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IMPROVING ON THE MLE OF A BOUNDED NORMAL MEAN[1]

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We consider the problem of estimating the mean of a p-variate normal distribution with identity covariance matrix when the mean lies in a ball of radius m. It follows from general theory that dominating estimators of the maximum likelihood estimator always exist when the loss is squared error. We provide and describe explicit classes of improvements for all problems (m, p). We show that, for small enough m, a wide class of estimators, including all Bayes estimators with respect to orthogonally invariant priors, dominate the maximum likelihood estimator. When m is not so small, we establish general sufficient conditions for dominance over the maximum likelihood estimator. These include, when $m \leq \sqrt{p}$, the Bayes estimator with respect to a uniform prior on the boundary of the parameter space. We also study the resulting Bayes estimators for orthogonally invariant priors and obtain conditions of dominance involving the choice of the prior. Finally, these Bayesian dominance results are further discussed and illustrated with examples, which include (1) the Bayes estimator for a uniform prior on the whole parameter space and (2) a new Bayes estimator derived from an exponential family of priors.

1. Introduction. In many settings, there exists definite prior information concerning the values that a mean vector can take. In such settings, usual estimators for the unconstrained multivariate normal problem, such as the unbiased estimator $\delta_0(x) = x$, James–Stein type estimators and their derivatives are neither admissible nor minimax and a number of alternatives that capitalize on the prior information are available. We consider such a restricted parameter space problem, namely, the problem of estimating, based on an observation x, the mean θ under squared error loss of $X \sim N_p(\theta, I_p)$, with $\theta \in \Theta(m)$, $\Theta(m) = \{\theta \in \mathbb{R}^p\colon \|\theta\| \leq m\}$ for some m fixed, $m > 0$.

An immediate alternative to the unbiased estimator δ_0 is the maximum likelihood estimator δ_{mle} which is the truncation of δ_0 onto $\Theta(m)$ given by $\delta_{\mathrm{mle}}(x) = (m/\|x\| \wedge 1)x$. Although δ_{mle} is eminently preferable to δ_0, it has long been known that maximum likelihood estimators are often inadmissible under squared error loss for restricted parameter spaces [e.g., Sacks (1963)]. The inadmissibility of δ_{mle} for our problem also follows from the work of Charras and van Eeden (1991) who establish the inadmissibility of so called "boundary" estimators within a quite general framework of compact parameter spaces. Although their proof of the inadmissibility of δ_{mle} does not involve the

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determination of dominating estimators, they do provide implicitly for the case $p = 1$, dominating estimators. Casella and Strawderman (1981) showed for $p = 1$ and $m \le 1$ that the two-point boundary symmetric prior is least favorable and that the associated Bayes estimator is, not only minimax, but dominates δ_{mle} as well. Moors (1981, 1985) establishes in a general framework, the inadmissibility of estimators which take values on, or too close to, the boundary of the parameter space. For the univariate normal case, he sets $A_x = [-m \tanh(m|x|), m \tanh(m|x|)]$ and his results apply to orthogonally equivariant estimators δ such that $P_\theta[\{x\colon \delta(x) \notin A_x\}] > 0$. Moors (1981, 1985) shows that if, on the set $\{x\colon \delta(x) \notin A_x\}$, $\delta^*(x)$ is the truncation of $\delta(x)$ on the boundary of A_x and $\delta^*(x) = \delta(x)$ otherwise then δ^* dominates δ. In particular, this result applies to δ_{mle}.

Other alternatives worth exploring are Bayes estimators and their frequentist performance in the hope of determining interesting alternatives to δ_{mle}. Kempthorne (1988) presents, as a particular application of his general algorithm that yields an optimal estimator chosen according to the "maximin improvement" criterion, numerical examples of Bayes estimators that dominate the maximum likelihood estimator ($p = 1$). Two particular Bayes estimators of interest are those associated with (1) a uniform prior on the boundary of $\Theta(m)$, and (2) a fully uniform prior on $\Theta(m)$. Given equivariance considerations and the work of DasGupta (1985), the former is necessarily minimax for small enough m; and has been further studied with respect to minimaxity by Casella and Strawderman (1981) for $p = 1$, and Berry (1990) for $p > 1$. The latter is intuitively appealing and has, for the univariate case, been studied by Gatsonis, MacGibbon and Strawderman (1987) who showed that it performs satisfactorily, dominating δ_0 and improving on δ_{mle} over a large part of the parameter space. As well, the Bayes estimator with respect to the fully uniform prior is, for small enough m, optimal according to a Γ-minimax criterion with unimodal and symmetric priors [see Vidakovic and DasGupta (1996)].

Notwithstanding these above contributions, there remains few dominance results and the focus of this work is on providing dominating estimators to δ_{mle}. Section 3 presents a set of sufficient conditions which guarantee dominance for all problems (m, p) and clarifies the structure of possible improvements. As a consequence, we find that the Bayes estimator with respect to the boundary uniform prior dominates δ_{mle} whenever $m \le \sqrt{p}$. In Section 4, we work more generally on Bayes estimators δ_π with orthogonally invariant priors π. By using the results of Section 3, we establish that, for small enough m, the Bayes estimator δ_π, for any invariant of π, dominates δ_{mle}. For larger m, sufficient conditions on π for δ_π to dominate δ_{mle} are given. Finally, these Bayesian dominance results are further discussed and illustrated with examples. These include (1) the Bayes estimator with respect to a fully uniform prior on the whole parameter space, which is shown to dominate δ_{mle} for small enough m and (2) a new Bayes estimator derived from an exponential family of priors which is proposed for $m > \sqrt{p}$, and shown to dominate δ_{mle}

for $p > 1$ and small enough values of $m - \sqrt{p}$. We now proceed in Section 2 to collect some further notations, definitions and properties for later use.

2. Definitions and preliminaries. Throughout the paper we shall denote $\|X\|$ and $\|x\|$ by R and r, respectively. We will denote $\|\theta\|$ by λ when θ is viewed as a parameter, and by T, in a Bayesian context, when θ is viewed as a random variable. All the estimators considered below are orthogonally equivariant and it is convenient to express them as

$$\delta_g(x) = \frac{g(r)}{r} x.$$

We measure the performance of an estimator δ_g by its risk function $R(\theta, \delta_g) = \mathrm{E}_\theta[\|\delta_g(X) - \theta\|^2]$. Our dominance results are based on conditional risk decompositions. In short, if Y is a function of X and δ is an estimator of θ then $\mathrm{E}_\theta[\|\delta(X) - \theta\|^2 | Y]$ is the conditional risk of δ. Furthermore, the expectation of the conditional risk provides the risk. Clearly, if we show that $\mathrm{E}_\theta[\|\delta_1(X) - \theta\|^2 - \|\delta_2(X) - \theta\|^2 \mid Y] \le 0$ for all $\theta \in \Theta(m)$ and all possible values of Y, and if we find $\theta_0 \in \Theta(m)$ such that $\mathrm{P}_{\theta_0}[\mathrm{E}_{\theta_0}[\|\delta_1(X) - \theta_0\|^2 - \|\delta_2(X) - \theta_0\|^2 \mid Y] < 0] > 0$ then we obtain that δ_1 dominates δ_2.

We denote δ_{BU} as the Bayes estimator with respect to the boundary uniform prior. We obtain that

$$\delta_{\mathrm{BU}}(x) = \frac{\bar{g}_m(\|x\|)}{\|x\|} x,$$

where, for any $r > 0$, $\lambda \ge 0$,

$$\bar{g}_\lambda(r) = \mathrm{E}_\theta\left[\frac{\theta' X}{\|X\|} \middle| \|X\| = r\right].$$

LEMMA 1 [Berry (1990), Robert (1990)]. *An explicit expression for $\bar{g}_\lambda$ is given by*

$$\bar{g}_\lambda(r) = \lambda \rho_{(p/2)-1}(\lambda r),$$

where $\rho_\nu(t) = I_{\nu+1}(t)/I_\nu(t)$, $\nu > -1$; I_ν representing the modified Bessel function of order ν.

Thus, the ratio $\rho_{(p/2)-1}$ plays a key role and the next lemma recalls some useful properties, given by Watson (1983) for $p > 1$ and readily verified for $p = 1$ by using the representation $\rho_{-1/2} = \tanh$.

LEMMA 2. *For all $p \ge 1$:*

(a) *$\rho_{(p/2)-1}$ is increasing with $\rho_{(p/2)-1}(0) = 0$ and $\rho_{(p/2)-1}(t) \to 1$ as $t \to \infty$.*
(b) *$\rho_{(p/2)-1}(t)/t$ is decreasing in t with $\rho_{(p/2)-1}(t)/t \to 1/p$ as $t \to 0$.*

The function $f_p(\cdot, d)$ will denote the density of the square root of a random variable having a noncentral $\chi^2_p(d^2)$ distribution. In particular, the density of R is $f_p(\cdot,\lambda)$. The following properties, which will be useful, are essentially known, and follow from the Bessel representation of the noncentral chi-square distribution.

LEMMA 3. *Let r, $\lambda > 0$. For any measure σ we obtain*:

(a) $f_p(r,\lambda) = r(\frac{r}{\lambda})^{(p/2)-1} I_{(p/2)-1}(\lambda r)\exp -(\lambda^2 + r^2)/2$.
(b) $\int_0^\infty h(w) w \rho_{(p/2)-1}(w) f_p(w,\lambda)\sigma(dw) = \lambda \int_0^\infty h(w) f_{p+2}(w,\lambda)\sigma(dw)$.

Some of the risk function decompositions below will bring into play the conditional expectations

$$\alpha(m,\lambda) = \mathrm{E}_\theta[\rho_{(p/2)-1}(\lambda R) | R > m] \quad \text{and} \quad \beta(m,\lambda) = \mathrm{E}_\theta\left[\frac{\lambda}{R}\rho_{(p/2)-1}(\lambda R) | R \le m\right],$$

where $0 < \lambda \le m$, as well as the functions

$$\bar{\alpha}(m) = \sup_{0<\lambda\le m} \alpha(m,\lambda) \quad \text{and} \quad \bar{\beta}(m) = \sup_{0<\lambda\le m} \beta(m,\lambda).$$

The following properties, which are proved in the Appendix, will be required.

LEMMA 4. (a) *$\alpha(\cdot,\cdot)$ is an increasing function in both arguments and, consequently, $\bar{\alpha}(m) = \alpha(m,m)$. Furthermore, $\bar{\alpha}(m) \to 0$ as $m \to 0$ and $\bar{\alpha}(m) \to 1$ as $m \to \infty$.*

(b) *$\beta(m,\cdot)$ is an increasing function and, consequently, $\bar{\beta}(m) = \beta(m,m)$. Furthermore, $0 \le \bar{\beta}(m) < m^2/p$ and $\lim_{m\to\infty} \bar{\beta}(m) \ge 1$.*

Finally, the solution in m of the equation $\bar{\alpha}(m) = 1/2$ will arise below and will be denoted m_1.

3. Dominance results. The dominance results of this section are organized as follows. Let Y be the indicator random variable of the event $\{\|X\| > m\}$. Theorem 1 gives sufficient conditions for δ_g to improve on δ_{mle} as measured by the difference in conditional risks, where the conditioning is on the value $Y = 1$. Theorem 2 does the same, but for $Y = 0$. Theorem 3 involves conditional risks based on the values of $\|X\|$, and Example 1 shows how Theorem 3 applies to the Bayes estimator with respect to the boundary uniform prior. Finally, Corollaries 1 and 2, which will be useful in Section 4, are further dominance results obtained by pooling some of the results of Theorems 1, 2 and 3.

THEOREM 1. *Let g be a nondecreasing function on (m,∞). If $(2\bar{\alpha}(m) - 1)m < g(r) < m$ for all $r \in (m,\infty)$ then $\mathrm{E}_\theta[L(\theta,\delta_{\mathrm{mle}}(X)) - L(\theta,\delta_g(X)) \mid R > m] > 0$ for all $\theta \in \Theta(m)$.*

PROOF. First, Lemma 4 shows that $(2\bar{\alpha}(m) - 1) < 1$, for all $m > 0$, which implies that the given condition on g is not vacuous. Decomposing the difference in conditional risks, we obtain

$$\begin{aligned}
&\mathrm{E}_\theta[L(\theta, \delta_{\mathrm{mle}}(X)) - L(\theta, \delta_g(X)) \mid R > m] \\
&\quad = \mathrm{E}_\theta[(m - g(R))(g(R) - \{2\lambda\rho_{(p/2)-1}(\lambda R) - m\}) \mid R > m] \\
&\quad \geq \mathrm{E}_\theta[(m - g(R))(g(R) - \{2\lambda\alpha(m, \lambda) - m\}) \mid R > m] \\
&\quad \geq \mathrm{E}_\theta[(m - g(R))(g(R) - (2\bar{\alpha}(m) - 1)m) \mid R > m] \\
&\quad > 0,
\end{aligned}$$

where (1) the equality comes from a conditional expectation given R and Lemma 1, (2) the first inequality holds by virtue of the inequality $\mathrm{Cov}_\theta[\rho_{(p/2)-1}(\lambda R), g(R) \mid R > m] \geq 0$, which in turn is valid since both $\rho_{(p/2)-1}$ and g are nondecreasing on (m, ∞), and (3) the second inequality follows from Lemma 4.

THEOREM 2. *Let $g(r)$ and $r(r - g(r))$ be nondecreasing in r on $(0, m]$. If $\bar{\beta}(m) < 1$ and $(2\bar{\beta}(m) - 1)r < g(r) < r$ for all $r \in (0, m]$ then $\mathrm{E}_\theta[L(\theta, \delta_{\mathrm{mle}}(X)) - L(\theta, \delta_g(X)) \mid R \leq m] > 0$ for all $\theta \in \Theta(m)$.*

PROOF. First, note that $\{m\colon \bar{\beta}(m) < 1\} \neq \varnothing$ since the properties of $\bar{\beta}(m)$ in Lemma 4 imply that $(0, \sqrt{p}] \subset \{m\colon \bar{\beta}(m) < 1\}$. Moreover, we obtain

$$\begin{aligned}
&\mathrm{E}_\theta[L(\theta, \delta_{\mathrm{mle}}(X)) - L(\theta, \delta_g(X)) \mid R \leq m] \\
&\quad = \mathrm{E}_\theta\left[(R - g(R))\left(g(R) - \left(2\lambda\frac{\rho_{(p/2)-1}(\lambda R)}{R} - 1\right)R\right)\middle| R \leq m\right] \\
&\quad \geq \mathrm{E}_\theta[(R - g(R))(g(R) - \{2\beta(m, \lambda) - 1\}R) \mid R \leq m] \\
&\quad \geq \mathrm{E}_\theta[(R - g(R))(g(R) - \{2\bar{\beta}(m) - 1\}R) \mid R \leq m] \\
&\quad > 0,
\end{aligned}$$

where (1) the first inequality holds because $r(r - g(r))$ is nondecreasing in r on $(0, m]$ and $\frac{\rho_{(p/2)-1}(\lambda r)}{r}$ is nonincreasing in r for $r \in (0, m]$ which implies that $\mathrm{Cov}_\theta[R(R - g(R)), \frac{\rho_{(p/2)-1}(\lambda R)}{R} \mid R \leq m] \leq 0$, and (2) the second inequality comes from Lemma 4.

COROLLARY 1. *Let $0 \leq g(r) \leq r \wedge m$ for all $r > 0$. If g is nondecreasing, $g(r)/r$ is nonincreasing in r for $r > 0$ and $g(m)/m \geq 2\{\bar{\alpha}(m) \vee \bar{\beta}(m)\} - 1$, then $R(\theta, \delta_g) \leq R(\theta, \delta_{\mathrm{mle}})$ for all $\theta \in \Theta(m)$.*

PROOF. We have $(2\bar{\alpha}(m) - 1)m \leq g(r) \leq m$ for all $r > m$ so the conditions of Theorem 1 are verified. Since $g(r)/r$ is nonincreasing in r for $r > 0$ we obtain that $r(r - g(r))$ is nondecreasing in r and $(2\bar{\beta}(m) - 1)r \leq g(r) \leq r$ for $0 < r \leq m$ so the conditions of Theorem 2 are verified.

THEOREM 3. *Let* $r > r_1$ *with* $r_1 = \inf\{r\colon \bar{g}_m(r)/r < 1,\ r > 0\}$. *If* $2\bar{g}_m(r) - r \wedge m < g(r) < r \wedge m$ *then* $E_\theta[L(\theta, \delta_{\mathrm{mle}}(X)) - L(\theta, \delta_g(X)) \mid R = r] > 0$ *for all* $\theta \in \Theta(m)$.

PROOF. Since $\bar{g}_m(r) < m$, and $r_1 < m$ by virtue of Lemma 2, we have $2\bar{g}_m(r) - r \wedge m < r \wedge m$ if and only if $r > r_1$. Moreover, we have

$$\begin{aligned} &E_\theta[L(\theta, \delta_{\mathrm{mle}}(X)) - L(\theta, \delta_g(X)) \mid R = r] \\ &\quad = (r \wedge m - g(r))(g(r) - \{2\lambda\rho_{(p/2)-1}(\lambda r) - r \wedge m\}) \\ &\quad \geq (r \wedge m - g(r))(g(r) - \{2\bar{g}_m(r) - r \wedge m\}) \\ &\quad > 0, \end{aligned}$$

where the first inequality comes from the monotonicity of the function $\rho_{(p/2)-1}$.

EXAMPLE 1. When $m \leq \sqrt{p}$, it follows from Lemma 2 that $r_1 = 0$ and, consequently, δ_{BU} dominates the maximum likelihood estimator by virtue of Theorem 3. When $m > \sqrt{p}$, $r_1 > 0$ and Theorem 3 does not apply to δ_{BU} directly. However, Theorem 3 does apply to the truncated version δ_g with $g(r) = \bar{g}_m(r) \wedge r$ which dominates the maximum likelihood estimator.

COROLLARY 2. *Let* $0 \leq g(r) \leq r \wedge m$ *for* $r > 0$. *If* g *is nondecreasing and* $m \leq m_1 \wedge \sqrt{p/2}$, *then* $R(\theta, \delta_g) \leq R(\theta, \delta_{\mathrm{mle}})$ *for all* $\theta \in \Theta(m)$.

PROOF. This proof is based on verifying the conditions of Theorem 1 on $r \in (m, \infty)$ and the conditions of Theorem 3 on $r \in (0, m]$. With Theorem 1 we need to verify that $(2\bar{\alpha}(m) - 1)m < g(r) < m$ for all $r > m$. The conditions of Corollary 2 imply that $2\bar{\alpha}(m) - 1 \leq 0$. Therefore, if g is nondecreasing and $0 \leq g(r) \leq m$ for $r > m$, the conditions of Theorem 1 will be satisfied. With Theorem 3, we need to verify that $(2\bar{g}_m(r) - r) < g(r) < r$ for $r \in (0, m]$. Theorem 3 applies whenever $r > r_1$. Part (b) of Lemma 2 tells us that $\bar{g}_m(r)/r \to m^2/p$ as $r \to 0$ and $\bar{g}_m(r)/r$ is decreasing in r. Since $m \leq \sqrt{p/2}$, we have $r_1 = 0$, and $2\bar{g}_m(r) - r \leq 0$ for all $r > 0$. Therefore, if $0 \leq g(r) \leq r$ for $r \in (0, m]$, the conditions of Theorem 3 will be satisfied as well. □

We conclude this section by remarking upon the fact that, analogously to Moors (1981, 1985), the methods above may be applied to determine improvements over other estimators that take values on, or too close to, the boundary of the parameter space $\Theta(m)$. For further details and related results, the reader is referred to Marchand and Perron (1999).

4. Bayesian estimators. In this section, we consider Bayes estimators δ_π associated with orthogonally invariant prior distributions π on $\Theta(m)$. The Bayesian dominance results below involve the specification of priors π that lead to the applicability of Corollaries 1 and 2. In Section 4.1, Theorems 4 and 5 give useful characterizations and properties, while Corollaries 3 and 4 are Bayesian versions of Corollaries 2 and 1, respectively. Subsection 4.2. is devoted to examples, illustrations and further comments.

4.1. *Bayesian dominance results.*

THEOREM 4. *Let $T = \|\theta\|$. For a given orthogonally invariant prior π on $\Theta(m)$ with $\pi(\{T = 0\}) < 1$, the Bayes estimator δ_π is given by $\delta_\pi(x) = (g_\pi(r)/r)x$ where $g_\pi(r) = \mathrm{E}[\bar{g}_T(r) \mid R = r]$. In other words, $g_\pi(r)$ is the expectation of $\bar{g}_T(r)$ with respect to the posterior distribution of T. An alternative representation of g_π is given by*

$$g_\pi(r) = \frac{\int_0^m t^{-(p/2-2)} I_{p/2}(rt) \exp(-t^2/2)\sigma(dt)}{\int_0^m t^{-(p/2-1)} I_{p/2-1}(rt) \exp(-t^2/2)\sigma(dt)}.$$

Moreover, g_π is increasing with $g_\pi(r) \to 0$ as $r \to 0$, $0 \le g_\pi \le \bar{g}_m$ and $g_\pi(r)/r \le 1$ whenever $m \le \sqrt{p}$.

PROOF. Assume, without loss of generality, that $\|x\| > 0$. We use the representation $\theta = TU$ in distribution, where T and U are independent, T is distributed according to a probability measure σ on $[0, m]$ and U is uniformly distributed on the unit sphere on $\mathbb{R}^p$. This representation now implies that the posterior distribution of T has a density, with respect to the measure σ, proportional to $t^{1-p} f_p(t, r)$, that is,

$$\sigma(dt \mid x) = \frac{t^{1-p} f_p(t, r)}{\int_0^m u^{1-p} f_p(u, r)\sigma(du)} \sigma(dt) \quad \text{for } t \in [0, m],$$

and that, conditionally on the event $T = t$, the posterior distribution of U is a Langevin distribution with parameters (κ, μ) and mean $\rho_{(p/2)-1}(\kappa)\mu$ where $\kappa = rt$ and $\mu = (1/r)x$ [see Watson (1983) for details]. Since the Bayes estimator is given by $\mathrm{E}[\theta \mid X = x]$ we obtain

$$\begin{aligned}
\mathrm{E}[\theta \mid X = x] &= \mathrm{E}[TU \mid X = x] \\
&= \mathrm{E}[T\mathrm{E}[U \mid T, X = x] \mid X = x] \\
&= \mathrm{E}[T(\rho_{(p/2)-1}(rT)/r)x \mid X = x] \\
&= \mathrm{E}[T\rho_{(p/2)-1}(rT)/r \mid R = r]x \\
&= (\mathrm{E}[\bar{g}_T(r) \mid R = r]/r)x,
\end{aligned}$$

where the fourth equality holds because the posterior distribution of T depends on x through r only. The alternative representation of δ_π is derived from part (b) of Lemma 3. Since $0 \le \bar{g}_{\|\theta\|} \le \bar{g}_m$ for all $\theta \in \Theta(m)$ and $\bar{g}_m(r) \to 0$ as $r \to 0$ we obtain $0 \le g_\pi \le \bar{g}_m$ and $g_\pi(r) \to 0$ as $r \to 0$. Whenever $m \le \sqrt{p}$, part (b) of Lemma 2 tells us that $\bar{g}_m(r) \le r$, which coupled with the property

$g_\pi \le \bar{g}_m$, implies $g_\pi(r) \le r$. Finally, for $0 < r_1 < r_2$ we obtain

$$\begin{aligned} g_\pi(r_2) &= \mathrm{E}[T\rho_{(p/2)-1}(r_2 T) \mid R = r_2] \\ &> \mathrm{E}[T\rho_{(p/2)-1}(r_1 T) \mid R = r_2] \\ &\ge \mathrm{E}[T\rho_{(p/2)-1}(r_1 T) \mid R = r_1] \\ &= g_\pi(r_1), \end{aligned}$$

where the inequalities come from the monotonicity property of $\rho_{(p/2)-1}$ and the fact that the conditional distribution of T given that $R = r$ has monotone likelihood ratio in T, r being viewed as the parameter. □

REMARK 1. The above characterization of δ_π is quite general. For instance, it applies to the cases where T is degenerate, in particular at m yielding the boundary uniform prior Bayes estimator δ_{BU}. It also applies to the Bayes estimator associated with a fully uniform prior on $\Theta(m)$, for which $\sigma(dt) = (p/m^p)t^{p-1}\,dt$ and $g_\pi(r) = r\mathrm{P}[\chi^2_{p+2}(r^2) \le m^2]/\mathrm{P}[\chi^2_p(r^2) \le m^2]$.

COROLLARY 3. *If δ_π is any Bayes estimator with respect to an orthogonally invariant prior, then δ_π dominates δ_{mle} whenever $m \le m_1 \wedge \sqrt{p/2}$.*

PROOF. It will suffice to verify that, with such g_π's, the conditions of Corollary 2 are met. First, Theorem 4 shows that g_π is nondecreasing and $0 \le g_\pi \le \bar{g}_m \le m$. Second, since by assumption $m \le \sqrt{p/2}$, we have by virtue of part (b) of Lemma 2, $g_\pi(r) \le r/2 \le r$ for $r > 0$ which completes the proof. □

Note that, with this last result, once the conditions on (m, p) are fulfilled, dominance applies to all orthogonally invariant π. However, when $m > m_1 \wedge \sqrt{p/2}$ conditions on the choice of π will be required. As an intermediate step towards an application of Corollary 1 in a Bayesian context, which follows in Corollary 4, Theorem 5 introduces a subfamily of orthogonally invariant priors and describes conditions under which (i) $g_\pi(r)/r$ is nonincreasing, (ii) $g_\pi(r)/r$ is bounded from above by 1.

THEOREM 5. *Suppose that the prior π has a density of the form $K\exp(-h\times(\|\theta\|^2))$ where K is the normalizing constant.*

(a) *If h is nondecreasing then $g_\pi(r)/r$ is bounded from above by 1.*

(b) *If h is convex then $g_\pi(r)/r$ in nonincreasing in r. Moreover, $g_\pi(r)/r$ is bounded from above by 1 if and only if $\int_0^m t^{p+1}\exp(-\{h(t^2) + t^2/2\})\,dt \le p\int_0^m t^{p-1}\exp(-\{h(t^2) + t^2/2\})\,dt$.*

PROOF. From Theorem 4 we have

$$\begin{aligned}
\frac{g_\pi(r)}{r} &= \frac{\int_0^m f_{p+2}(t,r)\exp(-h(t^2))\,dt}{\int_0^m f_p(t,r)\exp(-h(t^2))\,dt}\\
&= \frac{\int_0^m \left\{\int_0^t f_p(\sqrt{t^2-v^2},r) f_2(v,0)(t/\sqrt{t^2-v^2})\,dv\right\}\exp(-h(t^2))\,dt}{\int_0^m f_p(t,r)\exp(-h(t^2))\,dt}\\
&= \frac{\int_0^m \left\{\int_v^m f_p(\sqrt{t^2-v^2},r)\exp(-h(t^2))(t/\sqrt{t^2-v^2})\,dt\right\} f_2(v,0)\,dv}{\int_0^m f_p(t,r)\exp(-h(t^2))\,dt}\\
&= \frac{\int_0^m \left\{\int_0^{\sqrt{m^2-v^2}} f_p(w,r)\exp(-\{h(w^2+v^2)-h(w^2)\})\exp(-h(w^2))\,dw\right\} f_2(v,0)\,dv}{\int_0^m f_p(w,r)\exp(-h(w^2))\,dw}\\
&= \mathrm{E}[\exp(-\{h(W^2+V^2)-h(W^2)\})I(W^2\le m^2-V^2)],
\end{aligned}$$

where V and W are independent random variables, V^2 has a χ_2^2 distribution and the density of W is the posterior density of $\|\theta\|$. The last expression shows that if h is nondecreasing then $g_\pi(r)/r$ is bounded from above by 1. The density of W has monotone likelihood ratio in W with r being the parameter. If h is convex then, $h(w^2+v^2)-h(w^2)$ is nondecreasing in w for fixed v, v, $w>0$. This shows that, as a function of w with $v>0$ fixed, the function $\exp(-\{h(w^2+v^2)-h(w^2)\})\mathrm{I}(w^2\le m^2-v^2)$ is nonincreasing for $w>0$. Therefore, $\mathrm{E}[\exp(-\{h(W^2+V^2)-h(W^2)\})I(W^2\le m^2-V^2)|V=v]$ is nonincreasing in r for all $v>0$ which implies that $g_\pi(r)/r$ in nonincreasing in r. Finally, Theorem 4 shows that $g_\pi(r)/r \to \int_0^m t^{p+1}\exp(-\{h(t^2)+t^2/2\})\,dt/p\int_0^m t^{p-1}\exp(-\{h(t^2)+t^2/2\})\,dt$ as $r\to 0$.

COROLLARY 4. *Suppose that the prior π has a density of the form $K\exp(-h(\|\theta\|^2))$ where* (i) *h is convex*, (ii) *$g_\pi(r)/r$ is bounded from above by* 1 *and* (iii) *$g_\pi(m)/m\ge 2(\bar\alpha(m)\vee\bar\beta(m))-1$, then the Bayes estimator δ_π dominates δ_{mle}.*

The proof is a direct consequence of Corollary 1 and Theorem 5.

4.2. *Examples, illustrations and further comments.*

EXAMPLE 2. Consider δ_U, the Bayes estimator with respect to the fully uniform prior on $\Theta(m)$, whose functional form was given in Remark 1. Part (a) of Theorem 5 applies with $h=0$. Consequently, conditions (i) and (ii) of Corollary 4 are met, and Table 1 reports, for $1\le p\le 10$, on the sufficient condition (iii) of Corollary 4 for δ_U to dominate δ_{mle}.

Ideally, we would like to find a prior π such that δ_π dominates δ_{mle}. When $m\le\sqrt{p}$, δ_{BU} dominates δ_{mle}. However, since $\mathrm{R}(0,\delta_{\mathrm{BU}})=\mathrm{E}_0[\{\bar g_m(R)\}^2]$ increases in m with $R(0,\delta_{\mathrm{BU}})/m^2\to 1$ as $m\to\infty$ while $R(0,\delta_{\mathrm{mle}})\le p$, δ_{BU} will not dominate δ_{mle} for large values of m. Similarly, numerical results suggest that $R(\theta,\delta_U)>R(\theta,\delta_{\mathrm{mle}})$ at $\|\theta\|=m$ for large m. As shown by Gatsonis,

TABLE 1
Sufficient conditions for dominance of δ_{mle} by different Bayesian estimators

	Type of prior			
	Orthogonally invariant	**Boundary uniform**	**Uniform**	**Exponential family**
p	$(m \le m_1 \wedge \sqrt{p/2})$	$(m \le \sqrt{p})$	$(m \in \Delta_U(p))$	$(m \in \Delta_E(p))$
1	$m \le 0.4837$	$m \le 1.0000$	$m \le 0.5230$	$\varnothing$
2	$m \le 0.7487$	$m \le 1.4142$	$m \le 0.8949$	$1.4142 < m \le 1.4365$
3	$m \le 0.9540$	$m \le 1.7320$	$m \le 1.2731$	$1.7320 < m \le 1.8274$
4	$m \le 1.1251$	$m \le 2.0000$	$m \le 1.6968$	$2.0000 < m \le 2.1900$
5	$m \le 1.2734$	$m \le 2.2361$	$m \le 2.2003$	$2.2361 < m \le 2.5369$
6	$m \le 1.4053$	$m \le 2.4495$	$m \le 2.6679$	$2.4495 < m \le 2.8744$
7	$m \le 1.5248$	$m \le 2.6457$	$m \le 3.0289$	$2.6457 < m \le 3.2068$
8	$m \le 1.6346$	$m \le 2.8284$	$m \le 3.3908$	$2.8284 < m \le 3.5372$
9	$m \le 1.7367$	$m \le 3.0000$	$m \le 3.7534$	$3.0000 < m \le 3.8685$
10	$m \le 1.8325$	$m \le 3.1623$	$m \le 4.1167$	$3.1623 < m \le 4.2025$

MacGibbon and Strawderman (1987) for the case $p = 1$, it seems plausible that comparison of the risks at $\|\theta\| = m$ leads to a sufficient (and of course necessary) condition to decide whether or not δ_U dominates δ_{mle}. If this is so then, as implied by the following Lemma, δ_U may very well be viewed as a benchmark since unimodal prior densities will lead to Bayes estimators with more shrinkage and larger risk at $\|\theta\| = m$, while bowl-shaped prior densities will lead to Bayes estimators with less shrinkage and smaller risk at $\|\theta\| = m$.

LEMMA 5. *Let π and π' be two orthogonally invariant priors on $\Theta(m)$ and consider σ and σ' as being their induced probability measures on $\|\theta\|$.*

(a) *If $\frac{d\sigma}{d\sigma'}$ is nondecreasing then $g_\pi \ge g_{\pi'}$.*
(b) *If $g_\pi \ge g_{\pi'}$ then $R(\theta, \delta_\pi) \le R(\theta, \delta_{\pi'})$ when $\|\theta\| = m$ and $R(0, \delta_\pi) \ge R(0, \delta_{\pi'})$.*

PROOF. (a) Let $r > 0$ be fixed and let $T = \|\theta\|$. We obtain

$$g_\pi(r) - g_{\pi'}(r) = \operatorname{Cov}_{\pi'}\left(\bar{g}_T(r), K(r)\frac{d\sigma}{d\sigma'}(T) - 1 \mid R = r\right) \ge 0$$

with $K(r) = \int_0^m u^{1-p} f_p(u, r)\sigma'(du) / \int_0^m u^{1-p} f_p(u, r)\sigma(du)$. In other words, if we assume that π' is the prior then $g_\pi(r) - g_{\pi'}(r)$ is equal to the covariance between $\bar{g}_T(r)$ and $K(r)\frac{d\sigma}{d\sigma'}(T) - 1$ based on the posterior distribution of T. The inequality holds because it is the covariance between two nondecreasing functions of T.

(b) We have

$$R(\theta, \delta_g) = \mathrm{E}_\theta[(g(R) - \bar{g}_\lambda(R))^2] + \mathrm{E}_\theta[\lambda^2 - \bar{g}_\lambda(R)^2].$$

Since $0 \le g_{\pi'} \le g_\pi \le \bar{g}_m$, the results follow by setting $\lambda = m$ and $\lambda = 0$. $\square$

To continue, assume that $m > \sqrt{p}$ and consider an arbitrary prior π_0 which satisfies conditions (i) and (ii) of Corollary 4. Consider further an exponential family where π_a is defined as $\frac{d\pi_a}{d\pi_0}(\theta) = \exp\{a\|\theta\|^2/2 - \kappa(a)\}$ and a is chosen such that $g_{\pi_a}(r)/r \to 1$ as $r \to 0$. Let δ_E be the Bayes estimator with respect to π_a. The prior π_a also satisfies conditions (i) and (ii) of Corollary 4. Although there is no guarantee that δ_E will dominate δ_{mle}, Lemma 5 shows that $g_{\pi_a}(m) > g_{\pi_0}(m)$ if $a > 0$ which is desirable since we need $g(m)$ to be large in order to satisfy condition (iii) of Corollary 4.

EXAMPLE 3. Assume that π_0 is the fully uniform prior and that $m > \sqrt{p}$. It follows from above that a is the solution to the equation $\int_0^m t^{p+1} \exp\{(a-1)t^2/2\}\,dt = p\int_0^m t^{p-1} \exp\{(a-1)t^2/2\}\,dt$. Moreover, the new estimator δ_E has multiplier

$$g_{\pi_a}(r) = \frac{\int_0^m t^{p/2+1} I_{p/2}(rt) \exp\{(a-1)t^2/2\}\,dt}{\int_0^m t^{p/2} I_{p/2-1}(rt) \exp\{(a-1)t^2/2\}\,dt}.$$

REMARK 2. It follows from Corollary 3 that, for an orthogonally invariant prior π, δ_π dominates δ_{mle} if $m \le m_1 \wedge \sqrt{p/2}$. It follows from Theorem 3 that δ_{BU} dominates δ_{mle} if $m \le \sqrt{p}$. It follows from Corollary 4 that δ_U dominates δ_{mle} if $m \in \Delta_U(p)$ with $\Delta_U(p) = \{m\colon g_{\pi_0}(m)/m \ge 2(\bar{\alpha}(m) \vee \bar{\beta}(m)) - 1\}$ and, δ_E dominates δ_{mle} if $m \in \Delta_E(p)$ with $\Delta_E(p) = \{m\colon m > \sqrt{p}$ and $g_{\pi_a}(m)/m \ge 2(\bar{\alpha}(m) \vee \bar{\beta}(m)) - 1\}$. Table 1 gives the values of m satisfying these conditions for $p = 1, 2, \ldots, 10$.

REMARK 3. Figures 1 to 3 permit a comparison of the risk function of δ_{mle} versus the risk functions of δ_{BU}, δ_U and δ_E. When m is small relative to p, the risk functions of δ_{BU} and δ_U are much smaller than the one of δ_{mle} and it is preferable to use δ_U as we can see in Figure 1. However, there are cases $[(m, p) = (1, 1)$, e.g.$]$ where δ_{BU} dominates δ_{mle} but δ_U does not. Several different scenarios can happen when we want to verify if the estimators δ_{BU} or δ_U dominate δ_{mle} as m and p vary. Figure 3 illustrates that the sufficient condition $m \le \sqrt{p}$ for δ_{BU} to dominate δ_{mle} is too restrictive. Similarly, the same can be said about the sufficient condition $m \in \Delta_U(5) = (0, 2.2003]$ for δ_U to dominate δ_{mle} (numerical evaluations of the risk functions, as in Marchand and MacGibbon (2000) give for $p = 5$ that δ_U dominates δ_{mle} if and only if $m \le m_0$, with $m_0 \approx 3.07497$). Finally, the estimator δ_E provides a good

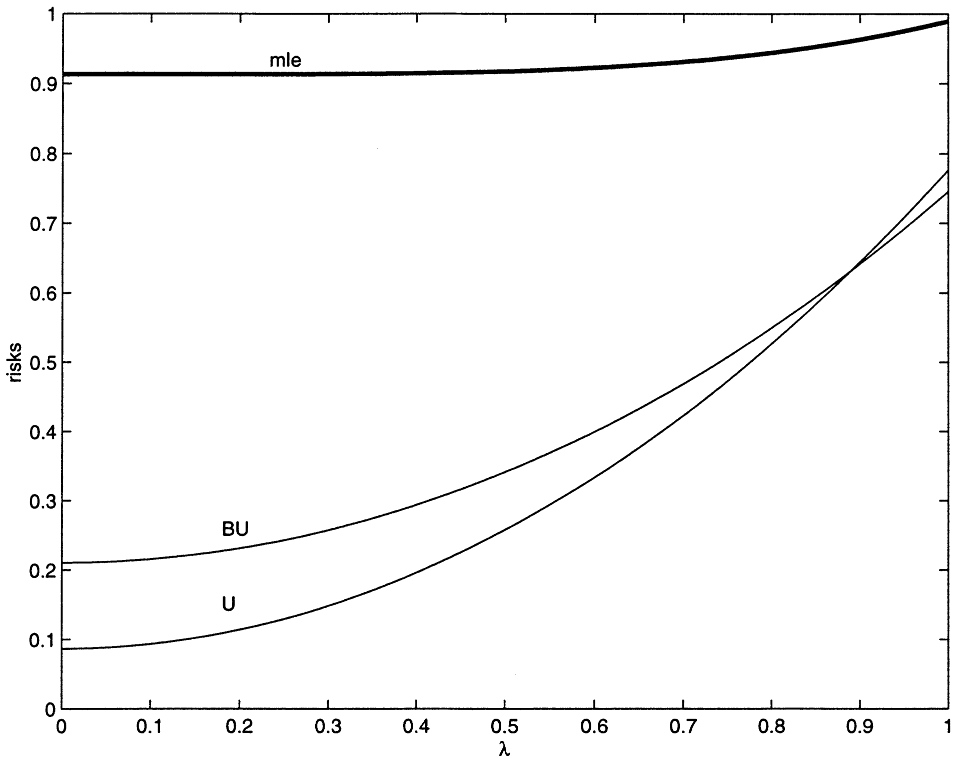

FIG. 1. *Risk of the* mle *versus the risks of other Bayes estimators for* $(m, p) = (1,3)$. mle: *maximum likelihood estimators*, BU: *boundary uniform prior*, U: *fully uniform prior*.

compromise between δ_{BU} and δ_U. We also observed that its risk function tends to be flatter and that δ_E tends to have smaller maximum risk than δ_{mle}.

REMARK 4. Corollary 4 has been applied to the Bayesian estimators δ_U and δ_E only. However, the introduction of Lemma 5 coupled with the numerical evaluations in Table 1 of the benchmarks $m \le \sqrt{p}$, $m \in \Delta_U(p)$ and $m \in \Delta_E(p)$ yield further implications for other Bayesian dominating estimators. We conclude by describing these implications which refer to Bayesian estimators associated with priors of the form $K \exp(-h(\|\theta\|^2))$. Please note that the conditions on h for dominance in (1) and (2) below are very generous, yielding many Bayesian dominating estimators.

1. Whenever $m \le \sqrt{p}$ and $m \in \Delta_U(p)$, then any prior with convex and nonincreasing h will yield a dominating Bayes estimator. To see why this is so, we apply Corollary 4. Shrinkage [condition (ii)] occurs necessarily when $m \le \sqrt{p}$ and Lemma 5 ensures us that the corresponding $g_\pi(m)$ will exceed the corresponding multiplier $g_{\pi_0}(m)$ for the fully uniform Bayes estimator.

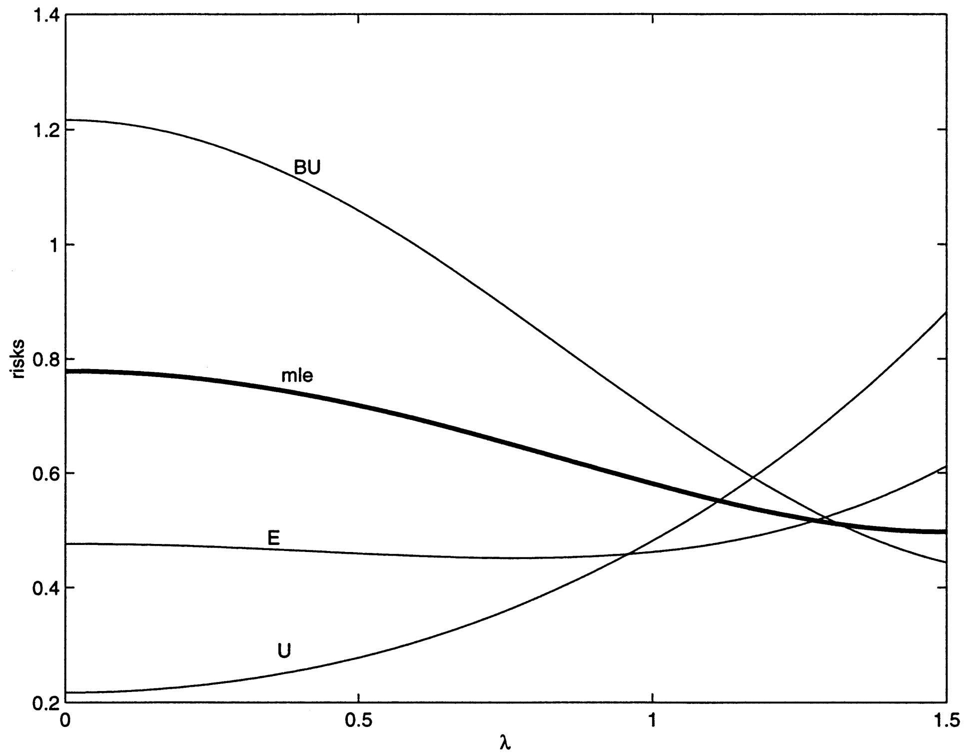

FIG. 2. *Risk of the* mle *versus the risks of other Bayes estimators for* $(m, p) = (1.5,1)$. mle: *maximum likelihood estimators*, BU: *boundary uniform prior*, *U*: *fully uniform prior*, E: *exponential family prior*.

2. Whenever $p \geq 6$, $m > \sqrt{p}$ and $m \in \Delta_U(p)$, then any prior with convex and nonincreasing h, such that $h(t^2) + at^2/2$ is nondecreasing in t^2, will yield a dominating Bayes estimator (this includes δ_E). Here the additional assumption guarantees that the resulting Bayes estimator is a shrinkage estimator [via Lemma 5 and by definition of (a)]. Please note that from Table 1, the conditions $m > \sqrt{p}$ and $m \in \Delta_U(p)$ are incompatible for $p \leq 5$, but compatible for $6 \leq p \leq 10$ (as well, extended computations reveal compatibility for $11 \leq p \leq 30$).
3. Whenever $p \geq 2$, $m > \sqrt{p}$ and $m \in \Delta_E(p)$, δ_E dominates δ_{mle}. Furthermore, if $m \notin \Delta_U(p)$, Lemma 5 tells us that no unimodal orthogonally invariant prior density will satisfy condition (iii) of Corollary 4, since its multiplier g_π will be less than δ_U's multiplier, which is itself not large enough to meet this same condition (iii) of Corollary 4. Please note that from Table 1, the set $\Delta_E(p)$ is empty for $p = 1$, but not for $2 \leq p \leq 10$ [as well, extended computations mentioned in (2) above imply that $\Delta_E(p) \neq \emptyset$ for $11 \leq p \leq 30$].
4. Finally, whenever $m > \sqrt{p}$ and $m \notin \Delta_E(p)$, δ_E does not satisfy the conditions of Corollary 4.

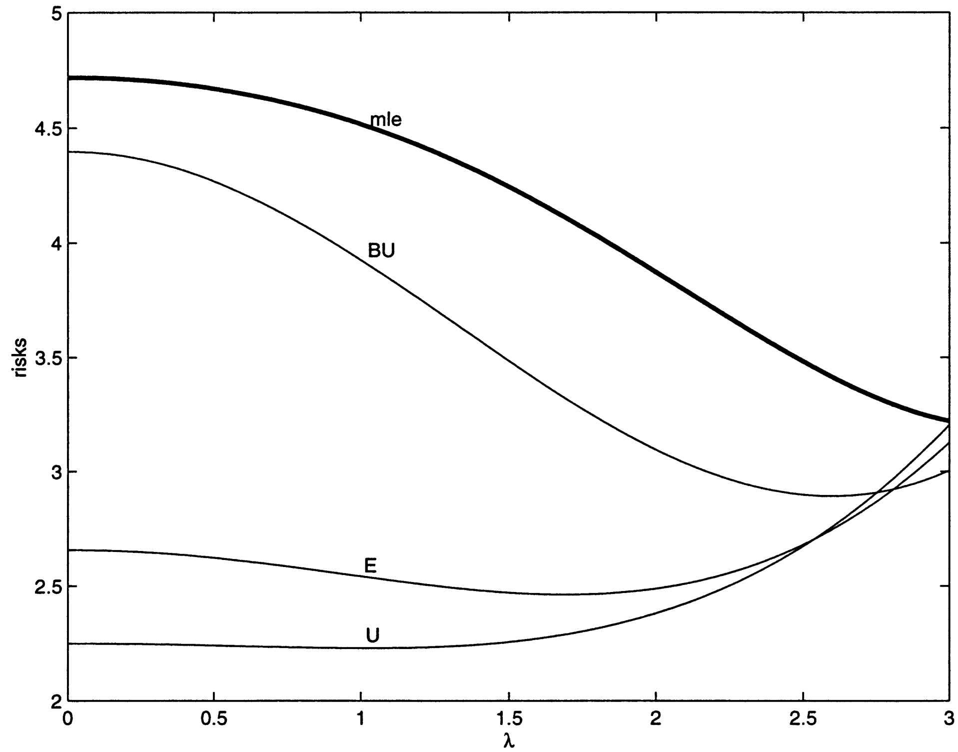

FIG. 3. *Risk of the* mle *versus the risks of other Bayes estimators for* $(m, p) = (3, 5)$. mle: *maximum likelihood estimators*, BU: *boundary uniform prior*, *U*: *fully uniform prior*, E: *exponential family prior*.

APPENDIX

PROOF OF LEMMA 4. (a) Let $0 < \lambda_1 < \lambda_2$. From Lemma 1 we obtain

$$\begin{aligned}\alpha(m, \lambda_1) &= \mathrm{E}_{\theta_1}[\rho_{(p/2)-1}(\lambda_1 R) \mid R > m] \\ &< \mathrm{E}_{\theta_1}[\rho_{(p/2)-1}(\lambda_2 R)] \mid R > m] \\ &\le \mathrm{E}_{\theta_2}[\rho_{(p/2)-1}(\lambda_2 R)] \mid R > m] \\ &= \alpha(m, \lambda_2),\end{aligned}$$

where (i) the first inequality follows from the monotone increasing property of $\rho_{(p/2)-1}$, and (ii) the second inequality follows from the fact that the density of R has monotone likelihood ratio in R. Also, if $0 < \lambda \le m_1 < m_2$ then

$$\begin{aligned}\alpha(m_1, \lambda) = {}& \frac{\mathrm{P}_\theta[R > m_2]}{\mathrm{P}_\theta[R > m_1]}\alpha(m_2, \lambda) + \left(1 - \frac{\mathrm{P}_\theta[R > m_2]}{\mathrm{P}_\theta[R > m_1]}\right) \\ & \times \mathrm{E}_\theta[\rho_{(p/2)-1}(\lambda R) \mid m_1 < R \le m_2] < \alpha(m_2, \lambda),\end{aligned}$$

since the increasing property of $\rho_{(p/2)-1}$ leads to $\rho_{(p/2)-1}(\lambda r) < \alpha(m_2, \lambda)$ on $\{r\colon m_1 < r \le m_2\}$. Finally, from Lemma 2 with $\|\theta\| = m$, we obtain

$$0 \le \bar{\alpha}(m) = \mathrm{E}_\theta[mR\frac{\rho_{(p/2)-1}(mR)}{mR} \mid R > m] \\ \le \frac{m}{p}\mathrm{E}_\theta[R \mid R > m] \to 0 \text{ as } m \to 0$$

and

$$1 \ge \bar{\alpha}(m) \ge \rho_{(p/2)-1}(m^2) \to 1 \text{ as } m \to \infty.$$

(b) When $p = 1$ and m is fixed, straightforward computations give

$$\frac{\partial\beta(m,\lambda)}{\partial\lambda} = \mathrm{E}_\theta\left[\frac{\rho_{-1/2}(\lambda R)}{R}\middle| R \le m\right] \\ + \lambda^2\left\{\mathrm{E}_\theta\left[\frac{1}{R}\middle| R \le m\right] - \mathrm{E}_\theta\left[\frac{\rho_{-1/2}(\lambda R)}{R}\middle| R \le m\right] \right. \\ \left. \times \mathrm{E}_\theta[\rho_{-1/2}(\lambda R) \mid R \le m]\right\} \\ \ge 0,$$

with $|\theta| = \lambda$. When $p > 1$ and m is fixed, calculations using Lemma 3 and the Poisson mixture representation of the noncentral chi-square distribution lead to $\beta(m, \lambda) = 2\mathrm{E}_\lambda[\gamma(K)]$ where K is a random variable with

$$\mathrm{P}_\lambda[K = l] \propto \frac{(\frac{\lambda^2}{2})^l}{l!2^{(p/2)+l}\Gamma((p/2)+l)}\int_0^{m^2} v^{p/2+l-1}e^{-v/2}\,dv, \qquad l = 0, 1, \ldots,$$

and

$$\gamma(k) = k\frac{\int_0^{m^2} u^{(p/2)+k-2}e^{-u/2}\,du}{\int_0^{m^2} v^{p/2+k-1}e^{-v/2}\,dv}.$$

Since K has monotone likelihood ratio in K, it is sufficient to show that γ is nondecreasing to prove that $\beta(m, \lambda)$ is nondecreasing in λ for fixed m. Integration by parts gives

$$\gamma(k) = \frac{k}{\frac{p}{2} + k - 1}\left[\frac{e^{-m^2/2}}{\int_0^{m^2}(\frac{v}{m^2})^{(p/2)+k-1}e^{-v/2}\,dv} + \frac{1}{2}\right]$$

and, from this expression, we see that, when $p > 1$, γ is the product of two nondecreasing functions. Finally, from Lemma 2 with $\|\theta\| = m$, we obtain

$$0 \le \bar{\beta}(m) < \frac{m^2}{p} \quad \text{for } m > 0$$

and

$$\bar{\beta}(m) \geq \mathrm{E}_\theta[\rho_{(p/2)-1}(mR) \mid R \leq m] \to 1 \quad \text{as } m \to \infty.$$

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