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DEGENERACY PROPERTIES OF SUBCRITICAL BRANCHING PROCESSES

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This paper describes the limit behavior of sub-critical, age-dependent branching processes for which the Malthusian parameter does not exist.

1. Introduction. In this paper we apply several results about "functions of measures"—a subject we discussed in [5]—to the limit theory of subcritical branching processes. We consider a class of processes for which the socalled "Malthusian parameter" does not exist, and present results about the corresponding limit distributions for population size, conditioned on nonextinction. We include variants and simplifications of the relevant material from [5], adapted to the present setting.

Let $f(s) = \sum_{k=0}^{\infty} p_k s^k$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution function $G(\cdot)$ (see [1] or [6] for background). We take the process to be subcritical, i.e., $f'(1) = m < 1$. Let $Z(t)$ denote the number of particles at time t, and $F(s, t) = \sum P\{Z(t) = k\}s^k$, $|s| \leqq 1$. Let $\hat{G}(\alpha) = \int_0^\infty e^{-\alpha t}\, dG(t)$ denote the Laplace–Stieltjes transform of G. Then one defines the Malthusian parameter of the process as the (unique) root, call it $\alpha = \alpha(m, G)$, of the equation $m\hat{G}(\alpha) = 1$, provided such a root exists. Roughly speaking, the Malthusian parameter will exist if the tail of G decreases faster than exponentially as $t \to \infty$; and will fail to exist if the tail decreases slower than exponentially.

In case $\alpha(m, G)$ does exist, the limit distribution

$$\lim_{t\to\infty} P\{Z(t) = k \mid Z(t) > 0\} = b_k \qquad k \geqq 0 \tag{1.1}$$

is nondegenerate. Indeed it was proved by Ryan [8], and in Athreya–Ney [1], that

$$\lim_{t\to\infty} e^{-\alpha t}[1 - F(s, t)] \equiv Q(s)$$

exists for $0 \leqq s < 1$, and that $Q(s) \equiv 0$ if and only if $\sum p_j j \log j = \infty$. On the other hand if $\alpha(m, G)$ fails to exist by virtue of "slower than exponential" decrease in the tail of G, the limit distribution (1.1) will be degenerate ($b_1 = 1$). This was shown by Chistyakov [3] for small m and by ourselves [4] for general $m < 1$ (see also Athreya–Ney [1]). The existence/non-existence of α is, however, a crude index for the nature of the limit in (1.1). There are borderline classes of distributions for which α fails to exist but whose tails decrease faster than exponentially.

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Our main results (in Section 4) answer the degeneracy question for a large class of distributions.

In Section 2 we introduce the classes of distributions which we shall study, and present a main lemma regarding them. In Section 3 we discuss asymptotic properties of $F(s, t)$ and of the means $\mu(t) \equiv EZ(t)$. In Section 6 we append a brief description of an alternative approach to the main theorem, using a contraction principle.

2. Distributions with large tails. Let $G(\cdot)$ denote a probability distribution function on $[0, \infty)$, $G(0) = 0$; and let $G_n(\cdot)$ denote the n-fold convolution of G with itself. We shall consider the following conditions for G:

$$\text{(2.1)} \qquad \lim_{t\to\infty} \frac{1 - G_2(t)}{1 - G(t)} \quad \text{exists} = c\,(< \infty)\,,$$

and

$$\text{(2.2)} \qquad \lim_{t\to\infty} \frac{1 - G(t - b)}{1 - G(t)} \quad \text{exists} = \psi(b)$$

for all real b. From (2.2) it follows easily that $\psi(b) \equiv e^{\rho b}$ for some $\rho \geqq 0$, and the convergence is uniform for b in compact sets. In terms of ρ, we formulate a third condition:

$$\text{(2.3)} \qquad \textstyle\int_0^\infty e^{\rho t}\, dG(t) \quad \text{exists} = d\,(< \infty)\,.$$

For a fixed $d \geqq 1$, let $\mathscr{S}(d)$ denote the set of distributions G which satisfy (2.1), (2.2), and (2.3). The constants c in (2.1) and d in (2.3) are related: necessarily

$$\text{(2.4)} \qquad c = 2d\,.$$

We proved this equality in [5] for the case $d = 1$, and for the case $d \geqq 1$ when G is a lattice or an absolutely continuous distribution. The methods of proof for Theorems 1 and 4 of [5] in fact extend without change to all G on $[0, \infty)$ and $d \geqq 1$, yielding (2.4). We shall not repeat the steps here. (A further note: It is easy to show directly that if (2.1) holds, then $c \geqq 2$ necessarily; and if $c = 2$, then necessarily $\rho = 0$ in (2.2)—see [1] or [3]. Also (2.2) does not imply (2.1). For a related counterexample, and an elementary proof of (2.4) under other hypotheses, see Rudin [7].)

Here are several examples of densities whose distributions are in $\mathscr{S}(1)$:

$$g(t) \sim at^{-b}\,, \qquad a > 0,\ b > 1\,.$$
$$g(t) \sim \exp\{-at^{\alpha}\} \qquad a > 0,\ 0 < \alpha < 1\,.$$
$$g(t) \sim \exp\{-t/\log^2 t\}\,.$$

One way to construct densities whose distributions are in $\mathscr{S}(d)$ for $d > 1$ is to multiply densities whose distributions are in $\mathscr{S}(1)$ by negative exponentials. Thus if

$$\text{(2.5)} \qquad G \quad \text{is absolutely continuous and} \quad G'(t) \sim t^{-b}e^{-t} \qquad b > 1\,,$$

then G is such a distribution. One can easily compute d and see that $d > 1$.

(Distributions of the form (2.5) in fact lie in the intersection between our classes $\mathscr{S}(d)$ and the class of G with faster than exponential tail decay.)

The following lemma describes the most important property of distributions in the class $\mathscr{S}(d)$. It could be derived—just as the equality $c = 2d$—by the general methods of [5]; however, we include here a more elementary proof which takes advantage of our using only tails of distributions.

For any γ, $0 \leqq \gamma < 1$ define

$$U_\gamma(t) = \sum_{n=0}^{\infty} \gamma^n G_n(t) . \tag{2.6}$$

LEMMA 1. *If $G \in \mathscr{S}(d)$ and $\gamma d < 1$, then*

$$\lim_{t\to\infty} \frac{(1-\gamma)^{-1} - U_\gamma(t)}{1 - G(t)} = \frac{\gamma}{(1-\gamma d)^2} . \tag{2.7}$$

PROOF.[1] Observe that

$$\frac{(1-\gamma)^{-1} - U_\gamma(t)}{1 - G(t)} = \sum_n \gamma^n \frac{1 - G_n(t)}{1 - G(t)} . \tag{2.8}$$

We will show (by induction) that

$$\frac{1 - G_n(t)}{1 - G(t)} \to n d^{n-1} \qquad \text{as } t \to \infty , \tag{2.9}$$

and that given any $\varepsilon > 0$ there is a $K < \infty$ such that

$$\frac{1 - G_n(t)}{1 - G(t)} \leqq K(d+\varepsilon)^n \qquad \text{for all } t \geqq 0 . \tag{2.10}$$

This allows us to take the limit as $t \to \infty$ through the right side of (2.8), and then (2.9) implies (2.7).

Suppose first that (2.9) holds for some fixed n. Then for any $0 < A < t$

$$\begin{aligned} \frac{1 - G_{n+1}(t)}{1 - G(t)} = 1 &+ \int_0^{t-A} \frac{1 - G_n(t-y)}{1 - G(t-y)} \frac{1 - G(t-y)}{1 - G(t)} dG(y) \\ &+ \int_{t-A}^{t} \frac{1 - G_n(t-y)}{1 - G(t)} dG(y) . \end{aligned} \tag{2.11}$$

Integrating by parts,

$$\begin{aligned} &\lim_{t\to\infty} \int_{t-A}^{t} \frac{1 - G_n(t-y)}{1 - G(t)} dG(y) \\ &= -1 + [1 - G_n(A)]e^{\rho A} + \int_0^A e^{\rho y} dG_n(y) \\ &\to d^n - 1 \qquad \text{as } A \to \infty \end{aligned} \tag{2.12}$$

[1] In the case when $d = 1$ and $\gamma \leqq C^{-1}$, where

$$C = \sup_t \frac{G(t) - G_2(t)}{1 - G(t)} ,$$

a proof somewhat along the following lines is contained in Chistyakov [3].

(since $\int e^{\rho y}\, dG_n(y) = d^n$). Thus also

$$\lim_{t\to\infty} \int_0^{t-A} \frac{1 - G(t-y)}{1 - G(t)}\, dG(y)$$

$$= \lim_{t\to\infty} \frac{G(t) - G_2(t)}{1 - G(t)} - \lim_{t\to\infty} \int_{t-A}^{t} \frac{1 - G(t-y)}{1 - G(t)}\, dG(y) \tag{2.13}$$

$\to d$ as $A \to \infty$. (We have used (2.12) here.)

The induction hypothesis together with (2.11), (2.12), (2.13) implies (2.9) for $n + 1$ and hence for all n.

Next assume (2.10) is true for fixed n, pick any $\varepsilon > 0$, and choose $0 < A < T < \infty$ so that

$$\sup_{t \geq T} \int_0^{t-A} \frac{1 - G(t-y)}{1 - G(t)}\, dG(y) < d + \frac{\varepsilon}{2}, \tag{2.14}$$

and choose K so that

$$\frac{2B_T}{\varepsilon d} < K < \infty, \tag{2.15}$$

where

$$B_T = 1 + \frac{1}{1 - G(T)} + \sup_{t \geq T} \frac{G(t) - G(t-A)}{1 - G(t)}.$$

Then

$$\begin{aligned}\sup_{t \geq 0} \frac{1 - G_{n+1}(t)}{1 - G(t)} &\leq 1 + \sup_{0 \leq t \leq T} \int_0^t \frac{1 - G_n(t-y)}{1 - G(t)}\, dG(y) \\ &\quad + \sup_{t \geq T} \int_{t-A}^{t} \frac{1 - G_n(t-y)}{1 - G(t)}\, dG(y) \\ &\quad + \sup_{t \geq T} \int_0^{t-A} \frac{1 - G_n(t-y)}{1 - G(t-y)} \cdot \frac{1 - G(t-y)}{1 - G(t)}\, dG(y) \\ &\leq B_T + \left[\sup_{t \geq 0} \frac{1 - G_n(t)}{1 - G(t)}\right] \\ &\quad \times \left[\sup_{t \geq T} \int_0^{t-A} \frac{1 - G(t-y)}{1 - G(t)}\, dG(y)\right],\end{aligned}$$

which, by (2.13) and the induction hypothesis, $\leq B_T + K(d + \varepsilon)^n(d + \varepsilon/2)$ for A and T sufficiently large. Via (2.15) this implies (2.10), with n replaced by $(n + 1)$. This completes the proof.

COROLLARY 1. *If $G \in \mathscr{S}(d)$ and $\gamma d < 1$, then*

$$\lim_{t\to\infty} \frac{(1-\gamma)^{-1} - U_\gamma(t-b)}{(1-\gamma)^{-1} - U_\gamma(t)} = e^{\rho b}.$$

COROLLARY 2. *If $G \in \mathscr{S}(d)$ and $\gamma d < 1$ then*

$$\lim_{t\to\infty} \frac{[1 - G(t)] * U_\gamma(t)}{1 - G(t)} = \frac{1-\gamma}{(1-\gamma d)^2}.$$

PROOF. The left side above

$$= \lim_{t\to\infty} \frac{1}{\gamma} \sum \gamma^n \frac{1 - G_n(t)}{1 - G(t)} - \lim_{t\to\infty} \sum \gamma^n \frac{1 - G_n(t)}{1 - G(t)} .$$

Apply Lemma 1.

COROLLARY 3. *If $h(t)$ is of bounded variation and right continuous, $G \in \mathscr{S}(d)$, $\gamma d < 1$ and $h(t) \sim 1 - G(t)$ then*

$$\lim_{t\to\infty} \frac{h(t) * U_\gamma(t)}{1 - G(t)} = \frac{1 + \gamma[\rho\hat{\eta}(-\rho) - 1]}{(1 - \gamma d)^2} ,$$

where $\eta(t) = h(t) - [1 - G(t)]$ and $\hat{}$ denotes the Laplace transform. (Note that $\hat{\eta}(-\rho)$ exists by (2.3).)

PROOF. For any $0 < A < t$

$$h(t) * U_\gamma(t) = [1 - G(t)] * U_\gamma(t) + J_1(t) + J_2(t) , \tag{2.16}$$

where

$$J_1(t) = \int_0^{t-A} \left\{\frac{h(t-y)}{1 - G(t-y)} - 1\right\} [1 - G(t-y)]\, dU_\gamma(y)$$

$$J_2(t) = \int_{t-A}^{t} \{h(t-y) - [1 - G(t-y)]\}\, dU_\gamma(y) .$$

Since by hypothesis $h \sim 1 - G$, Corollary 2 implies that

$$J_1(t) = o[1 - G(t)] . \tag{2.17}$$

(The little o is as A and $t \to \infty$.) Since η is of bounded variation, we can integrate J_2 by parts to obtain

$$J_2(t) = \eta(A)\left[\frac{1}{1-\gamma}\right] - U_\gamma(t - A)] - \eta(0)\left[\frac{1}{1-\gamma} - U_\gamma(t)\right]$$

$$- \int_0^A \left[\frac{1}{1-\gamma} - U_\gamma(t-y)\right] d\eta(y) .$$

Now apply Lemma 1 and Corollary 1 to conclude that

$$\lim_{t\to\infty} \frac{J_2(t)}{1 - G(t)} = \eta(A)e^{\rho A} \frac{\gamma}{(1-\gamma d)^2} - \eta(0) \frac{\gamma}{(1-\gamma d)^2}$$

$$- \int_0^A \frac{\gamma}{(1-\gamma d)^2} e^{\rho y}\, d\eta(y) .$$

By integrating by parts and letting $A \to \infty$, we see

$$\lim_{t\to\infty} \frac{J_2(t)}{1 - G(t)} = \frac{\rho\gamma\hat{\eta}(-\rho)}{(1-\gamma d)^2} .$$

The result now follows by (2.16) and Corollary 2.

3. Asymptotic behavior of $Z(t)$. The generating function $F(s, t)$ is the unique bounded solution of the equation

$$F(s, t) = s[1 - G(t)] + \int_0^t f[F(s, t - y)]\, dG(y) . \tag{3.1}$$

From this one shows that $\mu(t) = EZ(t)$ is the unique bounded solution of

$$\mu(t) = 1 - G(t) + m \int_0^t \mu(t-y)\, dG(y)\,; \tag{3.2}$$

and hence that

$$\mu(t) = [1 - G(t)] * U_m(t)\,, \tag{3.3}$$

where

$$U_m(t) = \sum_{n=0}^{\infty} m^n G_n(t) \tag{3.4}$$

and $*$ denotes convolution. For proofs of the above facts see [1] or [6].

THEOREM 1. *If $G \in \mathscr{S}(d)$ and $md < 1$, then*

$$\mu(t) \sim \frac{1-m}{(1-md)^2}[1 - G(t)] \qquad \text{as } t \to \infty\,. \tag{3.5}$$

PROOF. Corollary 2.

THEOREM 2. *If $G \in \mathscr{S}(d)$ and $md < 1$ then*

$$\lim_{t\to\infty} \frac{1 - F(s,t)}{1 - G(t)} \equiv L(s) \qquad \text{exists and is} \quad \geqq 1 - s\,. \tag{3.6}$$

PROOF. Given any $\varepsilon > 0$, there is a u_0 such that

$$m - \varepsilon < \frac{1 - f(u)}{1 - u} \leqq m \qquad \text{for } 1 \geqq u > u_0\,.$$

Fix s. Since $F(s,t) \nearrow 1$ (see [1]), there exists u_0 such that t_0 is a continuity point for G and such that $F(s,t) > u_0$ for $t \geqq t_0$. Then by decomposing the integral in (3.1) from 0 to $t - t_0$ and $t - t_0$ to t one can show that

$$\begin{aligned} R_{m-\varepsilon}(t) + (m-\varepsilon) \textstyle\int_0^t [1 - F(s, t-y)]\, dG(y) & \\ \leqq 1 - F(s,t) & \\ \leqq R_m(t) + m \textstyle\int_0^t [1 - F(s,t-y)]\, dG(y)\,, & \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} R_m(t) = (1-s)[1 - G(t)] + \textstyle\int_{(t-t_0)^+}^t \{1 - f[F(s,t-y)]\}\, dG(y) \\ - m \textstyle\int_{(t-t_0)^+}^t [1 - F(s,t-y)]\, dG(y)\,. \end{aligned} \tag{3.8}$$

$((x)^+ \equiv \max(0, x))$.
Iterating (3.7) we obtain

$$R_{m-\varepsilon}(t) * U_{m-\varepsilon}(t) \leqq 1 - F(s,t) \leqq R_m(t) * U_m(t)\,,$$

and since $R_m \leqq R_{m-\varepsilon}$

$$R_m(t) * U_{m-\varepsilon}(t) \leqq 1 - F(s,t) \leqq R_m(t) * U_m(t)\,. \tag{3.9}$$

Applying Corollary 3 with $\gamma = m - \varepsilon$ and m, the existence of the limit in (3.6) is immediate from

LEMMA 2. *If $G \in \mathscr{S}(d)$, $d > 1$, $md < 1$, then $R_m(t)$ is of bounded variation, $\sim$* const. $[1 - G(t)]$, *and right continuous.*

PROOF OF LEMMA 2. With a little manipulation in (3.8), one can write $R_m(t)$ as sums and differences of monotone functions; hence it is of bounded variation. The right continuity is trivial.

For the asymptotic behavior, it is sufficient to show that

$$\int_{t-t_0}^{t} \xi(t-y)\,dG(y) \sim \text{const.}\,[1-G(t)]\,, \qquad t \to \infty\,, \tag{3.10}$$

for any monotone, bounded right continuous $\xi(\cdot)$. Via integration by parts, the left side of (3.10)

$$= \xi(t_0)[1-G(t-t_0)] - \xi(0)[1-G(t)] - \int_0^{t_0}[1-G(t-y)]\,d\xi(y)\,.$$

The conclusion follows from the defining properties of $\mathscr{S}(d)$.

The fact that the limit in (3.6) is $\geqq 1-s$ follows from the fact that $1-F(s,t) > (1-s)[1-G(t)]$, which is clear from (3.1).

4. The limit law for $Z(t)$. We have seen that when $d=1$, $G(t)-G(t-b) = o[1-G(t)]$ as $t \to \infty$. Using this fact in (3.8), we can conclude that

$$R_m(t) = (1-s)[1-G(t)] + o[1-G(t)]\,.$$

Hence by (3.9) and Corollary 3

$$L(s) = \frac{1-s}{1-m}\,. \tag{4.1}$$

Thus

$$\lim_{t\to\infty} Es^{\bar{Z}(t)} = \lim \frac{F(s,t)-F(0,t)}{1-F(0,t)} = s\,.$$

($\bar{Z}(t) = Z(t)$ conditioned on non-extinction.)

This fact was proved for $f''(1) < \infty$ and $m < c^{-1}$ (see the footnote to the proof of Lemma 1) by Chistyakov [3]. Thus

THEOREM 3. *If $G \in \mathscr{S}(1)$, then*

$$\lim_{t\to\infty} P\{Z(t) = 1 \mid Z(t) \neq 0\} = 1\,, \tag{4.2}$$

i.e. the Yaglom limit law is degenerate.

This contrasts with the case $d > 1$, for which we have the following result.

THEOREM 4. *If $G \in \mathscr{S}(d)$, $d > 1$, $md < 1$, then*

$$\lim_{t\to\infty} P\{Z(t) = k \mid Z(t) > 0\} \equiv b_k \quad \textit{exists} \tag{4.3}$$

and $0 < b_1 < 1$, i.e. the limit law is nondegenerate.

REMARK. If $d > 1$ and $md \geqq 1$, then the Malthusian parameter exists and (4.3) is known to be nondegenerate.

PROOF OF THEOREM 4. The existence of the limit in (3.1) follows from Theorem 2. To prove that $b_1 < 1$ it is sufficient to show that $L(s)$ is not a linear function of s. To this end the following formula is useful.

LEMMA 3. *If* $G \in \mathscr{S}(d)$, $d > 1$, $md < 1$, $0 \leqq s \leqq 1$, *then*

$$L(s) = (1 - md)^{-1}\{(1 - s) + \textstyle\int_0^\infty (1 - f[F(s, t)])\rho e^{\rho t}\, dt\}\,, \tag{4.4}$$

where the integral in (4.4) *converges.*

PROOF OF LEMMA 3. Let $\lambda(s, t) = f[F(s, t)]$, and observe that

$$\lim_{t\to\infty} \frac{1 - \lambda(s, t)}{1 - G(t)} = \lim_{t\to\infty} \frac{1 - f[F(s, t)]}{1 - F(s, t)} \frac{1 - F(s, t)}{1 - G(t)} = mL(s)\,, \tag{4.5}$$
$$\lambda(s, 0) = f(s)\,,$$

and $\lambda(s, t)$ is increasing in t, with $\lambda(s, t) \nearrow 1$ as $t \to \infty$. Thus, integrating by parts we have

$$\begin{aligned}&\textstyle\int_{t-T}^{t} \frac{1 - \lambda(s, t - y)}{1 - G(t)}\, dG(y)\\ &\quad = [1 - \lambda(s, T)] \frac{1 - G(t - T)}{1 - G(t)} - [1 - f(s)] + \textstyle\int_0^T \frac{1 - G(t - y)}{1 - G(t)} \lambda(s, dy)\\ &\quad \to e^{\rho T}[1 - \lambda(s, T)] - [1 - f(s)] + \textstyle\int_0^T e^{\rho y}\lambda(s, dy)\end{aligned}$$

as $t \to \infty$. Integrating by parts again we see that

$$\lim_{t\to\infty} \textstyle\int_{t-T}^{t} \frac{1 - \lambda(s, t - y)}{1 - G(t)}\, dG(y) = \textstyle\int_0^T \rho e^{\rho y}[1 - \lambda(s, y)]\, dy\,. \tag{4.6}$$

Next, since

$$1 - \lambda(s, t) = O(1 - G(t))$$

observe that for some constant $K < \infty$,

$$\lim_{t\to\infty} \textstyle\int_T^{t-T} \frac{1 - \lambda(s, t - y)}{1 - G(t)}\, dG(y) \leqq \lim_{t\to\infty} K \textstyle\int_T^{t-T} \frac{1 - G(t - y)}{1 - G(t)}\, dG(y)\,, \tag{4.7}$$

where by (2.1) and (2.3) the right side goes to 0 as $T \to \infty$.

Finally, by (4.5)

$$\begin{aligned}&\lim_{t\to\infty} \textstyle\int_0^T \frac{1 - \lambda(s, t - y)}{1 - G(t)}\, dG(y)\\ &\quad = mL(s) \textstyle\int_0^T e^{\rho y}\, dG(y) \to mdL(s) \qquad \text{as} \quad T \to \infty\,.\end{aligned} \tag{4.8}$$

Combining (4.6), (4.7) and (4.8).

$$\lim_{t\to\infty} \textstyle\int_0^t \frac{1 - f[F(s, t - y)]}{1 - G(t)}\, dG(y) = mdL(s) + \textstyle\int_0^\infty \rho e^{\rho y}[1 - F(s, y)]\, dy\,. \tag{4.9}$$

Combining (4.9) with (3.1) yields (4.4).

REMARK. If $d = 1$, then the limit in (4.6) is zero, implying (4.1).

Returing to the proof of Theorem 3, it is sufficient to show that

$$I(s) \equiv \textstyle\int_0^\infty (1 - f[F(s, t)])\rho e^{\rho t}\, dt$$

is not linear in s. If it were, then we would have

$$I(s+\delta)+I(s-\delta)-2I(s)=0$$

or

$$\int_0^\infty \rho e^{\rho t}\{2f[F(s,t)]-f[F(s+\delta,t)]-f[F(s-\delta,t)]\}\,dt=0\,.$$

But due to the convexity of f and F the integrand is $\leqq 0$, and hence must be $=0$. This implies that f and F are linear, a contradiction.

To see that $b_1>0$ note that $b_1=-L'(0)/L(0)$, and it is clear that $L(0)-L(h)\geqq h(1-md)^{-1}$ and $L(0)$ is positive.

5. Some remarks. Consider an arbitrary one of the $Z(t)$ particles existing at time t, and consider its "generation number," i.e. the number of ancestors it has. It can be shown that in a certain average sense

(i) if $G(t)\in\mathscr{S}(d)$ for $d\geqq 1$, then this generation number converges to a finite limit as $t\to\infty$;

(ii) if the Malthusian parameter for m, G exists, then the generation number goes to ∞ as $t\to\infty$.

These ideas are discussed in detail in Athreya and Ney [2].

It is interesting to observe here, however, that the class $\{\mathscr{S}(d);\, d>1\}$ plays a borderline role for branching processes. Namely

(a) If $G\in\mathscr{S}(1)$ then the process is degenerate in two senses: the conditioned limit law is concentrated at 1, and the (conditional) distribution of the generation number of live particles converges to a proper distribution (see [2]).

(b) If the Malthusian parameter exists, then the limit law in the Yaglom theorem is nondegenerate, and the generation number goes to ∞ as $t\to\infty$ (as one would expect).

(c) If $G\in\{\mathscr{S}(d),\, d>1\}$, then the conditioned limit law is nondegenerate as in (b), but the generation numbers are finite as in (a).

6. An alternate method. In this section we briefly sketch an alternate method for obtaining limit results such as in Theorem 2, which exposes the nature of the integral in

$$F(s,t)=s[1-G(t)]+\int_0^t f[F(s,t-y))\,dG(y) \tag{6.0}$$

as inducing a *contraction* transformation in an appropriate function space, when f is a subcritical generating function. We consider here only smooth G, with continuous nonzero density g, and write out assumptions about convolution $(*)$ in terms of g.

THEOREM 2′. *Let the continuous nonzero density g on $[0,\infty)$ satisfy the following conditions*:

$$\lim_{t\to\infty}\frac{g*g(t)}{g(t)}\quad \textit{exists}=c\;(<\infty) \tag{6.1}$$

$$\lim_{t\to\infty}\frac{g(t-b)}{g(t)}\quad \textit{exists}=\psi(b) \tag{6.2}$$

for all real b. Necessarily $\psi(b) \equiv e^{\rho b}$ *for some* $\rho \geqq 0$. *Assume that* $\rho > 0$ *and that*

$$(6.3) \qquad \int_0^\infty e^{\rho t} g(t)\, dt \quad \text{exists} = d\,(< \infty)\,.$$

(*Necessarily* $c = 2d$). *Suppose moreover that*

$$(6.4) \qquad \lim_{t\to\infty} \frac{g(t)}{1 - G(t)} \qquad \text{exists} = \sigma \quad (0 < \sigma < \infty)\,,$$

and finally, that $md < 1$. *Then*

$$(6.5) \qquad \lim_{t\to\infty} \frac{1 - F(s, t)}{g(t)} \quad \text{exists} = L(s) > 0\,.$$

OUTLINE OF PROOF. The main idea—similar to that in [5]—is to construct a (metric) space W of functions w which have a required limiting property ($\lim_{t\to\infty} w(t)/g(t)$ exists), and to show that for fixed s the function $1 - F(s, t)$ lies in W—or at least "close enough" to some element $w_0 \in W$. To this end we fix s and rewrite (6.0) in the form

$$(6.6) \qquad z(t) = \eta_A(t) + \int_0^{t-A} h(z(t-y)) g(y)\, dy\,,$$

where $z(t) \equiv 1 - F(s, t)$, A is a positive constant to be chosen later, $h(x) \equiv 1 - f(1 - x)$, and η_A is a remainder term. (Equation (6.6) will have a unique solution subject to $0 \leqq z(t) \leqq 1$.) Note that $h'(0) = f'(1) = m < 1$. Since we expect the solution $z(t)$ in (6.6) to approach 0 as $t \to \infty$, the fact that $h' < 1$ for small arguments suggests that the right-hand side of (6.6) may represent a contraction operation on z. If we can show this operation to be acting in the constructed space W, we shall have a fixed point w_0—a solution of (6.6) with the desired limit property. Actually we cannot achieve this goal exactly, but we can carry the program through for solutions w of a slightly modified version (6.6′) of (6.6), and then show that the fixed point w_0 of (6.6′) can be taken so close to the solution z of (6.6) that its limit properties carry over to the latter.

For the required modification, choose $\varepsilon > 0$, and choose some continuous increasing function $\gamma(t) = \gamma(t; A, \varepsilon)$ with $\gamma(t) \equiv 1$ for $t \geqq$ some $t_1(A, \varepsilon)$, and satisfying the technical requirement

$$(6.7) \qquad \frac{\gamma(t)}{g(t)} \int_0^A g(t-y) g(y)\, dy \leqq (1 + \varepsilon) \int_0^\infty e^{\rho y} g(y)\, dy\,, \qquad t \geqq 0\,.$$

Now let W consist of all functions w on $[0, \infty)$ such that $0 \leqq w(t) \leqq 1$ for $t \geqq 0$ and $\lim_{t\to\infty} w(t)/(1 - G(t))$ exists; and define a metric

$$\rho(w_1, w_2) = \sup_{t\geq 0} \frac{\gamma(t)}{g(t)} |w_1(t) - w_2(t)|\,, \qquad w_1, w_2 \in W\,.$$

W will be complete with respect to ρ. We use $\gamma(t)$ also to modify (6.6) for small values of the argument $t - y$, namely:

$$(6.6') \qquad w(t) = \eta_A(t) + \int_0^{t-A} h(\gamma(t-y) w(t-y)) g(y)\, dy \equiv Tw\,.$$

(A solution to (6.6′) must also be unique, subject to the requirement $0 \leqq w(t) \leqq 1$ for $t \geqq 0$.) The main tasks now are (a) to show that the transformation T defined by the right hand side of (6.6′) maps W into W; and (b) to show that T is a contraction, that is, $\rho(Tw_1, Tw_2) < \theta\rho(w_1, w_2)$ for some constant θ with $0 < \theta < 1$, and $w_1, w_2 \in W$. Actually (b) is easier to accomplish than (a); and for the proof, we must choose ε sufficiently small in (6.7). Once we know T to be a contraction in W, we have a (unique) function $w_0 \in W$ satisfying (6.6′)—and for which $\lim_{t\to\infty} w_0(t)/g(t)$ exists $= \lambda(A, \varepsilon)$. Then, we can return to the original solution $z(t) = 1 - F(s, t)$ of (6.6), compare it with $w_0(t)$, and get an inequality

$$\left|\frac{z(t)}{1 - G(t)} - \lambda(A, \varepsilon)\right| \leqq K\varepsilon \qquad \text{for } t \geqq t_0(A),$$

for some fixed K. We can show that $\lambda(A_n, \varepsilon_n) \to L = L(s) > 0$ for suitable sequences $\varepsilon_n \to 0$ and $A_n \to \infty$. Thus it follows finally that $\lim_{t\to\infty} z(t)/(1 - G(t) = L(s)$.

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