The Annals of Probability
1973, Vol. 1, No. 1, 183–185

A CHARACTERIZATION OF THE EXPONENTIAL LAW[1]

BY K. BRUCE ERICKSON AND HARRY GUESS

Stanford University

Under a slight regularity condition we prove that if the spent and residual waiting times at t in a renewal process are independent random variables for one value of $t = t_0$, then the process is Poisson.

In a recent talk, see [1], K. L. Chung gave various characterizations of the exponential law and the related Poisson process. By placing various natural restrictions on the distributions of the spent and residual waiting times associated with a renewal process, he proved that the underlying distribution F of the process is exponential. In a different direction we show, under a slight regularity condition, that if the spent and residual waiting times at t are independent for one $t = t_0$ then F is exponential. For two further characterizations inspired by Professor Chung's talk see [3].

Let F be a probability distribution on $[0, \infty)$ which satisfies

$$F(x) > 0 = F(0)\,, \qquad x > 0 \tag{1}$$

where $F(x) = F\{(-\infty, x]\} = F\{[0, x]\}$, and let $\{S_n\}$, $n \geqq 0$ be a renewal process associated with F: $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$ where the X_n are independent random variables with $P(X_i \leqq t) = F(t)$ (see [2] VI, for basic facts). For each t there is a unique integer N_t determined by $S_{N_t} \leqq t < S_{N_t+1}$ and we call $Y_t = t - S_{N_t}$, $Z_t = S_{N_t+1} - t$, respectively, the spent and residual waiting time at t. If F is an exponential distribution; that is, for some constant λ, $0 < \lambda < \infty$

$$1 - F(x) = e^{-\lambda x}\,, \qquad 0 \leqq x < \infty$$

then Y_t and Z_t are independent random variables for all t. Our purpose is to prove the following converse.

THEOREM. *Assume* (1) *holds for the inter-event distribution of a renewal process* $\{S_n\}$. *If for one* $t_0 > 0$, Y_{t_0} *and* Z_{t_0} *are independent, then* F *is an exponential distribution* (*and hence* Y_t *and* Z_t *are independent for all* t).

PROOF. Let U be the renewal measure associated with F, i.e.,

$$U(t) = U\{[0, x]\} = \textstyle\sum_{n=0}^{\infty} F^{n*}(t)\,, \qquad t \geqq 0\,,$$

where F^{n*} is the distribution of S_n. Then for any $\varepsilon > 0$, $b \geqq 0$ we have

$$P\{Z_{t_0} > b, Y_{t_0} \leqq \varepsilon\} = \textstyle\int_{t_0-\varepsilon}^{t_0} [1 - F(t_0 + b - y)]U\{dy\} \tag{2}$$

by the usual renewal argument (see for example, [2] page 369). Now the points

Received July 28, 1971; revised May 11, 1972.

[1] Research supported in part by National Science Foundation Grant GP 28876, Stanford University.

of increase of U, i.e., all x such that $U\{I\} > 0$ for every open interval containing x, coincide with the closure of the set of all finite sums of points of increase of F. Assumption (1) implies that every interval $(0, \delta)$ contains points of increase of F and it follows that $U\{I\} > 0$ for every nonempty interval I in $[0, \infty)$. This remark together with (1) shows that

$$P\{Y_{t_0} \leqq \varepsilon\} = \int_{t_0-\varepsilon}^{t_0} [1 - F(t_0 - y)]U\{dy\} > 0$$

for every $\varepsilon > 0$. Dividing (2) by $P\{Y_{t_0} \leqq \varepsilon\}$ and using the monotonicity of F we obtain

$$1 - F(b + \varepsilon) \leqq P\{Z_{t_0} > b \mid Y_{t_0} \leqq \varepsilon\} \leqq \frac{1 - F(b)}{1 - F(\varepsilon)} .$$

But Y_{t_0} and Z_{t_0} are independent random variables, hence

(3) $$1 - F(b) = \lim_{\varepsilon \to 0+} P\{Z_{t_0} > b \mid Y_{t_0} \leqq \varepsilon\} = P\{Z_{t_0} > b\} .$$

Now $\alpha = N_{t_0} + 1$ is optional relative to the process $\{S_n\}$. This and (3) imply that the variables $\{X_n'\}_{n=1}^{\infty}$ where $X_1' = Z_{t_0}$, $X_k' = X_{\alpha+k-1}$ for $k \geqq 2$ constitute the inter-arrival times for a renewal process having the distribution F. Thus Z_{t_0}' and Y_{t_0}' are independent where Z_{t_0}' and Y_{t_0}' are the spent and residual times at t_0 associated with the renewal process $S_n' = X_1' + \cdots + X_n'$. The above argument applied to the process $\{S_n'\}$ implies that Z_{t_0}' has the same distribution as X_1' which in turn has the same distribution as X_1. An easy calculation shows that $Z_{t_0}' = Z_{2t_0} = S_{N_{2t_0}+1} - 2t_0$. Hence Z_{2t_0} has the same distribution as X_1. By induction, therefore,

$$F(b) = P\{Z_{mt_0} \leqq b\}$$

for each $m = 1, 2, \cdots, b \geqq 0$. Now F is not arithmetic by (1) so by the renewal theorem ([2] page 370)

(4) $$F(b) = \lim_{m \to \infty} P\{Z_{mt_0} \leqq b\} = \frac{1}{\mu} \int_0^b [1 - F(y)]\, dy$$

where $\mu = EX_1 \leqq \infty$. Since $F(0) = 0$ and $F(b) > 0$ for $b > 0$, (4) implies $0 < \mu < \infty$ and $F(x) = 1 - e^{-x/\mu}$ for all $x \geqq 0$.

REMARK 1. One can give a more analytic proof of the theorem which avoids the key renewal theorem and optional random variables. Here is a brief sketch. The random variables $S_{N_{t_0}} = t_0 - Y_{t_0}$ and $S_{N_{t_0}+1} = Z_{t_0} + t_0$ are independent and have the joint distribution

$$P\{S_{N_{t_0}} \leqq y, S_{N_{t_0}+1} \leqq z\} = \int_{[0, t_0 \wedge y]} [F(z - x) - F(t_0 - x)]U\{dx\} .$$

Hence, if we put $R(z) = \int_0^{t_0} [F(z - x) - F(t_0 - x)]U\{dx\}$ then

$$R(z) \int_0^y [1 - F(t_0 - x)]U\{dx\} = \int_0^y [F(z - x) - F(t_0 - x)]U\{dx\} .$$

But this together with $U\{I\} > 0$ for all intervals $\phi \neq I \subset [0, \infty)$ implies

$$R(z)[1 - F(t_0 - y)] = F(z - y) - F(t_0 - y)$$

for all $0 \leqq y \leqq t_0$, $z \geqq t_0$. Hence $R(z) = F(z - t_0)$ and

(5) $$[1 - F(z)][1 - F(y)] = 1 - F(z + y) \qquad \text{for all} \quad z \geqq 0,\ 0 \leqq y \leqq t_0 .$$

It takes only a few more lines to show that the only bounded solution to (5) which satisfies (1) is $1 - F(x) = e^{-\lambda x}$, $0 < x < \infty$ for some constant λ; $0 < \lambda < \infty$.

REMARK 2. The covering interval $L_t = S_{N_t+1} - S_{N_t} = Y_t + Z_t$ is the sum of the independent random variables Y_t and Z_t in the exponential case. It would be of interest to know to what extent this characterizes the exponential distribution. That is, assume (1) holds and suppose for some t_0 (or all t in an interval, (t_1, t_2), $0 \leqq t_1 < t_2 \leqq \infty$ fixed) that $L_t = Y + Z$ (in distribution) where Y and Z are independent and Y is bounded. Can one then conclude that F is exponential?

REFERENCES

[1] CHUNG, K. L. (1971). The Poisson process as renewal process. *Periodica Mathematica* **1**. To appear.

[2] FELLER, W. (1970). *An Introduction to Probability Theory and Its Applications*, **2** (2nd ed.). Wiley, New York.

[3] JAGERS, P. (1971). Two mean values which characterize the Poisson Process. Technical Report No. 29, Department of Statistics, Stanford Univ.

DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305