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SAMPLE FUNCTIONS OF THE GAUSSIAN PROCESS[1]

BY R. M. DUDLEY

Massachusetts Institute of Technology

This is a survey on sample function properties of Gaussian processes with the main emphasis on boundedness and continuity, including Hölder conditions locally and globally. Many other sample function properties are briefly treated.

The main new results continue the program of reducing general Gaussian processes to "the" standard isonormal linear process L on a Hilbert space H, then applying metric entropy methods. In this paper Hölder conditions, optimal up to multiplicative constants, are found for wide classes of Gaussian processes.

If H is L^2 of Lebesgue measure on R^k, L is called "white noise." It is proved that we can write $L = P(D)[x]$ in the distribution sense where x has continuous sample functions if $P(D)$ is an elliptic operator of degree $> k/2$. Also L has continuous sample functions when restricted to indicator functions of sets whose boundaries are more than $k - 1$ times differentiable in a suitable sense.

Another new result is that for the Lévy(–Baxter) theorem $\int_0^1 (dx_t)^2 = 1$ on Brownian motion, almost sure convergence holds for any sequence of partitions of mesh $o(1/\log n)$. If partitions into measurable sets other than intervals are allowed, the above is best possible: $\mathcal{O}(1/\log n)$ is insufficient.

TABLE OF CONTENTS

0. Introduction. This paper is an attempt to survey what is known about sample functions of Gaussian processes. Emphasis is given to continuity and boundedness properties, as in most of the literature, although other topics are treated. On many topics, new results are presented.

The principal idea is to study one Gaussian process, defined as follows. A sequence $\{X_n\}$ of random variables will be called *orthogaussian* iff they are independent with mean 0 and variance 1, $\mathscr{L}(X_j) \equiv N(0, 1)$. Now let H be a real,

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infinite-dimensional Hilbert space. A Gaussian process L on H will be called *isonormal* iff L is a linear map from H into real Gaussian random variables with $EL(x) \equiv 0$ and $EL(x)L(y) \equiv (x, y)$ for all $x, y \in H$. (The term "isonormal" is due, I believe, to I. E. Segal, who originally called it the "normal distribution" (Segal (1954)).) Specifically if $\{\varphi_n\}$ is any orthonormal basis of H so that for $x \in H$, $x = \sum x_n \varphi_n$, let $L(x) = \sum x_n Y_n$ where the Y_n are orthogaussian.

As elaborated later on, most of the study of sample function continuity and boundedness of Gaussian processes reduces to the study of those sets $C \subset H$ on which the isonormal L has continuous or bounded sample functions, called GC-sets and GB-sets respectively. One measure of the size of a set C in H is the minimal number $N(C, \varepsilon)$ of sets of diameter $\leqq 2\varepsilon$ which cover it. GC-sets and GB-sets cannot quite be characterized in terms of $N(C, \varepsilon)$ (Dudley (1967) Proposition 6.10), but it still seems that such conditions are the most convenient general conditions now available. In Section 1 we give the weakest possible sufficient condition and the strongest possible necessary conditions on $N(C, \varepsilon)$ for the GC- and GB-properties. The best possible necessary condition for GB, namely $\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \log N(C, \varepsilon) < \infty$, is new. The example proving it best possible contradicts one statement in an announcement by Sudakov (1971) while, on the other hand, the best possible necessary condition is proved by methods suggested in Sudakov's earlier note (1969), for which the inequality of Slepian ((1962) Lemma 1 page 468) is fundamental.

Section 2 treats moduli of continuity (Hölder conditions) on sample functions. If $\log N(C, \varepsilon)$ is a reasonably smooth function of ε asymptotically as $\varepsilon \downarrow 0$ (of course, it only changes by jumps!) we find uniform moduli of continuity which are optimal within a constant factor (Theorem 2.6 below). N. Kôno (1970) earlier found some results on moduli of continuity in terms of $N(C, \varepsilon)$.

Section 3 treats processes with stationary increments, which are accessible both by metric entropy methods (3.1) and by Fourier analytic methods (3.2). For these methods, "stationary increments" is fundamentally no worse than "stationary." This is the main point of Section 3, which contains no new hard results.

Given a measure space $(X, \mathscr{S}, \mu)$, the isonormal process L on $L^2(X, \mathscr{S}, \mu)$ is called *μ-noise*. Section 4 treats these processes. If μ is Lebesgue measure on R^k, we have what is called *white* noise W. Sub-section 4.1 shows that if $P(D)$ is an elliptic operator with constant coefficients of order m, then there is a process x_t, $t \in R^k$, with continuous sample functions and $P(D)[x_t] = W$ (in the distribution sense) iff $m > k/2$. In Sub-section 4.1 we restrict W to indicator functions χ_A of sets A whose boundaries are α times differentiable, obtaining continuous sample functions of the process $A \to W(\chi_A)$ if $\alpha > k - 1$ and not if $\alpha < k - 1$, conjecturally also not if $\alpha = k - 1$. Here α is not necessarily an integer. In this respect convex sets behave like sets with exactly twice differentiable boundaries. The proofs of these results depend on another paper (Dudley (1972b)). They answer questions raised by R. Pyke in Oberwolfach, March 1971.

Our final result on noise processes is an extension of the "Lévy–Baxter" theorem, on almost sure quadratic Brownian variation $\int_0^1 (dx_t)^2 = 1$, to partitions of measurable sets with mesh $o(1/\log n)$. In P. Lévy's original theorem the partitions were formed by successive refinement, allowing the mesh to approach 0 arbitrarily slowly; thus our results are independent.

Most of the rest of the paper surveys the literature. In Section 7 are two propositions showing that sample-continuity conditions for non-Gaussian processes are much different from Gaussian ones; sufficient conditions for the non-Gaussian case must be much stronger (7.1), while necessary conditions are weaker (7.2).

The bibliography at the end is no doubt uneven and incomplete. References to it are made with the date after the author's name. For a longer list, up to 1968, see J. Neveu (1968).

DEFINITION. A stochastic process $\{x_t, t \in T\}$ over a topological space T is *sample-continuous* if there is a version of the process with continuous sample functions, i.e., there is a countably additive probability measure on the space of continuous functions on T with the same joint distributions as x_t on finite subsets of T.

A real-valued process $\{x_t, t \in T\}$ will be called *sample-bounded* iff it has a version with bounded sample functions, i.e., there is a countably additive probability measure on the space of all bounded functions from T into X with the same joint distributions as x_t on finite subsets of T.

In this paper, we shall only consider real-valued Gaussian processes, although the results would carry over to complex processes easily. Gaussian measures can be defined on vector spaces, on locally compact Abelian groups (Urbanik (1960), Corwin (1970)) and on suitable homogeneous spaces.

The isonormal process L can be regarded as the only real Gaussian process in view of the theorem that Gaussian distributions are uniquely determined by their means and covariances. Thus let $\{x_t, t \in T\}$ be any real Gaussian process with mean $Ex_t = m_t$. Then $L(x_t - m_t) + m_t$ is another version of the same process, where we take H as the Hilbert space $L^2(\Omega, P)$. We can "forget" the specific joint distributions of $x_t - m_t$ over (Ω, P) and remember only the abstract, geometric Hilbert space structure of the function $t \to x_t - m_t \in H$. Then L will "remember" the joint distributions for us.

A set $C \subset H$ is called a GC-set iff L restricted to C is sample-continuous. C is called a GB-set iff L on C is sample-bounded. The following theorems from Dudley (1967) and Feldman (1971) show how these notions apply.

THEOREM 0.1. *Let $\{x_t, t \in T\}$ be a real Gaussian process over a metrizable space T with $Ex_t = m_t$. Then the following are equivalent.*

(a) *x_t is sample-continuous.*

(b) *$t \to L(x_t - m_t) + m_t$ is sample-continuous.*

(c) *$t \to m_t$ is continuous and $t \to L(x_t - m_t)$ is sample-continuous which implies $t \to x_t - m_t$ continuous $T \to H = L^2(\Omega, P)$.*

THEOREM 0.2. *Theorem* 0.1 *remains true if "continuous" is replaced by "bounded" throughout, and "metrizable space" replaced by "set." Also,* $L(x_t - m_t)$ *is sample-bounded if and only if* $\{x_t - m_t : t \in T\}$ *is a* GB*-set. A set* B *is* GB iff *its closed, convex, symmetric hull in* H *is* GB.

If B is GB then $\bar{L}(B) \equiv \sup\{|L(x)| : x \in A\}$, for any countable dense subset A of B, is a finite random variable which changes only with 0 probability if A is changed. Fernique (1970) and Landau and Shepp (1971) have shown that

$$\Pr(\bar{L}(B) > t) \leqq C \exp\{-t^2/2\alpha^2\}$$

for some $C < \infty$ and $\alpha < \infty$, specifically for any $\alpha > \sup\{\|x\| : x \in B\}$ (Marcus and Shepp (1971), Fernique (1971)). The latter's claim of an error in Landau–Shepp (1971) has been retracted. In any case Fernique's proofs are simpler. An error in an earlier preprint by Shepp alone was found by D. Cohn. Sudakov (1971) says his proof is based on a lemma in the preprint of Shepp. Thus Sudakov's theorem is open to doubt (see also the counter-example to a corollary, Remark after Theorem 1.1 below).

Turning now to GC-sets, we have the following theorem, due in present generality to J. Feldman (1971) with miscellaneous contributions by others (Jain and Kallianpur (1970); Dudley (1967) Theorem 4.6).

THEOREM 0.3. *Let* T *be a compact metric space and* $\{x_t, t \in T\}$ *a real Gaussian process with* $Ex_t \equiv 0$. *Let* $C \equiv \{x_t : t \in T\} \subset L^2(\Omega, P) = H$. *Assume* $t \to x_t$ *is continuous* $T \to H$. *Then the following are equivalent*:

(a) x_t *is sample-continuous.*

(b) C *is a* GC*-set.*

(c) *The closed, convex, symmetric hull of* C *is a* GC*-set.*

(d) *For every* $\varepsilon > 0$, $P(\bar{L}(C) < \varepsilon) > 0$.

(e) *For every orthornomal basis* $\{\varphi_n\}$ *of the linear span of* C, *the series* $\sum (x, \varphi_n)L(\varphi_n)$ *converges uniformly for* $x \in C$ *with probability* 1.

The Karhunen–Loève expansion of a Gaussian process $\{x_t, a \leqq t \leqq b\}$ is a special case of the orthogonal series in (e). That expansion involves the eigenfunctions of the covariance kernel $K(s, t) = Ex_s x_t$, with respect to L^2 of Lebesgue measure. But for processes without stationarity properties, Lebesgue measure is not especially natural, and there is no good reason to single out the Karhunen–Loève expansion from other orthogonal expansions.

Dudley, Feldman and LeCam (1971) have shown, among other things, that the class of GC-sets is stable under vector sum (as is obvious for GB-sets, as well as homothetic invariance of both classes). It is also known that the compact GC-sets are precisely those compact GB-sets which are not "maximal" in a sense defined by compact operators (J. Feldman (1971)).

Fernique (1971) has given rather sharp sufficient conditions for the GB- and GC-properties, as follows. Let (K, μ) be a probability space. Let $G(K, \mu) = \{f : \exists \alpha > 0, \int_K \exp(\alpha f^2(x))\, d\mu(x) < \infty\}$. Then the μ-equivalence classes of

functions in $G(K, \mu)$ form a Banach space for either of the norms

$$N(f) = \sup_{p \geq 1} (p!)^{-\frac{1}{2}p} \|f\|_{L^{2p}},$$
$$N'(f) = \inf \{\alpha > 0 : \int \exp(f^2/\alpha^2) - 1 \, d\mu \leqq 1\}.$$

Here $N \leqq N' \leqq 2N$. Let $G_C(K, \mu)$ be the closure of the bounded functions in $G(K, \mu)$, a proper closed linear subspace in general. Then the dual space $G_C^*(K, \mu)$ consists of the μ-equivalence classes of functions f such that

$$\int |f|(\max (0, \log |f|)^{\frac{1}{2}} \, d\mu < \infty .$$

The following is a reformulation of Théorème 5 of Fernique (1971).

THEOREM (Fernique). *Suppose $C \subset H$ and B is a compact subset of H. Let μ be a finite measure on B. Let f be a real-valued function on $B \times C$ such that the map $x \to f(\cdot, x)$ is bounded from C into $G_C^*(B, \mu)$. Suppose that for all $x, y \in C$,*

$$(x, y) = \int\int (s, t) f(s, x) f(s, y) \, d\mu(s) \, d\mu(t) .$$

Then C is a GB-*set. Furthermore if $x \to f(\cdot, x)$ is continuous from C into G_C^* with weak-star topology, then C is a* GC-*set.*

If in the original Théorème 5 of Fernique (1971), one lets $K = [0, 1]$, $\Gamma(s, t) = 1$ for $s = t$ and 0 for $s \neq t$, then there are difficulties.

Fernique shows that his sufficient condition is also necessary in many cases, but to find f and B still requires ingenuity in different cases.

1. Continuity and boundedness; metric entropy. Let C be a subset of a metric space (S, d). Given $\varepsilon > 0$, let $N(C, \varepsilon)$ be the smallest n such that there exist sets $A_1, \cdots, A_n : C \subset \bigcup_{j=1}^n A_j$, and for each j, $d(x, y) \leqq 2\varepsilon$ for all $x, y \in A_j$. Let $H(C, \varepsilon) = \log N(C, \varepsilon)$. $H(C, \varepsilon)$ is called the *metric entropy* of C, following G. G. Lorentz (1966). Kolmogorov (1956), who invented this notion, called it "ε-entropy," as have most others who used it. On the other hand Posner, Rodemich and Rumsey (1969) use "ε-entropy" to mean infimum of information-theoretic entropy of partitions of C into sets of diameter $\leqq \varepsilon$, for a probability measure on C. Lorentz's term *metric entropy* seems well adapted as a name for the purely metric $H(C, \varepsilon)$ (there is no given or natural probability measure P on C here).

The *exponent of entropy* $r(C)$ is defined by

$$r(C) = \limsup_{\varepsilon \downarrow 0} \log H(C, \varepsilon)/\log (1/\varepsilon) .$$

It is known that C is a GB-set in H if $r(C) < 2$ and not if $r(C) > 2$ (Sudakov (1969), Chevet (1970)). It is also known, however, (Dudley (1967) Proposition 6.10) that we may have $H(E, \varepsilon)/H(Oc, \varepsilon) \to 0$ as $\varepsilon \downarrow 0$ where Oc is a GC-set and GB-set while E is neither, $r(E) = r(Oc) = 2$. Thus inside $r(C) = 2$, there is an ambiguous range where $H(C, \varepsilon)$ does not determine whether C is a GB-set. V. N. Sudakov (1971) has recently announced a necessary and sufficient geometric condition on C for the GB-property, but this condition seems difficult to apply in practice, no proof is yet published, and the result is in doubt. Thus, despite

the ambiguity at $r(C) = 2$, the metric entropy conditions seem still the most useful. Here is a summary of the best possible such conditions.

THEOREM 1.1. *Let C be a compact set in H. Then*

(a) *C is always a* GC-*set if $\int_0^1 H(C, x)^{\frac{1}{2}}\,dx < \infty$, in particular if for some $\delta > 0$ and all small enough ε, $H(C,\varepsilon) \leqq 1/\varepsilon^2|\log \varepsilon|^2[\log|\log \varepsilon|]^2 \cdots [\log \cdots \log|\log \varepsilon|]^{2+\delta}$.*

(b) *There are non-*GB*-sets C with*

$$H(C, \varepsilon) \leqq 1/\varepsilon^2|\log \varepsilon|^2[\log|\log \varepsilon|]^2 \cdots [\log \cdots \log|\log \varepsilon|]^2 .$$

(c) *C is never a* GB*-set if $\limsup_{\varepsilon \downarrow 0} \varepsilon^2 H(C, \varepsilon) = +\infty$, in particular if $H(C, \varepsilon) \geqq \varepsilon^{-2} \log \cdots \log|\log \varepsilon|$ for a sequence of values of $\varepsilon \downarrow 0$.*

(d) *There is a* GB*-set C with $\limsup_{\varepsilon \downarrow 0} \varepsilon^2 H(C, \varepsilon) > 0$.*

(e) *There are* GC*-sets C such that $\sup_{0<\varepsilon\leqq\delta} \varepsilon^2 H(C, \varepsilon) \downarrow 0$ arbitrarily slowly as $\delta \downarrow 0$.*

PROOF. For (a) note that $\int_0^1 H(C, x)^{\frac{1}{2}}\,dx < \infty$ iff $\sum_{n=1}^{\infty} H(C, 2^{-n})^{\frac{1}{2}}/2^n < \infty$. Then the result is stated in Dudley ((1967) Theorem 3.1). (A small error in the proof, noted by J. Neveu, is corrected in the proof of Theorem 2.1 below.) Sudakov ((1971) Theorem 5) has another approach to this fact. A related fact for processes $\{x_t, 0 \leqq t \leqq 1\}$ is due to Delporte (1964) with a neater formulation by Fernique (1964) and another recent proof by Garsia, Rodemich and Rumsey (1970).

For (b) we apply the examples of Fernique ((1964) Théorème 3, Remarque); for a more detailed discussion see Marcus and Shepp (1970), (1971).

For (c) we shall apply Slepian's inequality (1962) as in Sudakov's proof (1969) that $r(C) > 2$ implies C is not GB. I am grateful to S. Chevet, from whose presentation (1970b) the following lemma and proof are adapted, and to E. Giné who gave another exposition. Let Φ be the standard normal distribution function

$$\Phi(t) \equiv (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp(-x^2/2)\,dx \equiv 1 - F(t) .$$

LEMMA 1.2 (Sudakov–Chevet). *Let $a_1, \cdots, a_n \in H$, $1 \leqq M < \infty$, $0 < \varepsilon \leqq 1$, $||a_j|| \leqq M$ for all j, and $||a_i - a_j|| \geqq \varepsilon$ for $i \neq j$. Then*

$$F(1) \Pr\{L(a_j) \leqq 1, j = 1, \cdots, n\} < 2^{-n-1} + (2\pi)^{-\frac{1}{2}} \int_0^{\infty} \exp(-t^2/2)\Phi(Kt/\varepsilon)^n\,dt$$

where $K = (2(M^2 + 1))^{\frac{1}{2}}$.

PROOF. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of H such that $a_1, \cdots, a_n$ belong to the linear span of $e_1, \cdots, e_n$. Let H_{n+1} be the linear span of $e_1, \cdots, e_{n+1}$. Let G be the standard Gaussian measure on H_{n+1}, and $b_i = a_i - e_{n+1}$, $i = 1, \cdots, n$. Then

$$\begin{aligned} F(1)G\{z : (z, a_i) \leqq 1, i = 1, \cdots, n\} &= G\{z : (z, e_{n+1}) \geqq 1, (z, a_i) \leqq 1, i = 1, \cdots, n\} \\ &\leqq G\{z : (z, b_i) \leqq 0, i = 1, \cdots, n\} . \end{aligned}$$

Let $b_{ij} = (b_i, b_j)/\|b_i\|\,\|b_j\|$. Let θ be the angle between b_i and b_j, so that $b_{ij} = \cos\theta$. Then b_{ij} is largest for $i \neq j$ when $\|a_i\| = \|a_j\| = M$. Hence

$$\begin{aligned} b_{ij} &= 2\cos^2(\tfrac{1}{2}\theta) - 1 \\ &\leqq 2(M^2 + 1 - \varepsilon^2/4)/(M^2 + 1) - 1 \leqq 1 - \varepsilon^2/K^2 \leqq 1/(1 + \varepsilon^2/K^2)\,. \end{aligned}$$

Let $f_i = \varepsilon e_i/K - e_{n+1}$, $f_{ij} = (f_i, f_j)/\|f_i\|\,\|f_j\| = 1/(1 + \varepsilon^2/K^2)$ for $i \neq j$, $b_{ii} = f_{ii} = 1$. Thus Slepian's inequality ((1962) Lemma 1 page 468) can be applied to $b_i/\|b_i\|$ and $f_i/\|f_i\|$, giving

$$\begin{aligned} G\{z\colon (z, b_i) \leqq 0,\ i = 1, \cdots, n\} &\leqq G\{z\colon (z, f_i) \leqq 0,\ i = 1, \cdots, n\} \\ &= G\{z\colon \varepsilon(z, e_i)/K \leqq (z, e_{n+1}),\ i = 1, \cdots, n\} \\ &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{\infty} G\{z\colon (z, e_i) \leqq tK/\varepsilon,\ i = 1, \cdots, n\} \exp(-t^2/2)\,dt \\ &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{\infty} \Phi(Kt/\varepsilon)^n \exp(-t^2/2)\,dt \\ &= (2\pi)^{-\frac{1}{2}} \textstyle\int_{0}^{\infty} \exp(-t^2/2)[\Phi^n(Kt/\varepsilon) + \Phi^n(-Kt/\varepsilon)]\,dt \\ &\leqq 2^{-n-1} + (2\pi)^{-\frac{1}{2}} \textstyle\int_{0}^{\infty} \exp(-t^2/2)\Phi^n(Kt/\varepsilon)\,dt\,. \end{aligned}$$

□

To prove (c) from the lemma, we choose $\varepsilon_k \downarrow 0$ such that $\varepsilon_k{}^2 H(C, \varepsilon_k) \geqq k$, $k = 1, 2, \cdots$. In the lemma let $\varepsilon = \varepsilon_k$, $n = N(C, \varepsilon_k)$. Let $F(t) \equiv 1 - \Phi(t)$. Since $(1 - F)^n \leqq e^{-nF}$ it is enough to prove that $N(C, \varepsilon_k)F(s/\varepsilon_k) \to +\infty$ as $k \to \infty$ for each $s > 0$ (then we apply the dominated convergence theorem). We have

$$F(x) \geqq [\exp(-\tfrac{1}{2}x^2)]/6x \qquad \text{for } x \geqq 1\,.$$

Thus it is enough to prove that

$$\lim_{k\to\infty} [H(C, \varepsilon_k) - |\log \varepsilon_k| - c\varepsilon_k{}^{-2}] = +\infty$$

for any $c > 0$, which is clear since $H(C, \varepsilon_k) \geqq k\varepsilon_k{}^{-2}$.

For (d) and (e),* let C_n be a cube of dimension n and side $2/n$ centered at 0. Let $n = n(k) = k^2$ and let the cubes $C_{n(k)}$ lie in orthogonal subspaces for $k = 1, 2, \cdots$.

Let $X_n \equiv \sup\{|L(x)|\colon x \in C_n\} = \sum_{j=1}^{n} |G_j|/n$ where G_j are orthogaussian. Then since $E|G_j| \equiv (2/\pi)^{\frac{1}{2}} < 1$, $EX_n < 1$ and $\sigma^2(X_n) < 1/n$. Thus by Chebyshev's inequality, $\Pr\{X_n \geqq 2\} < 1/n$. Hence $\sum_k \Pr(X_{n(k)} \geqq 2) < \infty$. If C is the closed convex hull of the $C_{n(k)}$, then C is a GB-set.

If $\varepsilon = \varepsilon_k \equiv (5n(k))^{-\frac{1}{2}}$ then

$$N(C, \varepsilon) \geqq N(C_{n(k)}, \varepsilon) \geqq e^{n(k)/8} = \exp(1/40\varepsilon^2)$$

by Lemma 3.6 of Dudley (1972b). Thus (d) is proved. This C is not a GC-set.

Let a_k be any sequence of positive numbers with $a_k \downarrow 0$. Then the convex hull of the sets $a_k C_{n(k)}$ is a GC-set, being a non-maximal GB-set (J. Feldman (1971)). Letting $a_k \downarrow 0$ slowly, we can make $\varepsilon^2 H(C, \varepsilon) \to 0$ as slowly as desired. □

REMARK. Sudakov ((1971) Theorem 5) asserts that if $\int_0^1 \varepsilon^2\,dH(C, \varepsilon) = -\infty$ then C is not a GB-set. The set C constructed in the proof of part (d) above is a counter-example to Sudakov's claim.

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* Note added in proof: J. Neveu suggests a simpler example: $C = \{\varphi_n(\log n)^{-\frac{1}{2}}, n \geqq 2\}$, φ_n orthonormal.

There is an $M < \infty$ such that $H(C, \varepsilon) \leqq M/\varepsilon^2$ for $0 < \varepsilon \leqq 1$, by part (c). Then $\int_0^1 \varepsilon^2\, dH(C, \varepsilon) = -\infty$ by integration by parts, or as follows.

Let $\varepsilon(j) = 1/5^{\frac{1}{2}} j^j$ for $j = 1, 2, \cdots$, and $m = 1/5\varepsilon(j)^2$, an integer. Then

$$\begin{aligned}\int_{\varepsilon(j+1)}^{\varepsilon(j)} \varepsilon^2\, dH(C, \varepsilon) &\leqq -\varepsilon(j+1)^2[H(C_m, \varepsilon(j+1)) - H(C, \varepsilon(j))] \\ &\leqq -\varepsilon(j+1)^2[40^{-1}\varepsilon(j+1)^{-2} - M\varepsilon(j)^{-2}] \\ &\leqq -1/50 \qquad \text{for } j \text{ large enough.}\end{aligned}$$

Hence $\int_0^1 \varepsilon^2\, dH(C, \varepsilon) = -\infty$ while C is a GB-set.

APPLICATION. P. T. Strait (1966) considered Gaussian processes $\{x_t, t \in B\}$ where B is a rectangular solid of sides 2^{-n} in Hilbert space, showing that if

$$E|x_s - x_t|^2 \leqq K/|\log ||s - t|||^{4+\delta}$$

for some $\delta > 0$, $K < \infty$, then $\{x_t, t \in B\}$ is sample-continuous. Here $4 + \delta$ can be improved to $2 + \delta$ using Theorem 1.1 above, since

$$H(B, \varepsilon) = \mathcal{O}(|\log \varepsilon|^2)\,, \qquad \varepsilon \downarrow 0\,.$$

Note that if C is a bounded set in a finite-dimensional Euclidean space then $H(C, \varepsilon) = \mathcal{O}(|\log \varepsilon|)$ so that the exponent $2 + \delta$ can be replaced by $1 + \delta$. Conversely for sets larger than B in the metric entropy sense, the bound on $E|x_s - x_t|^2$ would have to be correspondingly smaller to assure sample continuity.

2. Moduli of continuity.

DEFINITION. A function h from $[0, \infty)$ into $[0, \infty)$ is called a *modulus* (of continuity) iff both the following hold:

(a) h is continuous and $h(0) = 0$;
(b) $h(x) \leqq h(x + y) \leqq h(x) + h(y)$ for all $x, y \geqq 0$.

If is easily seen that h is a modulus iff there is a uniformly continuous function g with $h(t) = \sup\{|g(x + s) - g(x)| : x \in R, |s| \leqq t\}$ for all $t \geqq 0$. If x is restricted to lie in a bounded interval $I = [a, b]$, then there is a continuous g with

$$h(t) = \sup\{|g(x) - g(y)| : |x - y| \leqq t, x, y \in I\}$$

for $0 \leqq t \leqq b - a$; namely $g(x) = h(x - a)$.

2.1. *Uniform moduli.*

DEFINITION. Given a stochastic process $\{x_t; t \in T\}$ over a metric space (T, d), a modulus h will be called a (uniform) *sample modulus* for $\{x_t\}$ iff (for a suitable version of the process) for almost ω there is a $K_\omega < \infty$ such that for all $s, t \in T$, $|x_s(\omega) - x_t(\omega)| \leqq K_\omega h(d(s, t))$.

Since $h \circ d$ is a metric, one can say that the process x is a.s. Lipschitzian for $h \circ d$.

Now suppose x_t is a Gaussian process with mean 0. Then h is a sample modulus for x_t iff $\{(x_s - x_t)/h(||x_s - x_t||) : s, t \in T\}$ is a GB-set in H, the Hilbert space

$L^2(\Omega, P)$ with usual covariance inner product. For the isonormal process L on a set $C \subset H$, a sample modulus will be called simply a *modulus*.

The following theorem, perhaps the main result of this paper, will be proved by the method used for Theorem 3.1 of Dudley (1967). I thank J. Neveu for pointing out an error in that proof requiring some changes to prove the stronger result here. Rather surprisingly, the difficulties concern the case of "small" sets C such as a sequence converging rapidly to a point. For connected or convex sets the earlier proof essentially suffices. A slightly weaker bound was proved by C. Preston (1972).

THEOREM 2.1. *Suppose* $C \subset H$. *Let* $f(h) = \int_0^h H(C, x)^{\frac{1}{2}}\,dx$. *Then* f *is a modulus for* L *on* C.

PROOF. If $f(1) = +\infty$ there is nothing to show, so we assume $f(1) < \infty$. Also we may assume C is infinite. Then $H(C, \varepsilon) \to +\infty$ as $\varepsilon \downarrow 0$. Let $H(C, x) \equiv H(x)$.

We define sequences $\delta_n \downarrow 0$, $\varepsilon_n \downarrow 0$ inductively as follows. Let $\varepsilon_1 = 1$. Given $\varepsilon_1, \cdots, \varepsilon_n$, let

$$\delta_n = 2 \inf \{\varepsilon\colon H(\varepsilon) \leqq 2H(\varepsilon_n)\},$$

$$\varepsilon_{n+1} = \min (\varepsilon_n/3, \delta_n).$$

Then $\varepsilon_n \leqq 3(\varepsilon_n - \varepsilon_{n+1})/2$. Also if $\varepsilon_{n+1} = \delta_n$, then $\int_{\varepsilon_{n+1}}^{\varepsilon_n} H(x)^{\frac{1}{2}}\,dx \leqq 2H(\varepsilon_n)^{\frac{1}{2}}\varepsilon_n$, while otherwise $\varepsilon_{n+1} = \varepsilon_n/3$ and $\int_{\varepsilon_{n+1}}^{\varepsilon_n} H(x)^{\frac{1}{2}}\,dx \leqq 2\varepsilon_{n+1}H(\varepsilon_{n+1})^{\frac{1}{2}}$. Thus we have

$$\tfrac{2}{3}\sum_{m=n}^{\infty} H(\varepsilon_m)^{\frac{1}{2}}\varepsilon_m \leqq \sum_{m=n}^{\infty} (\varepsilon_m - \varepsilon_{m+1})H(\varepsilon_m)^{\frac{1}{2}} \leqq f(\varepsilon_n) \leqq 4\sum_{m=n}^{\infty} \varepsilon_m H(\varepsilon_m)^{\frac{1}{2}}.$$

So the convergence of the above integrals and sums is equivalent and they all converge.

Now for each n we can choose a set $A_n \subset C$ such that for any $x \in C$ there is a $y \in A_n$ with $\|x - y\| \leqq 2\delta_n$, and card $(A_n) \leqq \exp(2H(\varepsilon_n))$. Let $G_n = \{x - y\colon x, y \in A_{n-1} \cup A_n\}$. Then card $(G_n) \leqq 4\exp(4H(\varepsilon_n))$. Let

$$P_n = \Pr\{\max\{|L(z)|/\|z\|\colon z \in G_n\} \geqq 3H(\varepsilon_n)^{\frac{1}{2}}\}.$$

Now we use the standard Gaussian tail estimate; for $T > 0$, $1 - \Phi(T) < \exp(-\frac{1}{2}T^2)/T$. Then for n large enough so that $3H(\varepsilon_n)^{\frac{1}{2}} \geqq 1$, we have

$$P_n \leqq 4\exp\{4H(\varepsilon_n) - 9H(\varepsilon_n)/2\} \leqq 4\exp\{-\tfrac{1}{2}H(\varepsilon_n)\}.$$

Since $H(\varepsilon_{n+2}) \geqq H(\delta_n/3) \geqq 2H(\varepsilon_n)$, $\sum P_n$ is dominated by a geometric series and hence converges. So for almost all ω there is an $n_0(\omega)$ such that for all $n \geqq n_0(\omega)$ we have

$$|L(z)| < 3\|z\|H(\varepsilon_n)^{\frac{1}{2}} \qquad \text{for all } z \in G_n.$$

Now for any $x \in C$ choose $A_n(x)$ with $\|x - A_n(x)\| \leqq 2\delta_n$. Then for almost all ω, $L(A_n(x))(\omega)$ is a Cauchy sequence for all $x \in C$. We choose a version of L such that whenever $n_0(\omega)$ is defined, $L(A_n(x))(\omega)$ converges to $L(x)(\omega)$ as $n \to \infty$ for all $x \in C$.

Now if $n \geqq n_0(\omega)$ and $\varepsilon_{n+1} < \|s - t\| \leqq \varepsilon_n$, $s, t \in C$, we have $\|A_n(s) - A_n(t)\| \leqq \|s - t\| + 4\delta_n$, and

$$\begin{aligned}|L(s) - L(t)|(\omega) &\leqq |L(A_n(s)) - L(A_n(t))|(\omega) + \textstyle\sum_{m=n}^{\infty} (|L(A_m(s)) - L(A_{m+1}(s))| \\ &\quad + |L(A_m(t)) - L(A_{m+1}(t))|)(\omega) \\ &\leqq (\|s - t\| + 4\delta_n)3H(\varepsilon_n)^{\frac{1}{2}} + \textstyle\sum_{m=n}^{\infty} 8\delta_m \cdot 3H(\varepsilon_{m+1})^{\frac{1}{2}}.\end{aligned}$$

Now note that $\delta_m \leqq 6\varepsilon_{m+1}$, so

$$\begin{aligned}|L(s) - L(t)|(\omega) &\leqq 75\|s - t\|H(\|s - t\|)^{\frac{1}{2}} + 144 \textstyle\sum_{m=n+1}^{\infty} \varepsilon_m H(\varepsilon_m)^{\frac{1}{2}} \\ &\leqq 291 f(\|s - t\|).\end{aligned}$$

A modulus valid for small distances, on a totally bounded set, is also valid for all distances, possibly with a larger constant. □

EXAMPLE 2.2 (Brownian motion). For the usual Wiener process x_t, $0 \leqq t \leqq 1$, $\|x_s - x_t\| = |s - t|^{\frac{1}{2}}$. Thus $N(C, \varepsilon)$ is asymptotic to $1/4\varepsilon^2$ as $\varepsilon \downarrow 0$. Hence L on C has a modulus $f(x) = x|\log x|^{\frac{1}{2}}$. So x_t has a sample modulus $\varphi(h) = (h|\log h|)^{\frac{1}{2}}$. It is well known that, up to multiplicative constants which we are neglecting, φ is the best possible uniform sample modulus for Brownian motion (P. Lévy (1937), (1954)). Note that $y = o(f(y))$ as $y \downarrow 0$, and x_t is the convergent sum of independent functions of t with modulus $g(t) \equiv t$. Hence $\limsup_{|s-t| \downarrow 0} |x_s - x_t|/\varphi(\|x_s - x_t\|)$ is almost surely equal to a constant, which P. Lévy ((1954) Théorème 52.2 page 172) proved equal to $2^{\frac{1}{2}}$.

The following is also an easy consequence of Theorem 2.1.

COROLLARY 2.3. *Let $\{x_t, t \in K\}$ be a Gaussian process with mean 0 where (K, d) is a compact metric space. Let g be a modulus such that for all $s, t \in K$, $E(x_s - x_t)^2 \leqq g(d(s, t))^2$. Let $C = \{x_t : t \in K\}$ and assume that for some M, $\alpha < \infty$, $N(C, \varepsilon) \leqq M\varepsilon^{-\alpha}$ as $\varepsilon \downarrow 0$. Let $f(h) = |\log g(h)|^{\frac{1}{2}} g(h)$. Then f is a sample modulus for x_t.*

COROLLARY 2.4. *Suppose for some r, $0 < r < 2$, $\limsup_{\varepsilon \downarrow 0} \varepsilon^r H(C, \varepsilon) < \infty$. Then L on C has the modulus $f(x) = x^{1-\frac{1}{2}r}$.*

PROOF. For some $K < \infty$, $H(C, \varepsilon) \leqq K\varepsilon^{-r}$ for all small enough ε. Thus we have the modulus

$$f(x) = \textstyle\int_0^x t^{-\frac{1}{2}r}\, dt = x^{1-\frac{1}{2}r}/(1 - \frac{1}{2}r).$$ □

DEFINITION. We say f is a *weakly optimal* sample modulus of a process iff for any other modulus g of that process, $g \neq o(f)$, i.e.,

$$\liminf_{x \downarrow 0} f(x)/g(x) < \infty.$$

If the lim inf can be replaced by lim sup, we shall call f *strongly optimal.*

Even a strongly optimal modulus does not settle the finer question: for what moduli f is there for almost all ω a $\delta(\omega) > 0$ such that for $d(s, t) < \delta$, $|x_s(\omega) - x_t(\omega)| \leqq f(d(s, t))$. Such a modulus f is said to belong to the (uniform) *upper class*

$\mathscr{U}$ of the process x_t, while other moduli belong to the *lower class* $\mathscr{L}$. Usable, necessary and sufficient conditions for $f \in \mathscr{U}$ have been found for certain processes with stationary increments on a compact subset of R^k; see N. Kôno ((1970) Theorem 3).

DEFINITION. Given $0 < \delta < 1$, a function J is called *δ-slowly varying* as $x \downarrow 0$ iff $\lim_{x \downarrow 0} J(\delta x)/J(x) = 1$. Examples of δ-slowly varying functions include $J(x) = c|\log x|^{\alpha}(\log|\log x|)^{\beta} \cdots (\log \cdots \log|\log x|)^{\zeta}$ for any constants $c, \alpha, B, \cdots, \zeta$.

LEMMA 2.5. *If $H(x) = x^{-r}J(x)$ where $J \geqq 0$, $0 \leqq r < 2$, J is δ-slowly varying, $0 < \delta < 1$, and $H(x) \geqq H(y)$ for $0 < x \leqq y$, then for some $M < \infty$, $tH(t)^{\frac{1}{2}} \leqq \int_0^t H(x)^{\frac{1}{2}}\,dx \leqq MtH(t)^{\frac{1}{2}}$ for all small enough $t > 0$.*

PROOF. We have $tH(t)^{\frac{1}{2}} \leqq \int_0^t H^{\frac{1}{2}}$ by the monotonicity of H. Conversely, given $\varepsilon > 0$ such that $(1 + \varepsilon)\delta^{1-\frac{1}{2}r} < 1$, we choose $\gamma > 0$ such that $J(\delta x) \leqq (1 + \varepsilon)J(x)$ for $0 < x \leqq \gamma$. Then for $t \leqq \gamma$

$$\begin{aligned}\int_0^t H^{\frac{1}{2}} &= \sum_{n=0}^{\infty} \int_{\delta^{n+1}t}^{\delta^n t} H^{\frac{1}{2}} \\ &\leqq (1 - \delta)t(1 + \varepsilon)\delta^{-\frac{1}{2}r}H(t)^{\frac{1}{2}} \sum_{n=0}^{\infty} [(1 + \varepsilon)\delta^{1-\frac{1}{2}r}]^n \leqq MtH(t)^{\frac{1}{2}}\end{aligned}$$

for some $M < \infty$. □

We abbreviate $H(C, x)$ to $H(x)$ or H below, so long as the set C intended is clear.

THEOREM. 2.6. *Let $C \subset H$. Assume that $r(C) > 0$, $\int_0^1 H^{\frac{1}{2}} < \infty$, and for some $M < \infty$,*

$$f(t) \equiv \int_0^t H^{\frac{1}{2}} \leqq MtH(t)^{\frac{1}{2}}$$

for t small enough. Then f is a weakly optimal modulus for L on C. If for some $\delta > 0$, $H(\varepsilon) \leqq (1 - \delta)H(\delta\varepsilon)$ for ε small enough, then f is strongly optimal.

Before proving the above theorem, note that its hypotheses follow from those of Lemma 2.5 if $r > 0$.

DEFINITION. A set $C \subset H$ is called a GL-set iff the function $I(t) = t$ is a modulus for L on C.

PROPOSITION 2.7. *If $\limsup_{x \downarrow 0} H(C, x)/|\log x| = +\infty$, then C is not a* GL-*set.*

PROOF. I claim that $\limsup_{x \downarrow 0} N(C, \frac{1}{2}x)/N(C, x) = +\infty$. Otherwise, for some $M < \infty$, $N(C, 2^{-n}) \leqq M^n$ for all n, and for some k, $K < \infty$, $N(C, \varepsilon) \leqq K\varepsilon^{-k}$ for $0 < \varepsilon \leqq 1$, contrary to hypothesis. Given $\varepsilon > 0$, we choose a covering of C by a minimal number of sets $A_1, \cdots, A_{N(C,\varepsilon)}$, each of diameter $\leqq 2\varepsilon$. Let $x_1, \cdots, x_k$ be a maximal number of points such that $\|x_i - x_j\| > \varepsilon$ for $i \neq j$. Then $N(C, \varepsilon) \leqq k = k(\varepsilon)$ since the balls of radius ε and centers x_i cover C. One of the A_i contains at least $k(\varepsilon)/N(C, 2\varepsilon)$ of the x_j. Hence, for $n = 1, 2, \cdots$, there exist $\varepsilon_n > 0$ and a set $B_n \subset C$ with $\operatorname{card}(B_n) \geqq n$ and $\varepsilon_n < \|x - y\| \leqq 4\varepsilon_n$ for $x \neq y \in B_n$. Thus the union of the sets $\{(x - y)/4\varepsilon_n : x, y \in B_n\}$ is not totally bounded and hence not GB. Thus, $C_N \equiv \{(x - y)/\|x - y\| : x, y \in C\}$ is not GB

since if it were, the convex hull of $C_N \cup \{0\}$ would be GB. Thus C is not a GL-set. □

After this proof a little more will be said about GL-sets.

PROOF OF THEOREM 2.6. By Theorem 2.1, f is a modulus of L on C. Suppose there is another modulus $g = o(f)$. Then $g(t) = o(tH(t)^{\frac{1}{2}})$ as $t \downarrow 0$.

We have two cases:

Case I. $\limsup_{x \downarrow 0} x/g(x) > 0$, i.e., $\beta \equiv \liminf_{x \downarrow 0} g(x)/x < \infty$. Then, since g is a modulus, if $(n-1)x \leqq y \leqq nx$, n an integer, then $g(y) \leqq g(nx) \leqq ng(x)$, and $g(y)/y \leqq ng(x)/(n-1)x$. Letting $x \downarrow 0$, $n \to \infty$, gives $g(y)/y \leqq \beta$, so that the function $I(x) \equiv x$ is a modulus for L on C, contradicting Proposition 2.7 since $r(C) > 0$.

Case II. $\lim_{x \downarrow 0} x/g(x) = 0$. Then $x/g(x)$ extends to a continuous function on $[0, \infty)$.

If $\lim_{\varepsilon \downarrow 0} H(\varepsilon)/H(\frac{1}{2}\varepsilon) = 1$, then for $0 < \alpha < 1$ there is a $K < \infty$ such that $H(2^{-n-1}) \leqq K/(1-\alpha)^n$ for all n. Then for $1 > \varepsilon > 0$, $2^{-n-1} \leqq \varepsilon < 2^{-n}$ for some n, and

$$H(\varepsilon) \leqq K \exp\{|\log(1-\alpha)|\,|\log \varepsilon|/\log 2\},$$

a contradiction for $|\log(1-\alpha)|/\log 2 < r$. Thus for some $\alpha > 0$,

$$\liminf_{\varepsilon \downarrow 0} H(2\varepsilon)/H(\varepsilon) < 1 - \alpha .$$

Choose $\varepsilon_n \downarrow 0$ such that

$$H(2\varepsilon_n)/H(\varepsilon_n) < 1 - \alpha .$$

As in the proof of Proposition 2.7, choose sets $B_n \subset C$ such that $\varepsilon_n < \|x - y\| \leqq 4\varepsilon_n$ for $x \neq y \in B_n$ and

$$\begin{aligned} \operatorname{card}(B_n) \geqq N(C, \varepsilon_n)/N(C, 2\varepsilon_n) &= \exp\{H(C, \varepsilon_n) - H(C, 2\varepsilon_n)\} \\ &\geqq \exp\{\alpha H(C, \varepsilon_n)\} . \end{aligned}$$

Let $D_n = \{(x-y)/g(\varepsilon_n) : x, y \in B_n\}$. Then

$$N(D_n, \varepsilon_n/2g(\varepsilon_n)) \geqq \exp\{\alpha H(C, \varepsilon_n)\} .$$

For any $K < \infty$ we have for n large enough $H(C, \varepsilon_n) \geqq Kg(\varepsilon_n)^2/\alpha\varepsilon_n^2$. Letting $\kappa_n = \varepsilon_n/2g(\varepsilon_n)$ we have

$$H(D_n, \kappa_n) \geqq K\kappa_n^{-2} .$$

Thus by Theorem 1.1 (c), the union of all D_n is not a GB-set. Thus $C_g \equiv \{(x-y)/g(\|x-y\|) : x, y \in C\}$ is not a GB-set, since the convex hull of $4C_g$ includes each D_n. Thus g cannot be a modulus of L on C.

The above argument goes through as well for δ in place of $\frac{1}{2}$ if $0 < \delta < 1$. Then if $\limsup_{\varepsilon \downarrow 0} H(\varepsilon)/H(\delta\varepsilon) < 1$, we could choose $\varepsilon_n \downarrow 0$ to satisfy $g(\varepsilon_n)/\varepsilon_n H(C, \varepsilon_n)^{\frac{1}{2}} \downarrow 0$ and the same proof shows that f is a strongly optimal modulus. □

In one sense, Proposition 2.7 is best possible since any bounded open set C in

a k-dimensional linear subspace is a GL-set with $\limsup_{\varepsilon \downarrow 0} H(C, \varepsilon)/|\log \varepsilon| = k$. On the other hand if $C = \{x_t : 0 \leqq t \leqq 1\}$ for the usual Wiener process x_t, then the lim sup is 2 while C is not GL (cf. Example 2.2 above). Thus, the behavior of $N(C, \varepsilon)$ does not determine whether C is GL. More relevant is $N(D, \varepsilon)$, where $D = \{(x - y)/\|x - y\| : x, y \in C\}$. If C is convex and infinite dimensional, then D is not totally bounded and not GB. But the GL-property (unlike the GB- or compact GC-property) is not preserved by taking the convex hull. Here are some infinite-dimensional GL-sets.

PROPOSITION 2.8. *Suppose* $\{x_t, 0 \leqq t \leqq 1\}$ *is a Gaussian process with mean* 0, *and* $x_t = \int_0^t y_s\, ds$, *where* y_s *is another such process and the integral is a Bochner integral in* H. *Assume that for some* $\alpha > 0$ *and* $M < \infty$, $\|y_s - y_t\| \leqq M/|\log|s - t||^{\frac{1}{2}+\alpha}$ *for* $0 \leqq s, t \leqq 1$. *Finally assume that for some* $\delta > 0$, $\|x_s - x_t\| \geqq \delta|s - t|$. *Let* $C = \{x_t : 0 \leqq t \leqq 1\}$. *Then* C *is a* GL-*set*.

PROOF. Since $t \to x_t$ is a bi-Lipschitz map it suffices to prove that $I(t) = t$ is a sample modulus for x_t on $[0, 1]$. The hypothesis implies that y_t is sample-continuous. Then we can write

$$x_t(\omega) = \int_0^t y_s(\omega)\, ds$$

and the result follows. □

Clearly, it is not enough for $t \to y_t$ to be continuous $T \to H$. The above proposition could be extended, replacing $[0, 1]$ by finite-dimensional sets and using Fréchet derivatives; the condition $\|x_s - x_t\| \geqq \delta|s - t|$ need only hold locally, and no doubt further improvements are possible.

2.2. *Processes on* R^k. Let $\{x_t, t \in K\}$ be a Gaussian process where K is a compact set in R^k. Let $C = \{x_t : t \in K\} \subset L^2(\Omega, P)$. For $\varepsilon, h > 0$ let

$$\Psi(h) = \sup\{E|x_s - x_t|^2 : |s - t| \leqq h\}^{\frac{1}{2}}, \tag{2.9}$$

$$\eta(\varepsilon) = \sup\{h > 0 : \Psi(h) \leqq \varepsilon\}.$$

Then Ψ will be called the (uniform) QM *modulus* of $\{x_t, t \in K\}$. We assume x_t is continuous in quadratic mean (CQM), i.e., $\Psi(h) \downarrow 0$ as $h \downarrow 0$. Unless $\Psi \equiv 0$, $\eta(\varepsilon)$ is defined and non-decreasing for $\varepsilon > 0$ and small enough.

For some $M < \infty$ we have $N(K, \delta/2) \leqq M/\delta^k$ for $0 < \delta \leqq 1$. Hence $N(C, \Psi(\delta)) \leqq M/\delta^k$, $N(C, \varepsilon) \leqq M/\eta(\varepsilon)^k$. For $k = 1$ the following was shown by other methods in Garsia, Rodemich and Rumsey (1970), as far as β is concerned, and later extended to $k > 1$ by A. Garsia (1971).

THEOREM 2.10. *Let* $\{x_t, t \in K\}$ *be any* CQM *Gaussian process on a compact* $K \subset R^k$, *with* QM *modulus* Ψ, *mean* 0, *and* η *from* (2.9). *Let*

$$\alpha(h) = \int_0^{\Psi(h)} |\log \eta(x)|^{\frac{1}{2}}\, dx,$$

$$\beta(h) = \int_0^h |\log y|^{\frac{1}{2}}\, d\Psi(y),$$

$$\gamma(h) = \int_0^h \Psi(y)\, dy/y|\log y|^{\frac{1}{2}},$$

$$\kappa(h) = \Psi(h)|\log h|^{\frac{1}{2}}.$$

Then α, β and $\gamma + \kappa$ are always sample moduli of x_t; κ is a modulus if $\log \eta(x) \equiv x^{-r}J(x)$ *where* $0 \leqq r < 2$ *and for some* δ, $0 < \delta < 1$, J *is* δ*-slowly varying.*

PROOF. We represent $t \to x_t(\omega)$ by the composition $t \to x_t(\cdot) \to L(x_t(\cdot))(\omega)$, L isonormal. If f is a modulus for L on C, then $f \circ \Psi$ is a sample modulus for x_t. Thus by Theorem 2.1, α is a sample modulus for x_t.

To get β we substitute $x = \Psi(y)$. Note that $\eta(\Psi(y)) \geqq y$, so as $y \downarrow 0$, $|\log \eta(\Psi(y))|^{\frac{1}{2}} \leqq |\log y|^{\frac{1}{2}}$.

Then to get $\kappa + \gamma$ we use (Riemann–Stieltjes) integration by parts, yielding

$$\beta(h) \leqq \kappa(h) + \gamma(h) .$$

Lemma 2.5 proves the final statement. □

The modulus κ (with appropriate best possible constant multiples for the upper class, which we do not consider here) was first found for Brownian motion by P. Lévy ((1937), (1954) pages 168–172) and then for increasingly more general processes by Z. Ciesielski (1961), M. Marcus (1968 ff.) and N. Kôno (1970). The latter authors have also found other moduli under other conditions and further information not described here. The moduli in Theorem 2.10 are not always optimal (Marcus (1972)), but Theorem 2.6 says κ is optimal under the conditions on η stated at the end of Theorem 2.10 if $r > 0$.

M. Marcus ((1971) (4.4)) gives examples of Gaussian processes X, Y_1 and Y_2 such that

$$E(Y_1(t + h) - Y_1(t))^2 \leqq E(X(t + h) - X(t))^2 \leqq E(Y_2(t + h) - Y_2(t))^2$$

where Y_1 and Y_2 have the same weakly optimal sample modulus $[h|\log h|]^{\frac{1}{2}}$ and X has the weakly optimal modulus $[h \log|\log h|]^{\frac{1}{2}}$. Thus the modulus of the covariance map $t \to x_t \in H$ does not determine the optimal sample moduli. This is not surprising since we have seen that metric entropy does not determine the GL-, GB or GC-properties. See also Sub-section 3.1 below.

2.3. *Local moduli.* Now we consider $|x_s - x_t|$ as t approaches a fixed point s. A modulus f will be called a *local sample modulus* for x_t as s iff for almost all ω there is a $K_\omega < \infty$ such that $|x_s(\omega) - x_t(\omega)| \leqq K_\omega f(s - t|)$ for all t (in some neighborhood of s, hence for any compact set M, where K_ω also may depend on M).

Clearly any uniform sample modulus is also a local one everywhere. A modulus f is a local sample modulus of x_t at s iff $\{(x_t - x_s)/f(|s - t|) : t \in U\}$ is a GB-set for some open set $U \ni s$.

The proof of Theorem 2.6 above will show that the function $f(u) = uH(C, u)^{\frac{1}{2}}$ is an optimal local as well as uniform modulus for L on C, if in addition to the hypotheses of Theorem 2.6, we can choose a fixed point $x \in B_n$ for all large enough n (then the set D_n, also in that proof, can be formed using the fixed x). For $C = \{x_t : t \in U\}$ for example, where x_t has stationary increments (see Section

3) and U is open in R^k, local behavior at all points is the same so we can choose such an x.

In Theorem 2.6 we have $r(C) > 0$. For $r(C) = 0$ it is well known that we may have a local modulus smaller than the uniform one, with $|\log h|^{\frac{1}{2}}$ replaced by $(\log|\log h|)^{\frac{1}{2}}$. This is classical (due to Khinchin) for Brownian motion and has been extended to many other processes with stationary increments (see Section 3 below) by several workers including M. Marcus (1968 ff.), Sirao and H. Watanabe (1970), and N. Kôno (1970).

Sample behavior of x_t as $t \to \infty$ comes under the same rubric as behavior for $t \to 0$ for general processes, although not necessarily for processes with stationary increments.

For certain processes such that $E|x_s - x_t|^2$ behaves like $|s - t|^\alpha$, $0 < \alpha \leqq 2$, as $|s - t| \downarrow 0$, and $Ex_s x_t \to 0$ fast enough as $|s - t| \to \infty$, H. Watanabe (1970) found a necessary and sufficient condition on an increasing f so that $\limsup_{t\to\infty} x_t/f(t) \leqq 1$. (In his case the smallest such f are of the form $(Ex_t^2|\log t|)^{\frac{1}{2}}$ + terms of lower order of growth.) Such complete criteria have not yet been found for sets C in H, but it should be possible to get them for sets satisfying some good enough conditions. Such criteria cannot follow from Theorem 1.1 above although they might include conditions on metric entropy.

One can also consider local behavior along a given sequence $t_n \to s$ and ask how slowly t_n should converge to s for an optimal local modulus to be optimal also along that sequence. Here there are results for Brownian motion (Dudley (1972b)) with possibilities for generalization.

2.4. *Peano curves.* Consideration of stochastic processes with second moments as curves in Hilbert space goes back at least to Kolmogorov (1940). It is known that any suitably connected compact metric space is a continuous image of the unit interval. I shall give a specific construction for a compact, convex subset C of a Banach space, so that any compact convex GC-set or GB-set is represented as the range of a Gaussian process continuous in probability on $[0, 1]$, with some bound on the rate at which $E|x_s - x_t|^2 \to 0$ as $|s - t| \to 0$.

We construct a Peano curve C inductively. Choose $p \in C$ and let $f_0(t) = p$, $0 \leqq t \leqq 1$. At the nth stage we shall have a continuous function f_n from $[0, 1]$ into C. Here $[0, 1]$ is divided into k_n subintervals $I(n, 1), \cdots, I(n, k_n)$. These will be either "finished" or "unfinished" as defined below. On the closure of each interval $I(n, j)$, f_n will be linear. Let $A(n)$ be a set of minimal cardinality such that for each $x \in C$, $\|x - y\| \leqq 2^{1-n}$ for some $y \in A(n)$. Then card $A(n) \leqq N(C, 2^{-n})$. The values of f_n at the endpoints of the $I(n, j)$ will be precisely $\bigcup_{1\leq r\leq n} A(r)$.

Given f_n and an interval $I(n, j) = [a, b]$, we let $f_{n+1} \equiv f_n$ on $I(n, j)$ if $I(n, j)$ is finished; it then becomes one of the finished intervals $I(n + 1, k)$. If $I(n, j)$ is unfinished, we let $f_{n+1}(a) = f_{n+1}((a + b)/2) = f_n(a)$, $f_{n+1}(b) = f_n(b)$ and call $[(a + b)/2, b]$ a finished interval $I(n + 1, r)$. At the first step, $[0, 1]$ is unfinished.

We divide $[a, (a+b)/2]$ into at most $N(C, 2^{-n-1}) + 1$ equal unfinished subintervals $I(n+1, s)$. At the endpoints of these intervals, except for a and $(a+b)/2$, the values of f_{n+1} will include all points x of A_{n+1} such that $\|x - f_n(a)\| \leqq 2^{-n}$. This completes the inductive construction. Clearly the f_n converge uniformly to a continuous function f from $[0, 1]$ onto C, with $\|f - f_n\| \leqq 2^{2-n}$.

The intervals $I(n, j)$ have length at least $1/2^n \prod_{s=1}^{n} [N(C, 2^{-s}) + 1] = \delta_n$. On each unfinished or newly finished interval, the value of f changes by at most 2^{4-n}. By induction, for s, t in a previously finished interval

$$|f(s) - f(t)|/|s - t| \leqq 2^{4-(n-1)}/\delta_{n-1} \leqq 2^{4-n}/\delta_n .$$

Thus whenever $|s - t| \leqq \delta_n$, we have $|f(s) - f(t)| \leqq 2^{5-n}$.

For example, if $H(C, \varepsilon)$ is asymptotic to ε^{-r} for some $r > 0$, then for some constant K, $\delta_n \geqq \exp(-K \cdot 2^{nr})$, and a modulus of continuity of f is $h(\delta) \equiv |\log \delta|^{-1/r}$, which is best possible in this case.

3. Stationary increments. A process x_t, $t \in R^k$, is said to have *stationary increments* iff the joint distributions of the random variables $\{x_{t_2} - x_{t_1}, \cdots, x_{t_{2n}} - x_{t_{2n-1}}\}$ are unchanged if some vector $t \in R^k$ is simultaneously added to all the t_j, for any $t_1, \cdots, t_{2n} \in R^k$.

The additive group R has many automorphisms which are highly pathological. Thus the assumption of stationarity (of increments) can be expected to yield convenient results only under further conditions.

We call a process $\{x_t\}$ *continuous in quadratic mean* (CQM) iff for all s,

$$\lim_{t \to s} E|x_s - x_t|^2 = 0 .$$

A Gaussian process is CQM iff it is continuous in probability, as is well known.

If a stationary Gaussian process $\{x_t(\omega)\}$ is measurable jointly in t and ω, then its characteristic function φ, with $Ex_0 x_h = \varphi(h)$, is measurable and hence continuous (Loève (1963) page 209) so that x_t is CQM. Since joint measurability is a rather minimal regularity assumption, we consider only CQM processes in this section. Then the variance $\sigma^2(h) = E|x_{t+h} - x_t|^2 \to 0$ as $h \to 0$.

3.1. *Moduli of continuity.* If x_t is any Gaussian process and $C = \{x_t\} \subset L^2(\Omega, P)$, then a sample modulus f for L on C and the QM modulus Ψ of x_t always give a sample modulus $f \circ \Psi$ for x_t (cf. Sub-section 2.2 above). In general, however, Ψ may reflect local rather than uniform behavior of $\{x_t\}$, so that $f \circ \Psi$ may well not be optimal even if f is optimal for L on C. For processes with stationary increments the QM modulus Ψ is the same at all t so that there seems a better chance for $f \circ \Psi$ to be optimal, especially if σ is non-decreasing for small h.

In fact, the modulus κ in Theorem 2.10, $\kappa(h) = \sigma(h)|\log h|^{\frac{1}{2}}$ both as a uniform and for $r(C) > 0$ as a local modulus, has been found for many processes with stationary increments by M. Marcus (1968 ff.), Sirao and Watanabe (1970), and N. Kôno (1970), all of whom got further, more precise results which will not be restated here.

Khinchin's local law

$$\limsup_{t\to\infty} x_t/\sigma(t)[2\log\log t]^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

has been extended from Brownian motion to certain other processes with stationary increments; see S. Orey (1971), who used estimates of J. Pickands III (1967). T. Sirao (1960) found the upper and lower classes for Lévy's Brownian motion on R^k with $E|x_s - x_t|^2 = |s - t|$, locally at 0 and ∞.

S. M. Berman (1964) proved that if X_n is a stationary Gaussian sequence with $EX_n = 0$, $EX_n^2 = 1$, and $\lim_{n\to\infty}(\log n)EX_1X_n = 0$, then $\max(X_1, \cdots, X_n) - (2\log n)^{\frac{1}{2}} \to 0$ in probability. For orthogaussian X_j this is a theorem of Gnedenko. Berman (1962), in a closing remark, says that stationarity can be replaced by

$$\lim_{n\to\infty} nEX_iX_n = 0 \qquad \text{for all } i.$$

Cramér (1962) proved the analogous conclusion for real continuous-parameter stationary processes with a spectral density f of bounded variation with

$$\int_0^\infty \lambda^2(\log(1+\lambda))^a f(\lambda)\,d\lambda < \infty$$

for some $a > 1$. Such processes have a.s. continuously differentiable sample functions. M. G. Shur (1965) improved "in probability" to "with probability one."

More refined properties of sample functions, such as the Hausdorff dimension of level sets and local times, have been treated first for Brownian motion (see Itô and McKean (1965)) and later for other processes with $E(X_t - X_0)^2 \sim C|t|^\alpha$, $0 < \alpha < 2$ (S. M. Berman (1970), S. Orey (1970)).

3.2. *Fourier analysis.* A measure $\mu \geqq 0$ on $R^k \sim \{0\}$ will be called a *Lévy–Khinchin* (LK) measure iff

$$\int_{|x|\leq 1} |x|^2\,d\mu(x) + \mu\{|x| \geqq 1\} < \infty,$$

where $|x| = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$. (LK measures arise in the well-known formula for infinitely divisible characteristic functions.)

The following theorem was first stated by Kolmogorov (1940) for $k = 1$. In higher dimensions I do not know an explicit reference for it, although it is at least close to known results, e.g., A. M. Yaglom (1957), (1962).

THEOREM 3.1 (Kolmogorov–Yaglom *et al.*). *For any* CQM *complex-valued process* $\{x_t, t \in R^k\}$ *with mean* 0 *and stationary increments there is a unique* LK *measure* μ *on* $R^k \sim \{0\}$ *and a nonnegative definite real symmetric operator* A *on* R^k *such that for any* $s, t \in R$,

$$E(x_s - x_0)(x_t - x_0)^- = \int (e^{i\lambda\cdot s} - 1)(e^{-i\lambda\cdot t} - 1)\,d\mu(\lambda) + As\cdot t. \tag{3.2}$$

If x_t *is real-valued, then* μ *is symmetric:* $\mu(B) \equiv \mu(-B)$. *Conversely given any such* μ *and* A *there is a complex Gaussian* CQM *process* x_t *with stationary increments such that* (3.2) *holds. If* μ *is symmetric then there is such a real Gaussian* x_t. *For*

any random variable y such that $x_t + y$ is a Gaussian process, $x_t + y$ will also satisfy (3.2), *but no other Gaussian processes will satisfy* (3.2) *for a given μ and A.*

PROOF. Since $\{x_t\}$ is CQM, it has a jointly measurable version for which by Fubini's theorem $\int_K |x_t(\omega)|\, dt < \infty$ almost surely for any compact K. Thus x_t defines a random Schwartz distribution $X = [x_t]$ which has a gradient in the distribution sense $\operatorname{grad} X \equiv (\partial X/\partial t_1, \cdots, \partial X/\partial t_k)$, a stationary (also called homogeneous) Gaussian generalized random field as treated by K. Itô (1956) and A. M. Yaglom (1957). (Note that $\int x_t\, \partial f/\partial t_j\, dt$ is a limit of linear combinations of increments of x_t.)

Let $\mathscr{D}$ denote the space of C^∞ complex-valued functions on R^k with compact support. We need the following known fact which I shall prove for completeness.

LEMMA. *If $f \in \mathscr{D}$ and $\int_{R^k} f\, dt = 0$ then for some $f_1, \cdots, f_k \in \mathscr{D}$, $f = \sum_{1 \le j \le k} \partial f_j/\partial t_j$.*

PROOF. We use induction on k. For $k = 1$ let $f_1(x) = \int_{-\infty}^{x} f(t)\, dt$. In general, fix a function $\alpha \in \mathscr{D}(R^1)$ with $\int_{-\infty}^{\infty} \alpha = 1$. Let $g = \int_{-\infty}^{\infty} f(t)\, dt_k$. Then by induction assumption $g = \sum \partial g_j/\partial t_j$ in $\mathscr{D}(R^{k-1})$. Now $f(t) - g(t_1, \cdots, t_{k-1})\alpha(t_k) = \partial f_k/\partial t_k$ for some $f_k \in \mathscr{D}$, and let $f_j(t) = g_j(t_1, \cdots, t_{k-1})\alpha(t_k)$ for $j < k$. □

Thus we know that X restricted to $\mathscr{D}_0 = \{\varphi \in \mathscr{D} : \int \varphi = 0\}$ is stationary. We apply A. M. Yaglom ((1957) Theorems 6, 6′) to obtain a unique measure $\mu \geqq 0$ on R^k, tempered at ∞, $\int_{|\lambda|<1} |\lambda|^2\, d\mu(\lambda) < \infty$, and a unique nonnegative Hermitian matrix J such that for all $\varphi, \Psi \in \mathscr{D}_0$,

$$EX(\varphi)X(\Psi)^- = \int (\mathscr{F}\varphi)(\lambda)(\mathscr{F}\Psi)(\lambda)^-\, d\mu(\lambda) + J \operatorname{grad} \mathscr{F}\varphi(0) \cdot \operatorname{grad} (\mathscr{F}\Psi)^-(0), \tag{3.3}$$

where $\mathscr{F}$ denotes Fourier transform. (Itô (1956) Theorem 4.1 could also be applied to grad X to yield this result.)

Now let $\alpha_1 \in \mathscr{D}(R^1)$, $\int \alpha_1 = 1$, $\alpha_n(u) \equiv n\alpha_1(nu)$. Let $\beta_n \in \mathscr{D}(R^1)$, $\int \beta_n \equiv 1$, $|\beta_n(u)| \leqq 2e^{-2u}$, and $\lim_{n\to\infty} \beta_n(u) = e^{-2|u|}$ for all u.

X is a tempered random distribution since $\sup_{|t| \le M} E|x_t|^2 = \mathcal{O}(M^2)$ as $M \to \infty$. Let $\Psi_n(t) = \prod_{1 \le j \le k} [\alpha_n(t_j) - \beta_n(t_j)]$. Then $E|X(\Psi_n)|^2$ remains bounded while $\mathscr{F}\Psi_n(\lambda) \to 1$ as $n, \lambda \to \infty$. Hence $\mu\{|\lambda| \geqq 1\} < \infty$.

Now let $\gamma_n(u_1, \cdots, u_k) = \prod_{1 \le j \le k} \alpha_n(u_j)$. In (3.3) let $\varphi(u) = \gamma_n(u - s) - \gamma_n(u)$, $\Psi(u) = \gamma_n(u - t) - \gamma_n(u)$ and let $n \to \infty$. Then grad $\mathscr{F}\varphi(0) \to is$, grad $\mathscr{F}\Psi(0) \to it$. Thus we obtain (3.2) by letting A be the real part of J.

If x_t is real, then the change $d\mu(\lambda) \to d\mu(-\lambda)$ does not change the covariance, so μ is symmetric.

For the converse, we need only apply existence theorems for Gaussian processes with given covariance (Doob (1953) Chapter II, Section 3), since the covariances are clearly nonnegative definite, and real if μ is symmetric. The distributions of increments $x_t - x_0$ are uniquely determined, so that all distributions of x_t are determined given x_0. Hence x_t is unique in law up to an additive random variable y. □

Now suppose x_t is a stationary CQM process on R^k. Then, as is well known, there is a finite measure $\mu \geqq 0$ on R^k called the *spectral measure* of x_t, such that $Ex_s x_t = \int e^{i\lambda(s-t)}\, d\mu(\lambda)$. Clearly x_t also has stationary increments. Its LK measure is the same μ.

Conversely suppose x_t is Gaussian with mean 0 and that its LK measure μ is finite. Let y_t be a stationary Gaussian process with μ as spectral measure. Let L be the isonormal process on $H = L^2(\mu)$, or μ-noise process. Then versions of x_t and y_t are

$$y_t = L(\lambda \to e^{it\cdot\lambda}), \qquad x_t = L(\lambda \to e^{it\cdot\lambda} - 1) = y_t - L(1).$$

Since the increments of x_t and y_t have the same distributions, x_t and y_t have the same continuity properties, moduli etc. In other words, for any set A, $\{x_t : t \in A\}$ is a GB- or GC-set iff $\{y_t : t \in A\}$ is one.

Now let x_t be any CQM Gaussian process with stationary increments, mean 0, and LK measure μ. Let y_t, z_t be other such processes with LK measures $\mu|_{|\lambda|\leq 1}$ and $\mu|_{|\lambda|>1}$ respectively.

Let y_t and z_t be independent. Then $y_t + z_t - y_0 - z_0$ is a version of $x_t - x_0$. grad $[y_t]$ is a stationary random field. Its components have spectral measures with compact support. Thus (cf. Belyaev (1959)), grad $[y_t]$ and hence $[y_t]$ are processes whose sample functions can be written as entire functions of exponential type, by the Paley–Wiener–Schwartz (1950), (1966a) theorem. Also z_t has continuity properties equivalent to those of the stationary process with spectral measure $\mu|_{|\lambda|>1}$. The following has been proved.

THEOREM 3.4. *Let $\mathscr{G}$ be a linear space of functions on R^k including all entire functions of exponential type* 1. *Let x_t be a Gaussian process* CQM *with stationary increments and* LK *measure μ. Let y_t be the stationary process with spectral measure $\mu|_{|\lambda|>1}$. If y_t has a version with sample functions in $\mathscr{G}$, then so does x_t.*

For stationary Gaussian processes x_t, $t \in R$, Fourier methods have been in general use. Nearly best possible conditions for sample continuity in terms of spectral measure were found by Hunt (1951). Periodic processes, e.g., with $x_0 \equiv x_{2\pi}$, have simpler Fourier series rather than Fourier transforms, yet the dependence of sample properties on spectral asymptotics seems to be very similar. So, suppose we have a random Fourier series

$$x_t = a_0 Y_0 + \sum_{n=1}^{\infty} a_n (Y_n \cos nt + Z_n \sin nt) \tag{3.5}$$

where the Y_n and Z_n are all orthogaussian and $a_n > 0$. Let

$$s_n = [\sum \{a_k^2 : 2^n < k \leqq 2^{n+1}\}]^{\frac{1}{2}}.$$

Kahane (1960) proved that $\sum s_n < \infty$ is a necessary condition for sample-continuity of x_t, while $s_n \leqq t_n$, t_n decreasing, and $\sum t_n < \infty$ is a sufficient condition. M. Nisio (1969) extended these results to nonperiodic stationary processes, replacing the discrete set $\{2^n + 1, \cdots, 2^{n+1}\}$ by the interval $(2^n, 2^{n+1}]$ in Fourier transform space. Her condition $E \sup_{t\in[0,1]} |X(t)| < \infty$ is equivalent

to sample-boundedness by the Fernique–Landau–Shepp theorem and then to sample-continuity in view of stationarity (Belyaev (1961)). By Theorem 3.4 above, these results carry over to processes with stationary increments.

Marcus and Shepp ((1970) Section 5) showed that $\sum s_n < \infty$ is not sufficient for sample-continuity of (3.5) in general.

Lacunary random Fourier series, where in (3.5) $a_n = 0$ except for $n = n_j$, $n_{j+1}/n_j \geqq q > 1$, have been a useful source of examples for Gaussian processes (Fernique (1964), Marcus (1971)).

3.3. *Boundedness on R.* Yu. K. Belyaev (1958) proved that for every stationary Gaussian process $\{x_t, t \in R\}$ whose spectral measure μ is not purely atomic,

$$\sup\{|x_t| : t \in R\} = \infty \quad \text{almost surely.}$$

Suppose then that μ is purely atomic with masses μ_k at points λ_k, $k = 1, 2, \cdots$. Belyaev (1958) discovered that if $\sum \mu_k^{\frac{1}{2}} < \infty$ then the sample functions are bounded on R, while if $\sum \mu_k^{\frac{1}{2}} = \infty$ and the λ_k are all incommensurable then the sample functions are unbounded a.s. The reason is that for X_k orthogaussian, $\sum b_k|X_k|$ converges a.s. iff $\sum b_k < \infty$. If $\sum \mu_k^{\frac{1}{2}} < \infty$ then $\sum \mu_k^{\frac{1}{2}} X_k \exp(i\lambda_k t)$ a.s. converges uniformly on R. If $\sum \mu_k^{\frac{1}{2}} = +\infty$ and the λ_k are incommensurable, we can find t such that $\exp(i\lambda_k t) \simeq \operatorname{sgn} X_k$, $k = 1, \cdots, n$, $n \to \infty$.

For the λ_k all multiples of a fixed number, we have the random Fourier series of Kahane (1960) as discussed above; good necessary conditions and good sufficient conditions for boundedness are known but are not yet equivalent. If the λ_k are commensurable in a more complicated way, less seems to be known.

Let x_t be Gaussian continuous in probability with stationary increments and LK measure μ. If the sample functions of x_t are bounded on R, then for every $\varepsilon > 0$ so are the sample functions of the process with LK measure $\mu|_{|\lambda| \geqq \varepsilon}$, which is equivalent to sample-boundedness of the stationary process with spectral measure $\mu|_{|\lambda| \geqq \varepsilon}$. Hence μ is purely atomic, with masses μ_k at points λ_k. Again, $\sum \mu_k^{\frac{1}{2}} < \infty$ is sufficient for sample-boundedness on all of R, and necessary if the λ_k are incommensurable.

4. Noise processes. The isonormal linear process L will be called a *noise process* (for μ) in case the Hilbert space H is $L^2(\mu)$ for some measure μ. If μ is Lebesgue measure on a Euclidean space, the process is called *white noise*.

A linear process G on $L^2(\mu)$ will be called *μ-bounded* iff there is an $M < \infty$ such that $EG(f)^2 \leqq M \int |f|^2 \, d\mu$ for all $f \in L^2$. If G is Gaussian with mean 0 and μ-bounded, then we can write $G = L \circ A$ where A is a bounded linear operator. It is known that if L has continuous or bounded sample functions on a set $C \subset H$, then so does G (Dudley (1967) Proposition 4.1, Theorem 4.6; L. Gross (1962) Theorem 5).

A special case of interest is the "centered noise" L_c for μ where μ is a probability measure. Here

$$EL_c(f)^2 = \int |f|^2 \, d\mu - \left|\int f \, d\mu\right|^2 = \int \left|f - \int f \, d\mu\right|^2 d\mu \, .$$

In this case the operator A is projection onto the orthogonal complement of the constant functions. Since A differs from the identity only by the one-dimensional projection $f \to \int f\,d\mu$ onto constants, the asymptotic properties of L and L_c are essentially the same.

L_c arises as a limit (in some weak sense) of normalized empirical measures $n^{\frac{1}{2}}(\mu_n - \mu)$ where μ_n has mass $1/n$ at each of n points chosen independently with distribution μ. If we can prove continuity of L and hence of L_c on some classes of function or sets, then we can hope to prove some stronger central limit theorems.

If $\mu(A) < \infty$ then we define $L(A) = L(\chi_A)$, $L_c(A) = L_c(\chi_A)$.

4.1. *λ-bounded noise on R^k.* If μ is Lebesgue measure λ on R^k, then $L(A)$ has been studied mainly when A is a rectangle

$$A_t = [0, t_1] \times \cdots \times [0, t_k]\,.$$

Then $L(A_t)$ is a standard Wiener process for $k = 1$ and a generalized Brownian motion for $k > 1$ (Chentsov (1956)). The processes, of course, have sample functions continuous in t with probability 1. Here, at the suggestion of R. Pyke, more general sets will be considered. Let $Co(x) = Co(x_1, \cdots, x_m)$ be the convex hull of the points $x_1, \cdots, x_m \in R^k$.

THEOREM 4.1. *For a fixed k and $m > k$, let $g(x_1, \cdots, x_m, \omega) = G(Co(x), \omega)$, where G is a λ-bounded Gaussian process of mean 0. Then g has continuous sample functions on R^{mk} and, when restricted to x in a bounded set, has the sample modulus $h(u) = (u|\log u|)^{\frac{1}{2}}$.*

PROOF. See Theorem 2.1 and Corollary 2.3 above. Note that if $|x_j - y_j| \leqq \varepsilon$ for $j = 1, \cdots, m$, and x and y lie in a fixed bounded set B, then the Lebesgue measure of the symmetric difference, $\lambda(C(x) \triangle C(y))$, is $\mathcal{O}(\varepsilon)$ uniformly in x, $y \in B$ as $\varepsilon \downarrow 0$. The total surface area of the faces of $C(x)$ is bounded uniformly for $x \in B$. Now $E|G(C(x)) - G(C(y))|^2 = \mathcal{O}(\varepsilon)$ so the proof in Example 2.2 above applies to give the modulus, which incidentally is best possible. (Simple continuity was proved earlier, in Dudley (1965) Section 5.)

Now let $I(k, \alpha, M)$ be the class of all (indicator functions of) subsets of R^k with boundary functions having all derivatives of orders $\leqq \alpha$ bounded by M, as defined in Dudley (1972b).

THEOREM 4.2. *$I(k, \alpha, M)$ is a* GC*-set in $L^2(\lambda)$ if $\alpha > k - 1 \geqq 1$, and not a* GB*-set if $1 \leqq \alpha < k - 1$ or $0 < \alpha \leqq 1 < k - 1$ and $M > 0$.*

PROOF. This is a corollary of Theorems 1.1 above and 3.1 of Dudley (1972b). Note that the exponent of entropy of $I(k, \alpha, M)$ in H is twice its exponent of entropy in the d_λ metric, $d_\lambda(A, B) = \lambda(A \triangle B)$, since the metric induced from H is $d_\lambda^{\frac{1}{2}}$. □

REMARKS. If $\alpha < 1$ and $k = 2$ I conjecture that $I(2, \alpha, M)$ is not a GB-set for $M > 0$, based on the conjecture of equality in Dudley ((1972b) 3.4).

If $\alpha = k - 1$ I conjecture that $I(k, k - 1, M)$ is not a GB-set.

Let $C(U)$ be the class of all (indicator functions of) convex subsets of a given bounded open set $U \subset R^k$. An earlier investigation of λ-bounded processes on $C(U)$ was made by A. de Hoyos (1972), incorrectly.

THEOREM 4.3. *$C(U)$ is a* GB*-set in $L^2(\lambda)$ for $k = 1, 2$ and not* GB *for $k \geqq 4$.*

PROOF. This is a corollary of the Sudakov–Strassen theorem (1.1 above) and Dudley ((1972b) Theorem 4.1), with the same note as in the previous proof.

REMARK. For $k = 3$, I claim that $C(U)$ is not a GB-set. Although the theorems proved above do not apply directly, we can consider the spherical caps C_x at the end of the proof of Theorem 4.1 of Dudley (1972b). Then $\sum_{x \in A_\varepsilon} |L(C_x)|$ does not approach 0 as $\varepsilon \downarrow 0$, so $C(U)$ is not a GC-set. To prove it is not a GB-set, we could successively adjoin suitable unions of small caps (depending on ω) to a given convex set.

4.2. *Elliptic operators and white noise.* We know that the white noise process W on R^k can be written as $W = \partial^k[f]/\partial t_1 \cdots \partial t_k$ in the Schwartz distribution sense, where f is a process with continuous sample functions (Chentsov (1956), Dudley (1965)). It turns out that if we use elliptic operators a degree less than k will suffice.

For any polynomial P in k variables we let, as usual,

$$P(D) = P(-i\partial/\partial t_1, \cdots, -i\partial/\partial t_k) .$$

THEOREM 4.4. *Let $P(D)$ be an elliptic operator of degree m in k variables with constant coefficients. Let W be the white noise generalized random process on R^k. Then the following are equivalent:*

(i) *For almost all ω, the distribution solutions T of $P(D)T = W_\omega$, on any open set $U \subset R^k$, are continuous functions on U;*

(ii) *There is a process $T(t, \omega)$ with continuous sample functions on some open $U \subset R^k$ with $P(D)T = W$ in U;*

(iii) *Replace "continuous" by "L^2" in* (ii);

(iv) $k < 2m$.

PROOF. Let $\mathscr{D}(U)$ be the L. Schwartz space of test functions with support in U, $\mathscr{D} = \mathscr{D}(R^k)$ with usual topology and dual space $\mathscr{D}'$ of distributions (Schwartz (1966a)). By Minlos' theorem (Minlos (1959), Kolmogorov (1959)), W has some distributions W_ω as realizations. Then $P(D)T_\omega = W_\omega$ has distribution solutions T_ω (Hörmander (1964) Section 3.6). Thus

(i) implies (ii).

(ii) implies (iii) trivially.

(iii) $\Rightarrow$ (iv): we can assume $U = \{t: |t_j| < 4, j = 1, \cdots, k\}$. Let H_P be the inner product space of all $\varphi \in \mathscr{D}(U)$ with the norm

$$||\varphi||_P^2 = \int_U |P(-D)\varphi|^2 .$$

Then W restricted to H_P has a version with continuous sample functions for $||\cdot||_P$.

Hence, the identity from H_P into L^2 must be a Hilbert–Schmidt operator (Minlos (1959)).

Let $\eta \in \mathscr{D}(U)$ and $\eta = 1$ on $\pi U/4$. Let $\varphi_n(x) \equiv \eta(x)e^{in\cdot x}$, $n \in \mathbb{Z}^k$. Then $||\varphi_n||_P = \mathscr{O}(|n|^m)$ as $|n|^2 \equiv n_1^2 + \cdots + n_k^2 \to \infty$. Hence $\sum_{n\in\mathbb{Z}^k, n\neq 0} |n|^{-2m} < \infty$, so $k < 2m$.

(iv) $\Rightarrow$ (i): All distribution solutions T of $P(D)T = 0$ on open sets are continuous (in fact, real analytic) (see Hörmander (1964)). Thus we need only get a continuous solution process T in the cube $V = \pi U/4$. Here we have

$$W = \sum_{n\in\mathbb{Z}^k} X_n e^{in\cdot x}$$

where the X_n are orthogaussian. If φ is C^∞, all solutions S of $P(D)S = \varphi$ are C^∞. Also since P is elliptic, $P(n) = 0$ only for n in a finite set F. We assume $0 \in F$ even if $P(0) \neq 0$.

Now we need only prove that the Gaussian process

$$T_F(x, \omega) = \sum_{n\in\mathbb{Z}^k\sim F} X_n(\omega)e^{in\cdot x}/P(n)$$

has continuous sample functions. For some $c > 0$, $|P(n)| \geqq c|n|^m$ for all $n \notin F$. Thus for any $x, y \in V$,

$$E|T_F(x) - T_F(y)|^2 \leqq c^{-2} \sum_{n\notin F} |e^{in\cdot x} - e^{in\cdot y}|^2/|n|^{2m} .$$

For some $C_1 < \infty$, $\sum_{|n|\geqq M} |n|^{-2m} \leqq C_1/M$ for all $M > 0$, since $k - 1 - 2m \leqq -2$. Also

$$\sum_{0<|n|<M} \frac{|e^{in\cdot x} - e^{in\cdot y}|^2}{|n|^{2m}} \leqq C_2 M|x - y|^2$$

for some $C_2 < \infty$. Thus $E|T_F(x) - T_F(y)|^2 \leqq C_3|x - y|$. Hence T_F has continuous sample functions. □

Theorem 4.4, as just proved, offers possibilities of extensions from R^k to other k-dimensional manifolds X with a μ-noise process, where μ is a measure which on local coordinate patches has a sufficiently smooth density with respect to Lebesgue measure. If X is a Riemannian manifold, there is an invariantly defined Laplace–Beltrami operator whose powers give elliptic operators of as high even order as desired. In Theorem 4.4 presumably constant coefficients could be replaced by sufficiently smooth, nonsingular coefficients.

4.3. *Lévy–Baxter theorems.* P. Lévy (1940) proved that if $x(t)$ is a standard Brownian motion, then with probability 1

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} [x(k/2^n) - x((k-1)/2^n)]^2 = 1 .$$

This result was extended to some other Gaussian processes on $[0, 1]$ by G. Baxter (1956), E. G. Gladyshev (1961), and V. G. Alekseev (1963), and to Lévy's Brownian motion with multidimensional time by S. M. Berman (1967) and P. T. Strait (1969). (R. Borges (1966) showed that the "generalization" by F. Kozin (1957) only applies to Brownian motion.)

Here we give another extension. Let $(X, \mathscr{S}, \mu)$ be any finite measure space:

$\mu \geqq 0$, $\mu(X) < \infty$. A *partition* of X will be a finite collection π of disjoint measurable sets whose union is X. The mesh of π is defined by

$$m(\pi) = \max \{\mu(A) : A \in \pi\} .$$

Let L be the μ-noise. We consider the sums $L(\pi)^2 = \sum_{A \in \pi} L(A)^2$. As $m(\pi) \to 0$, $L(\pi)^2 \to \mu(X)$ in law and hence in probability. P. Lévy ((1940) Section 4, Théorème 5) proved that $L(\pi_n)^2 \to \mu(X)$ almost surely if the π_n are nested, i.e., for all $A \in \pi_{n+1}$ there is a $B \in \pi_n$ with $A \subset B$. F. Kozin (1957) proved $L(\pi_n)^2 \to \mu(X)$ almost surely if $m(\pi_n) = o(n^{-2})$, for interval partitions. Most other authors have used $m(\pi_n) \leqq 2^{-n}$, considering more general Gaussian processes. Here for μ-noise we prove $m(\pi_n) = o(1/\log n)$ suffices for a.s. convergence, and that this is best possible.

THEOREM 4.5. *Let L be μ-noise for a finite measure space (X, μ). If $\{\pi_n\}$ is any sequence of partitions with $m(\pi_n) = o(1/\log n)$, then $L(\pi_n)^2 \to \mu(X)$ almost surely. If $(X, \mu) = ([0, 1], \lambda)$ there exist partitions π_n (not consisting of intervals) such that $m(\pi_n) \leqq (1/\log n)$ and $L(\pi_n)^2$ does not converge* a.s. *to* 1.

PROOF. Given a partition $\pi = \{A_1, \cdots, A_k\}$, let $a_j = \mu(A_j)$. We can assume $\mu(X) = 1$. Then $\sum a_j^2 \leqq m(\pi)$. We have $L(A_j)^2 = a_j X_j^2$ where the X_j are ortho-gaussian. Hence by a theorem of D. L. Hanson and F. T. Wright (1971), there are constants C_1 and C_2 such that for any $\varepsilon > 0$,

$$\Pr \{|\textstyle\sum_{A \in \pi} L(A)^2 - 1| \geqq \varepsilon\} \leqq 2 \exp\{-\min (C_1 \varepsilon/m(\pi), C_2 \varepsilon^2/m(\pi))\} .$$

Let $m(\pi_k) = \varepsilon_k/2 \log k$, where $\varepsilon_k \to 0$. Then

$$\Pr \{|\textstyle\sum_{A \in \pi_k} L(A)^2 - 1| \geqq \varepsilon_k\} = \mathcal{O}(k^{-2}) ,$$

so $L(\pi_n)^2 \to 1$.

Now for Lebesgue measure λ on $[0, 1]$ we choose λ-independent partitions π_n consisting of k_n sets of equal measure where k_n is the least integer $> \log n$, $n = 2, 3, \cdots$. Then $k_n L(\pi_n)^2$ has a χ^2 distribution with k_n degrees of freedom. Letting $k = k_n$ we have for any fixed $\varepsilon > 0$

$$\begin{aligned} \Pr \{L(\pi_n)^2 < 1 - \varepsilon\} &= \Pr \{kL(\pi_n)^2 < (1 - \varepsilon)k\} \\ &= 2^{(2-k)/2}\Gamma(k/2)^{-1} \textstyle\int_0^{[(1-\varepsilon)k]^{\frac{1}{2}}} r^{k-1} \exp(-r^2/2)\, dr \\ &\geqq 2^{(2-k)/2}\Gamma(k/2)^{-1}\{[(1 - \varepsilon)k]^{\frac{1}{2}} - 1\}^{k-1} e^{-(1-\varepsilon)k/2} , \end{aligned}$$

which by Stirling's formula is asymptotic as $k \to \infty$ to

$$\begin{aligned} &(e/(k - 2))^{(k-2)/2}(\pi k)^{-\frac{1}{2}}\{[(1 - \varepsilon)k]^{\frac{1}{2}} - 1\}^{k-1} e^{-(1-\varepsilon)k/2} \\ &\qquad \sim c([1 - \varepsilon]^{\frac{1}{2}} - k^{-\frac{1}{2}})^{k-1}/(1 - 2k^{-1})^{(k-2)/2} e^{\varepsilon k/2} \geqq \alpha(1 - 2\varepsilon)^k , \end{aligned}$$

for some constants c, α, and k large. Letting $\varepsilon = \frac{1}{4}$ and noting $k \sim \log n$ we have $\sum_n \alpha(1 - 2\varepsilon)^k = +\infty$.

Now let $\pi_n = \{A_1, \cdots, A_k\}$ and $\pi_m = \{B_1, \cdots, B_r\}$, $r = k_m$, be two of our independent partitions, $\lambda(A_j \cap B_i) = \lambda(A_j)\lambda(B_i) = 1/kr$. Let $Z = L([0, 1])$. Then

$E(L(A_i) - Z/k)(L(B_j) - Z/r) = 0$ for all i, j. Thus the set of random variables $\{L(A_i) - Z/k\}$ is independent of the set $\{L(B_j) - Z/r\}$. Let

$$Q(n) = \sum_i [L(A_i) - Z/k]^2 = L(\pi_n)^2 - Z^2/k\ .$$

Then $Q(n)$ is independent of $Q(m)$ for $m \neq n$. Since $\sum_n P(Q(n) < \frac{3}{4}) = +\infty$, we have a.s. $Q(n) < \frac{3}{4}$ infinitely often by Borel–Cantelli. Then since $Z^2/k \to 0$ a.s. as $n \to \infty$, we have a.s. $L(\pi_n)^2 < \frac{7}{8}$ infinitely often. □

Kozin (1957) attributes to Lévy (1940) the assertion that $L(\pi_n)^2 \to 0$ almost surely if card $(\pi_n) \to \infty$, $\mu = \lambda$ on $[0, 1]$ and π_n consist of intervals. No doubt Kozin meant to include the assumption $m(\pi_n) \to 0$. Even then, the mere assumption that π_n consists of intervals cannot correctly replace Lévy's assumption of nested partitions, for reasons indicated later by Lévy ((1965) page 192).

Whether $o(1/\log n)$ is best possible for interval partitions seems to be an open question. Also open, and perhaps not difficult, are questions of the best possible assumptions on speed of $m(\pi_n) \to 0$ for the various generalizations of Lévy's work.

5. Discontinuous Gaussian processes. As we shall see in Section 7, conditions for sample-continuity of Gaussian processes are relatively mild as compared with conditions for general non-Gaussian processes. Nevertheless it is of some interest to consider how the sample functions of Gaussian processes may behave even when they are not continuous.

It may be asked whether, instead of restricting L to suitable subsets of H, we can obtain a version of L with good properties defined on all of H. Certainly L on H is not sample-continuous. If we take any particular finite Borel measure μ on H, then L has a version which is jointly $\mu \times P$-measurable and is linear on H for each fixed ω. However, we cannot replace μ-measurability here by simultaneous measurability for all Borel measures μ, or absolute measurability, since every absolutely measurable linear form on H is continuous by a theorem of Douady (Schwartz (1966b) Lemme 2).

By embedding the real line R in a compactification $\bar{R}$, we can obtain a regular Borel measure defining L on a compact Hausdorff space $\bar{R}^H$ (Kakutani (1943), E. Nelson (1959)) but, assuming the continuum hypothesis, this version of L is not jointly measurable, and such a large, non-metrizable compact has other bad properties (Dudley (1971a), (1972a)).

We have a probability measure P defining L on the algebraic dual space H^a of all linear forms on H, but this P has no extension regular for the weak topology $\sigma(H^a, H)$ (A. de Acosta (1971)).

Precisely because L is a universal model for all Gaussian processes, it is perhaps not surprising that L on the *entire* space H has the various pathological properties mentioned in the last few paragraphs.

Suppose $\{X_t, t \in T\}$ is a mean-zero Gaussian process, continuous in probability, on a separable metric space T. Then there exist orthogaussian random variables

Y_n and continuous functions f_n with $X_t = \sum Y_n f_n(t)$. Since finite partial sums are all continuous, such continuity properties as continuity at a fixed point, continuity everywhere, and even the existence of discontinuities of oscillation $\geqq \varepsilon$ for fixed $\varepsilon > 0$ on a fixed open set, all are "zero-one" properties. If X_t is not sample-continuous, then there exists a point t and an $\varepsilon > 0$ such that almost all sample functions oscillate by $\geqq \varepsilon$ in every neighborhood of t. Thus, unlike Markov processes with their isolated random jump discontinuities, Gaussian processes are discontinuous at fixed points. In fact, K. Itô and M. Nisio (1968) have proved given a Gaussian process x_t, $t \in [0, 1]$, there is a fixed function α such that almost every sample function $x_t(\omega)$ satisfies, for *all* t,

$$\limsup_{u,v \to t} |x_u - x_v| = \alpha(t) .$$

If a non-sample-continuous process on a separable metric T has probability laws invariant under a transitive group of homeomorphisms of T, then almost all sample functions oscillate by $\geqq \varepsilon > 0$ on every open set, since there is a countable base for the topology. Thus the sample functions are *everywhere* discontinuous. Further, the ε-oscillations pile up to produce infinite oscillations. On such matters see Yu. Belyaev (1961), S. M. Berman (1968), D. M. Eaves (1967), D. Cohn (1971), K. Itô and M. Nisio (1968), N. Jain and G. Kallianpur (1971).

Fernique (1971) proves that for any Gaussian process $\{x_t, t \in T\}$ with mean 0 and bounded measurable covariance, and any separable probability measure μ on T, the process has a version with almost all sample functions in the Banach space $G_C(\mu)$ (also defined in Section 1 above).

6. Infinitesimal σ-algebras and 0–1 laws. Let $\{x_t, t \in T\}$ be a mean-zero Gaussian process, continuous in probability, on a separable metric space (T, d). Then we can write $x_t(\omega) \equiv \sum f_n(t) X_n(\omega)$, f_n continuous, $\{X_n\}$ orthogaussian.

Suppose now that T is compact and $C \equiv \{x_t : t \in T\}$ is GC. A modulus f will be called a lim sup *modulus* for x_t iff for almost all ω,

$$\limsup \{x_t; f\} \equiv \limsup_{h \downarrow 0} \sup \{|x_s - x_t| : d(s, t) \leqq h\}/f(h) = 1 .$$

Clearly a lim sup modulus is a weakly optimal sample modulus as defined in Section 2 above. It is not clear whether it is strongly optimal.

Suppose g_n is a uniform modulus of continuity of f_n and assume that $\lim_{h \downarrow 0} g_n(h)/f(h) = 0$ for all n. This holds, for example, in case $T = C$, $t \to x_t$ is the identity, and $\lim_{h \downarrow 0} h/f(h) = 0$. Then $\limsup \{x_t, f\}$ is almost surely a constant (possibly infinite) by the zero-one law. If this constant is positive and finite, then some positive multiple of f is a lim sup modulus. For many processes, lim sup moduli are known, at least up to multiplicative constants. If f is a lim sup modulus, so is $f + o(f)$, but this apparently does not exhaust the class of lim sup moduli of the process. On the other hand it seems unclear whether lim sup moduli always exist in the cases we have been discussing:

QUESTION. Let C be a compact GC-set in H which is not a GL-set. Then is there always a lim sup modulus for L on C?

Corresponding questions can be asked for processes on [0, 1] and for local rather than uniform moduli, etc. For *non*-Gaussian processes it is known that local lim sup moduli need not exist: Gnedenko (1943), Rogozin (1968).

Let $\{x_t, 0 \leqq t \leqq 1\}$ be a Gaussian process with mean 0. Let $\mathscr{B}(\delta)$ be the smallest σ-algebra for which $\{x_t, 0 \leqq t \leqq \delta\}$ are all measurable, and let $\mathscr{B}(0^+) = \bigcap_{\delta>0} \mathscr{B}(\delta)$. If almost all sample functions of x_t have n derivatives at 0 then these derivatives are $\mathscr{B}(0^+)$-measurable functions which in general are non-trivial. Under assumptions of stationarity with spectral density behaving like an inverse power at ∞, Freidlin and Tutubalin (1962) showed that all $\mathscr{B}(0^+)$-measurable functions are measurable with respect to the derivatives which exist. Thus if x_t has non-differentiable sample functions then, under their conditions, every asymptotic property of the increments $x_s - x_t$ as $s \to t$ has probability 0 or 1. It seems reasonable to expect this zero-one law to hold under less restrictive conditions than those of Tutubalin and Freidlin. However, examples of a "gap" with $\mathscr{B}(0^+)$ not generated by derivatives are given by Levinson and McKean ((1964) pages 130–133) and Dym and McKean ((1970) page 1824). These papers also contain further relevant information on $\mathscr{B}(0^+)$.

At the other extreme, it may be asked which processes are entirely determined by their behavior in the neighborhood of one point. This question, for $T = L^p(\mu)$, has been considered by Bretagnolle and Dacunha-Castelle (1969). Such questions were also considered by P. Lévy (1948), (1965).

It is known rather generally that, given a Gaussian measure μ on a linear space X, the support of μ is a linear subspace of X (K. Itô (1970), G. Kallianpur (1970), (1971)). Here the support is defined as the smallest closed set whose complement has measure 0. There may also be some interest in considering non-closed supports. Let S be a nuclear Fréchet space with topology defined by a sequence of seminorms $|\cdot|_1 \leqq |\cdot|_2 \leqq \cdots$. For example, S may be L. Schwartz's test function space $\mathscr{S}$ or $\mathscr{D}(K)$, K compact. Let S' be the dual space of S, $S' = \bigcup S_n'$, where $S_n' = \{f \in S' : f \text{ continuous for } |\cdot|_n\}$. (Here S' is the inductive limit of the S_n', but in general this inductive limit is *not* strict, i.e., the embedding $S_n' \to S_{n+1}'$ is not a homeomorphism, contrary to an editorial insertion in my review (1969).) As noted by D. M. Eaves (1968), if μ is a Gaussian measure on S', then $\mu(S_n') = 1$ for some n. It happens often, for example if $S = \mathscr{S}$ or $\mathscr{D}(K)$, that S_n' is actually not closed and is dense in S', while $\mu(S_n') = 1$ is more interesting than $\mu(S') = 1$. For example, the smallest n such that $\mu(S_n') = 1$ may be the smallest n for which the sample generalized functions in S' are derivatives of order n of continuous functions, as in Section 4 above.

Further, many theorems assert that Gaussian measures live on subsets which are neither closed nor linear. For example, if $\{X_n\}$ are orthogaussian, $\lim_{n\to\infty} \sum_{j=1}^n X_j^2/n = 1$ a.s., and the set of sequences $\{X_j\}$ satisfying the condition is nonlinear. For an extension of this line of thought see, e.g., T. Hida and H. Nomoto (1964).

A zero-one law for Borel subgroups was proved by N. C. Jain (1971).

7. Non-Gaussian processes.

7.1. *Some counterexamples.* Let T be a bounded set in R^k. We know that if $\{x_t, t \in T\}$ is a Gaussian process with

$$E|x_s - x_t|^2 \leqq C/|\log|s-t||^{1+\delta} \qquad \text{for some } \delta > 0 \text{ and } C < \infty,$$

then x_t is sample-continuous. The Gaussian property is strongly used.

H. Totoki (1962) proved that if x_t is any process on $T \subset R^k$ (Gaussian or not) such that for some α and $p > 0$, $E|x_s - x_t|^p \leqq C|t-s|^{k+\alpha}$, then x_t is sample-continuous. This condition is of course much stronger than the one stated above for Gaussian processes.

P. Bernard (1970) weakened Totoki's condition to

$$E|x_{t+h} - x_t|^p \leqq C|h|^k/|\log|h||^s, \qquad s > p+1.$$

Earlier, J. Delporte (1966) gave this sufficient condition for sample-continuity if $k = 2$ and $s > p$.

Now here are examples where $s = -1$ and we do not have sample-continuity, so that the power k in $|h|^k$ is best possible. We still do not know the best possible power of the logarithm, between -1 and p or $p+1$.

PROPOSITION 7.1. *For $k = 1, 2, \cdots$, there is a stochastic process x_t, $t \in I^k$, where I^k is the unit cube in R^k, $I = [0, 1]$, such that*

$$E|x_s - x_t|^k \leqq |s-t|^k(1 + |\log|s-t||)$$

and (every version of) x_t has almost all sample functions unbounded.

PROOF. For each $n = 1, 2, \cdots$, we divide I^k into 2^{nk} equal, parallel cubes C_{nj}, $j = 1, \cdots, 2^{nk}$. Let f_{nj} be a function which is 1 at the center of C_{nj}, 0 outside C_{nj}, and linear on each line segment joining the center to the boundary of C_{nj}.

For each n let $j(n)$ be a random variable with $\Pr(j(n) = j) = 2^{-nk}$, $j = 1, \cdots, 2^{nk}$. Let the $j(n)$ be independent for different n.

Let

$$x_t(\omega) = \sum_{n=1}^{\infty} f_{n,j(n)(\omega)}(t), \qquad t \in I^k.$$

The series converges in $L^1(I^k, \lambda)$, and for each ω it converges for almost all t. The cubes $C_{n,j(n)}$ have accumulation points t such that in every neighborhood of such t, x_t is unbounded. (Actually, such t are almost surely dense in I^k.)

For $k^{\frac{1}{2}}/2^{n+1} \leqq |s-t| \leqq k^{\frac{1}{2}}/2^n$, we have

$$\begin{aligned} E|x_s - x_t|^k &\leqq \sum_{j=1}^{n} 2 \cdot (2^{(j+1)}|s-t|)^k/2^{jk} + \sum_{j>n} 2/2^{jk}; \\ &\leqq 2^{k+1}n|s-t|^k + 2^{-nk-k+2}. \end{aligned}$$

Dividing the process by a suitable constant we have the result. □

Most proofs of sample continuity of Gaussian processes use just upper bounds on tail probabilities. Thus similar results do hold for non-Gaussian processes

with tail bounds of Gaussian type, called "sub-Gaussian" processes; (see Kahane (1960), (1968), Kozačenko (1968), Hanson and Wright (1971)).

For processes stationary in quadratic mean, the assumptions on the spectrum for sample-continuity are again much stronger in general than they are for Gaussian processes; see, e.g., Kawata and Kubo (1970).

7.2. *Comparison of processes.* Marcus and Shepp ((1971) Lemma 1.5) proved that if X and Y are Gaussian processes with

$$E|Y(s) - Y(t)|^2 \leqq E|X(s) - X(t)|^2 , \qquad EX(t) = 0 ,$$

for $0 \leqq s \leqq t \leqq 1$, and X is sample-continuous, then so is Y. Their proof uses Slepian's inequality (1962). It may be amusing to note that the above result may fail for X non-Gaussian and Y Gaussian. Let H be the usual Hilbert space with an orthonormal basis $\{\varphi_n\}$. Let μ be a probability measure with mass 2^{-n-1} at $2^{n/2}\varphi_n$ and at $-2^{n/2}\varphi_n$ for $n = 1, 2, \cdots$. Let $N(f) = (f, x)$ where x has distribution μ. Then N has the same means and covariances as the isonormal process L: $EN(f) = 0$, $EN(f)N(g) = (f, g)$. Clearly N is sample-continuous on H. Hence:

PROPOSITION 7.2. *For any stochastic process $\{x_t, t \in T\}$ with $E|x_t|^2 < \infty$ and $E|x_s - x_t|^2 \to 0$ as $s \to t$, where T is any separable metric space, there is a process y_t with the same means and covariances as x_t and with continuous sample functions, namely $y_t = Ex_t + N(x_t - Ex_t)$.*

8. Miscellaneous topics on Gaussian sample functions. Several topics have not been treated at length in this survey, partly for lack of time and space. Here are brief remarks on a few of these topics.

8.1. *Multidimensional range.* Some questions arise for processes with multidimensional range which may be trivial for sample-continuous real-valued processes, such as the Hausdorff dimension or other measures of size of the range or trajectory, or of its intersection with a given set. D. Ray (1963) and S. J. Taylor (1964) proved sharp results for Brownian motion. See also Kahane (1968), Chapter 13 and S. Orey (1970). F. Spitzer (1958) considered asymptotic behavior of polar coordinates of plane-valued Brownian motion.

8.2. *Level crossings.* Let x_t be Gaussian CQM stationary with mean 0, covariance $Ex_s x_t = r(s - t)$, and spectral measure μ. Let $M(T, u)$ be the number of values of t with $x_t = u$ and $0 \leqq t \leqq T$. K. Itô (1964) proved that $EM(T, u) < \infty$ is equivalent to finiteness of the symmetric second derivative $r''(0)$ and hence to $\int \lambda^2 \, d\mu(\lambda) < \infty$. Thus such processes have first derivatives which are integrable to any finite power, but these derivatives are not necessarily sample-continuous. Itô proved rigorously a formula of S. O. Rice (1945) and V. Bunimovich (1951),

$$EM(T, u) = T\pi^{-1}(-r''(0)/r(0))^{\frac{1}{2}} \exp(-u^2/2r(0)) .$$

Others, e.g., Bulinskaya (1961), had proved the formula in increasing generality before Itô's final result.

If $EM(T, u) = +\infty$, can it happen that $M(T, u) < \infty$ almost surely? This question seems to be open.

As $u \to \infty$ for fixed T, $M(T, u)$ is asymptotically Poisson-distributed under suitable hypotheses (a recent reference is S. M. Berman (1971 b)).

On the other hand for fixed u, as $T \to \infty$ $M(T, u)$ is asymptotically normal (Malevich (1969)) under a list of conditions not reproduced here.

Yu. K. Belyaev and V. P. Nosko (1969) have several results on the asymptotic distribution of lengths of excursions above level u for stationary Gaussian (and other) processes. See also S. M. Berman (1971 a).

8.3. *Geometric properties in Hilbert space.* Let e_n be an orthonormal sequence in a Hilbert space H, $0 \leqq p < \infty$, and $\{a_n\}$ a decreasing sequence of positive real numbers. Let

$$B_p\{a_n\} = \{\sum x_n e_n : \sum |x_n/a_n|^p \leqq 1\},$$
$$B_\infty\{a_n\} = \{\sum x_n e_n : \sup |x_n/a_n| \leqq 1\}.$$

Then for $1 < p < \infty$, $B_p\{a_n\}$ is a GB-set iff $\sum |a_n Y_n|^q < \infty$ almost surely, where $1/p + 1/q = 1$ and Y_n are ortho-Gaussian. M. G. Sonis (1966) proved that this holds iff $\sum |a_n|^q < \infty$. For $p = q = 2$ this was classical.

Dudley (1967) proved $B_1\{a_n\}$ is a GB-set iff $a_n = \mathcal{O}(\log n)^{-\frac{1}{2}}$ and a GC-set iff $a_n = o(\log n)^{-\frac{1}{2}}$. Sonis (1966) had related, but more complicated conditions.

Let C be a closed, convex set in H. Let $V_n(C)$ be the supremum of n-dimensional Lebesgue measures of orthogonal projections of C into n-dimensional subspaces. Let

$$EV(C) = \limsup_{n\to\infty} (\log V_n)/n \log n .$$

My 1967 paper proved that $r(C) \geqq -2/(1 + 2EV(C))$ if $EV(C) < -\frac{1}{2}$ and conjectured that if $EV(C) < -1$, then $r(C) = -2/(1 + 2EV(C))$. It would follow that $r(C) < 2$, so that C is a GC-set. This conjecture remains open in general. My 1967 paper proved it for $B_p\{a_n\}$, $p = 1, 2, \infty$. S. Chevet (1969), (1970) proved if for $1 < p < 2$ and $2 < p < \infty$. For $1 < p \leqq \infty$, B_p is a GC-set iff it is a GB-set.

In my 1967 paper, Theorem 5.3 states that C is not a GB-set if

$$\sup_n [n^{-1} \log V_n(C) + \log n] = +\infty ,$$

in particular if $EV(C) > -1$. However, volumes, like $N(C, \varepsilon)$, cannot give a complete characterization of GC-sets or GB-sets (Dudley (1967) Proposition 6.10).

Sudakov (1969) announced that an ellipsoid $E = B_2\{a_n\}$ is a GB-set iff $\int_0^1 \varepsilon^2 \, dH(E, \varepsilon) > -\infty$. A proof of this can be based on Theorem 3 of B. S. Mityagin (1961) page 71. As we saw in Section 1 above, not every GB-set C satisfies $\int_0^1 \varepsilon^2 \, dH(C, \varepsilon) > -\infty$. There exist GB-sets, such as $B_1\{1/\log(n + 1)\}$, not included in any GB-ellipsoid.

8.4. *Differentiability.* A process has continuously differentiable sample functions iff it is the indefinite integral of a process with continuous sample functions.

For example, a Gaussian process with stationary increments and LK measure μ has m times continuously differentiable sample functions if for some $a > 1$,

$$\int_{|\lambda| \geq 1} \lambda^{2m} |\log \lambda|^a \, d\mu(\lambda) < \infty .$$

Yu. K. Belyaev (1959) proved that any stochastic process $\{x_t, 0 \leqq t \leqq 1\}$ with mean 0 and analytic covariance function $Ex_s x_t$ has a version with analytic sample functions. For Gaussian processes he proved that this sufficient condition is also necessary. For stationary processes with spectral measure μ he showed that sample functions x_t are analytic for $|t| \leqq r$ iff $\int_{-\infty}^{\infty} e^{r\lambda} \, d\mu(\lambda) < \infty$. Thus, for example, if μ has compact support or is itself Gaussian, then x_t has entire analytic sample functions. By Theorem 3.4 above these results on stationary processes carry over to processes with stationary increments, if we integrate over $|\lambda| > 1$.

8.5. *Examples of sample functions.* It may happen that almost all sample functions of a process are proved to have a certain property, yet it is difficult to construct any specific function with such a property. Thus, on the one hand, sample function theory can provide existence proofs simpler than constructive ones. On the other hand, construction of such functions is an interesting challenge; one such was met by K. Urbanik (1959).

8.6. *Local maxima.* Dvoretzky, Erdös and Kakutani (1961) proved that almost every Brownian path x_t has no points of increase, while it does have local maxima. In a side remark (page 105) inessential for their purposes, they state that "the set of points of maximum is, almost surely, of the power of the continuum in every open interval". But actually local maxima are a.s. countable, since strict local maxima form a countable union of discrete, hence countable sets, and for any rational $a < b < c$, we have a.s.

$$\max \{x_t : a < t < b\} \neq \max \{x_t : b < t < c\} ,$$

so a.s. all local maxima are strict. Also, a.s. the local maxima are dense. All this was noted by G. J. Foschini and R. K. Mueller (1970) with one lemma by L. Shepp.

Local behavior of Gaussian processes with smooth covariances near local maxima has been considered in several works by G. Lindgren (1971).

As to the possible values at local maxima, we have the following result, essentially due to D. Ylvisaker (1965), (1968).

THEOREM 8.1. *Let K be a compact metric space and $\{Y_t : t \in K\}$ any Gaussian process on K with continuous sample functions and such that for all $t \in K$,*

$$\sigma^2(t) \equiv E(Y_t - EY_t)^2 > 0 .$$

Then the distribution of $\max \{Y_t : t \in K\}$ *is absolutely continuous with respect to Lebesgue measure.*

PROOF. Ylvisaker (1965), (1968) proves that if in addition $\sigma^2(t) \equiv 1$ then the

law of $\max\{Y_t: t \in K\}$ has a density f with respect to Lebesgue measure where

$$f(x) = \exp(-x^2/2)G(x)$$

and G is non-decreasing and finite, so that f is bounded on bounded sets. Then for a general Y_t, $x \in R$, and $0 < \varepsilon \leqq 1$, we have

$$\begin{aligned}\Pr\{x - \varepsilon \leqq \max Y_t \leqq x + \varepsilon\} &= \Pr\{-\varepsilon \leqq \max (Y_t - x) \leqq \varepsilon\} \\ &\leqq \Pr\{-\varepsilon/\inf \sigma(t) \leqq \max (Y_t - x)/\sigma(t) \leqq \varepsilon/\inf \sigma(t)\},\end{aligned}$$

noting that by sample continuity, σ is continuous so that $\inf \sigma > 0$. Hence by Ylvisaker's results there is some $K < \infty$, depending on Y_t and x but not on ε, such that the above probabilities are less than $K\varepsilon$. This suffices to prove the absolute continuity. □

Considering $-Y_t$, we get the same result for $|Y_t|$. If $\sigma^2(t) = 0$ for some t, let F be the compact set $\{t: \sigma^2(t) = 0\}$, and $\alpha = \max\{EY_t: t \in F\}$. Then $P(\max Y_t = \alpha)$ may be positive, or may be 0. It seems plausible that the law of $\max Y_t$ is absolutely continuous except for a possible atom at its essential infimum.

Under the hypotheses of Theorem 8.1, if g is a fixed continuous real function on K, there is probability 0 that there exists an open set U such that $Y_t = g(t)$ for some $t \in U$ and $Y_t \leqq g(t)$ for all $t \in U$ (Ylvisaker (1968)), since we can take a countable base for the open sets, and apply Theorem 8.1 to $Y_t - g(t)$. (This allows some simplification of the proof of Lemma 5A of Dudley (1971 b).)

Yuditskaya (1970) considers maxima of stationary isotropic processes on R^k over sets with some regularity conditions.

9. Relations to other major subjects.

9.1. *Equivalence and singularity.* Study of sample function properties of processes can be viewed as seeking sets in function space which contain almost all the sample functions yet which are as small as possible. Thus, in principle, sample function properties should help in proving equivalence or singularity of Gaussian measures. But, in practice I. E. Segal ((1958) Theorem 3) and J. Feldman (1959) solved the problem of equivalence and perpendicularity in terms of covariances. Their condition in terms of Hilbert–Schmidt operators usually seems easier to apply than sample function properties. In situations where their theorem does not apply (e.g., only one of the measures is Gaussian) sample function properties may be useful; see, e.g., my paper (1971 b).

9.2. *Prediction.* Again, in principle, knowledge of sample function behavior of processes should help to predict the future of a given process, given all or part of its past. But the existing prediction theory, both in the classical Kolmogorov–Wiener–Masani form with infinite past, and in the prediction of Gaussian processes from a finite segment of the past (Levinson and McKean (1964), Dym and McKean (1970)), concerns itself with subspaces and projections

and hence with covariances rather than with sample functions. Of course, there are for example processes with analytic sample functions which can thus be perfectly predicted, but this intersection of prediction and sample function behavior seems relatively small and the theories seem to develop with little relation to each other.

9.3. *Diffusion.* Some interesting properties of Gaussian sample functions are connected with non-Gaussian diffusion processes such as Brownian motion with an absorbing or reflecting barrier, etc. There is an exposition by K. Itô and H. P. McKean, Jr. (1965).

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DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139