it follows that $\mathrm{Ran}_{p-1} \subseteq \mathrm{Ker}_p$. With $\wedge^{-1}(M) = \wedge^{n+1}(M) = \{0\}$, define the pth *de Rham cohomology group* of M to be the quotient space $H^p := \mathrm{Ker}_p/\mathrm{Ran}_{p-1}$, $0 \le p \le n$. It may be shown that H^p is a finite-dimensional vector space, and the numbers $b_p = \dim(H_p)$, $0 \le p \le n$, are called the *Betti numbers* of M. When M is a compact Lie group or a homogeneous space (e.g., a sphere), more precise information about the Betti numbers is given by Weyl (1946).

Let $f\colon M \to \mathbb{R}$ be a smooth function. A point $s \in M$ is a *critical point* for f if $df(s) = 0$; that is, in any local coordinate system $(s_1, \ldots, s_n)$ about s,

$$\frac{\partial f}{\partial s_1}(s) = \cdots = \frac{\partial f}{\partial s_n}(s) = 0.$$

A critical point s is called *nondegenerate* if the Hessian matrix $H(s)$, with (j, k)th entry $\partial^2 f/\partial s_j\, \partial s_k$, is nonsingular. The *index* $\lambda(s)$ of a critical point s is the number of negative eigenvalues of $H(s)$.

A smooth function $f\colon M \to \mathbb{R}$ is a *Morse function* if it has only a finite number of critical points and each critical point is nondegenerate. We denote by $m_p(f)$, $0 \le p \le n$, the number of critical points of f with index p. It is well known [Milnor (1963)] that the Betti numbers b_p and the numbers $m_p(f)$ satisfy the *Morse inequalities*

$$\text{(3)}\qquad \begin{aligned} \sum_{p=0}^{k} (-1)^{k-p} m_p(f) &\ge \sum_{p=0}^{k} (-1)^{k-p} b_p, \qquad 0 \le k \le n-1,\\ \sum_{p=0}^{n} (-1)^{n-p} m_p(f) &= \sum_{p=0}^{n} (-1)^{n-p} b_p. \end{aligned}$$

In the case when (3) consists entirely of equalities, the function f is called a *perfect* Morse function [Kirwan (1987)].

3. The denouement. Suppose $f\colon M \to \mathbb{R}$ is a nondegenerate Morse function with isolated critical points and distinct critical values. The stationary phase approximation to the characteristic function (1) may be described as follows [Hörmander (1983); Kirwan (1987)]. For any critical point s let

$$A_s = \sum_{j=0}^{\infty} (it)^{-j} a_{s,j},$$

where the real constants $a_{s,j}$ depend on the function f and its higher derivatives at the point s [Hörmander (1983), Section 7.7.5] and satisfy the following property: if $A_{s,k}(t)$ is the kth partial sum of A_s, then, as $t \to \infty$,

$$\int_M \exp(itf(s))\,dm(s) \tag{4}$$
$$\sim (2\pi)^{n/2} t^{-n/2} \sum_{s:\, df(s)=0} \exp\left(i\left[tf(s) + (n - 2\lambda(s))\frac{\pi}{4}\right]\right) A_{s,k}(t) + O(t^{-(n+2k)/2}).$$

Suppose that for all critical points s and all suitably large t the series A_s converges, say, to $A_s(t)$. The stationary phase approximation (4) is said to be *exact* if the $A_s(t)$ satisfy

$$\int_M \exp(itf(s))\,dm(s) = (2\pi)^{n/2} t^{-n/2} \times \sum_{s:\, df(s)=0} \exp\left(i\left[tf(s) + (n - 2\lambda(s))\frac{\pi}{4}\right]\right) A_s(t). \tag{5}$$

Then it is a remarkable theorem of Kirwan (1987) that *if the stationary phase approximation* (4) *is exact, then the following hold*: (i) n *is even*; (ii) *every critical point of f has even index*; *and* (iii) *f is a perfect Morse function.*

An interesting class of examples are those of the Langevin distributions [Watson (1983)] which generalize the Fisher distribution to higher-dimensional spheres. Here, for a given *concentration parameter* $\gamma > 0$ and *modal direction vector* $\nu \in S^{d-1}$, we have a probability density function, $c(\gamma)^{-1}\exp(\gamma\langle \nu, s\rangle)$, $s \in S^{d-1}$, relative to normalized surface measure $dm(s)$ on S^{d-1}. Then the method of stationary phase may be applied to the integral

$$c(i\gamma) = \int_{S^{d-1}} \exp(i\gamma\langle \nu, s\rangle)\,dm(s) \tag{6}$$

to obtain an asymptotic approximation as $\gamma \to \infty$. When d is odd it turns out that stationary phase provides the exact result, generalizing the example given in (2).

An alternative method of deriving the exact approximation is as follows. For even or odd d it follows from an invariance argument applied to (6) that [Watson (1983), page 187]

$$c(i\gamma) = \frac{\Gamma(d/2)}{\pi^{1/2}\Gamma((d-1)/2)} \int_{-1}^{1} \exp(i\gamma u)(1 - u^2)^{(d-3)/2}\,du.$$

When d is odd, we may expand the term $(1 - u^2)^{(d-3)/2}$ by the binomial theorem, and then $c(i\gamma)$ is obtained as a finite sum, generalizing the right-hand side of (2). This finite sum is identical with the results produced by the method of stationary phase. On the other hand, when d is even the method of stationary phase does not provide the exact value of $c(i\gamma)$, as predicted by Kirwan's theorem. This proves that the function $s \mapsto \langle \nu, s\rangle$, $s \in S^{d-1}$, is a perfect Morse function if and only if d is odd. In the special case $d = 3$, this result simply means that the function $(\theta, \phi) \mapsto \cos\theta$ on S^2 is a perfect Morse function.

Very importantly, in the case of odd-dimensional spheres, Kirwan's theorem implies the following result.

PROPOSITION. *Let the random variable X be uniformly distributed on the unit sphere S^{d-1}, and let $f(X)$ be a smooth, scalar-valued function of X. If d is even, then the characteristic function of $f(X)$ cannot be evaluated exactly by the method of stationary phase.*

As we have noted before, this result holds for any density function f with distinct critical values. However, this result remains valid [Kirwan (1987), page 40, Remark (4)] if the critical values of f are not distinct.

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