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LIMITING BEHAVIOR OF WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES

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In this paper, we study weighted sums $\sum_{i=1}^{n} c_{n-i} X_i$ of i.i.d. zero-mean random variables $X_1, X_2, \cdots$, under the condition that the sequence (c_n) is square summable. It is proved that such weighted sums are, with probability 1, of smaller order than $n^{1/\alpha}$ (respectively $\log n$, etc.) iff $E|X_1|^\alpha < \infty$ (respectively $Ee^{t|X_1|} < \infty$ for all $t < \infty$, etc.). Certain analogs of the law of the iterated logarithm for such weighted sums are also obtained.

1. Introduction. Let $X_1, X_2, \cdots$ be i.i.d. random variables with mean 0, and let $(c_n, n \geqq 0)$ be a sequence of real numbers such that $\sum_{n=0}^{\infty} c_n^2 < \infty$. In this paper, we study the limiting behavior of the sequence $(Y_n, n \geqq 1)$ of weighted sums, where $Y_n = \sum_{i=1}^{n} c_{n-i} X_i$. Such weighted sums of observations are used in [9] for detecting changes in the location of the distribution of a sequence of independent observations, such as in quality control problems. We find that in many respects, the limiting behavior of Y_n resembles that of a sequence of independent random variables.

Another kind of weighted sums of independent random variables has been considered by Gal [5], Stackleberg [13], Strassen [15], Gaposhkin [6] and Tomkins [16]. Let f be a continuous function on [0, 1] and consider $Z_n = \sum_{i=1}^{n} f(i/n) X_i$. Tomkins [16] proves that under certain conditions on f and X_i, the weighted sum Z_n satisfies the law of the iterated logarithm:

$$\limsup_{n\to\infty} \frac{Z_n}{(2n \log \log n)^{\frac{1}{2}}} = \left(\int_0^1 f^2(t)\, dt\right)^{\frac{1}{2}} \quad \text{a.e.}$$

The limiting behavior of the Z_n sequence therefore resembles that of partial sums of the X_i's.

A more general problem which includes both kinds of weighted sums above is to consider $\sum_{k=1}^{n} a_{nk} X_k$. Some convergence theorems for these general weighted sums have been obtained by Chow [2], Hanson and Koopman [7], Hill [8], Pruitt [12] and Stout [14]. In Section 4, we consider such general weighted sums under the condition $\limsup_{n\to\infty} \sum_1^n a_{nk}^2 < \infty$. Sections 2 and 3 are devoted to the study

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of weighted sums of the form $\sum_{i=1}^{n} c_{n-i} X_i$, and we obtain analogs of the strong law of large numbers and of the law of the iterated logarithm for such weighted sums.

2. Strong law of large numbers for weighted sums. The following theorem is an analog of Kolmogorov's strong law of large numbers. For $\alpha = 2$, it is a special case of a theorem of [2]. We shall say that a sequence (c_n) of real numbers is void if $c_n = 0$ for all n and that it is nonvoid if otherwise.

THEOREM 1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For any* $\alpha \geqq 1$, *the following statements are equivalent*:

(1) $$E|X_1|^\alpha < \infty$$

(2) $$\lim_{n\to\infty} n^{-1/\alpha} X_n = 0 \quad \text{a.e.}$$

(3) $\lim_{n\to\infty} n^{-1/\alpha} \sum_{i=1}^{n} c_{n-i} X_i = 0$ a.e. *for some* (*or equivalently for every*) *nonvoid sequence of real numbers* $(c_n, n \geqq 0)$ *such that* $\sum_{n=0}^{\infty} c_n^2 < \infty$.

LEMMA 1. *Suppose* $X_1, X_2, \cdots$ *are* i.i.d. *random variables such that* $EX_1 = 0$, $EX_1^2 < \infty$. *Let* $(a_n, n \geqq 1)$ *be a nonvoid sequence of real numbers such that* $\sum_{n=1}^{\infty} a_n^2 = A < \infty$, *and let*

$$Z_n = \sum_{i=1}^{n} a_i X_i\,, \qquad Z = \sum_{i=1}^{\infty} a_i X_i\,.$$

Then

(i) *for* $\alpha \geqq 2$, $E|X_1|^\alpha < \infty$ iff $E|Z|^\alpha < \infty$. *In this case,*

(4) $$\sup_{n\geqq 1} E|Z_n|^\alpha \leqq B_\alpha A^{\alpha/2} E|X_1|^\alpha$$

where B_α *is a constant depending only on* α.

(ii) $Ee^{t|X_1|} < \infty\ \forall t > 0$ iff $Ee^{t|Z|} < \infty\ \forall t > 0$. *In this case*, $\sup Ee^{t|Z_n|} < \infty$ $\forall t > 0$.

PROOF. To prove (i), we first assume that $E|Z|^\alpha < \infty$. Let $a_m \neq 0$. Then $Z = a_m X_m + (Z - a_m X_m)$. Since $a_m X_m$ and $Z - a_m X_m$ are independent, $E|Z|^\alpha < \infty$ implies that $E|a_m X_m|^\alpha < \infty$. Conversely, assume that $E|X_1|^\alpha < \infty$. By the Marcinkiewicz-Zygmund inequalities [11], there exists a constant B_α depending only on α such that

$$\begin{aligned} E|Z_n| &\leqq B_\alpha E(\textstyle\sum_1^n a_j^2 X_j^2)^{\alpha/2} \\ &= B_\alpha E(\textstyle\sum_1^n a_j^{(2\alpha-4)/\alpha} a_j^{4/\alpha} X_j^2)^{\alpha/2} \\ &\leqq B_\alpha E\{(\textstyle\sum_1^n a_j^2)^{\alpha/2-1} \sum_1^n a_j^2 |X_j|^\alpha\} \\ &\leqq B_\alpha A^{\alpha/2} E|X_1|^\alpha\,. \end{aligned}$$

Hence (4) holds and by Fatou's lemma, $E|Z|^\alpha \leqq \sup E|Z_n|^\alpha < \infty$.

To prove (ii), assume that $Ee^{t|X_1|} < \infty\ \forall t > 0$. Without loss of generality, we can assume that $EX_1^2 = 1$. Let $\psi(\theta) = Ee^{\theta X_1}$. Then ψ is an entire function, $\psi(0) = 1$, $\psi'(0) = 0$ and $\psi''(0) = 1$. Therefore there exists $\theta_0 > 0$ such that for $|\theta| \leqq \theta_0$, $\psi(\theta) \leqq 1 + \theta^2$. For any real t, we choose k such that $|a_n t| \leqq \theta_0$ for

$n \geqq k$. Then for $n \geqq k$,

$$Ee^{tZ_n} \leqq \{\prod_{i=1}^k \psi(a_i t)\} \prod_{i=k+1}^\infty (1 + a_i^2 t^2)$$
$$\leqq \{\prod_{i=1}^k \psi(a_i t)\} \exp(At^2) .$$

Since $Ee^{t|Z_n|} \leqq E(e^{tZ_n} + e^{-tZ_n})$, it follows that $\sup Ee^{t|Z_n|} < \infty$ and by Fatou's lemma, $Ee^{t|Z|} < \infty$.

Conversely, assume that $Ee^{t|Z|} < \infty \ \forall t > 0$. Let $a_m \neq 0$. Then $Z = a_m X_m + (Z - a_m X_m)$. Since $a_m X_m$ and $Z - a_m X_m$ are independent and $Ee^{tZ} < \infty$ for all real t, we have $E\exp(ta_m X_m) < \infty$ for all real t ([10], page 214). Hence $Ee^{t|X_1|} < \infty \ \forall t > 0$.

LEMMA 2. *Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of random variables such that $(X_1, \cdots, X_n)$ and Y_n are independent for each $n \geqq 1$. Suppose $f: R \times R \to R$ is a Borel function such that $f(X_n, Y_n) \to f(a, b)$* a.e. *and given any $\varepsilon > 0$, there exists $\delta > 0$ for which*

$$\inf\{|f(x, y) - f(a, b)| : |x - a| > \varepsilon, |y - b| < \delta\} > 0 .$$

If $Y_n \to_P b$, then $X_n \to_{\text{a.e.}} a$.

PROOF. Given any $\varepsilon > 0$, we have $\delta > 0$ and $\eta > 0$ such that $|f(x, y) - f(a, b)| \geqq \eta$ for all x, y with $|x - a| \geqq \varepsilon$ and $|y - b| \leqq \delta$. Let $\tau_m = \inf\{n \geqq m : |X_n - a| \geqq \varepsilon\}$. Then

$$\begin{aligned}
0 &= \lim_{m\to\infty} P(\textstyle\bigcup_{n=m}^\infty [|f(X_n, Y_n) - f(a, b)| \geqq \eta]) \\
&\geqq \lim_{m\to\infty} P(\textstyle\bigcup_{n=m}^\infty [\tau_m = n, |Y_n - b| \leqq \delta]) \\
&= \lim_{m\to\infty} \textstyle\sum_{n=m}^\infty P[\tau_m = n] P[|Y_n - b| \leqq \delta] , \\
&\qquad\qquad \text{by the independence of } Y_n \text{ and } (X_1, \cdots, X_n) \\
&\geqq \tfrac{1}{2} \lim_{m\to\infty} \textstyle\sum_{n=m}^\infty P[\tau_m = n] , \qquad \text{since } Y_n \to_P b \\
&= \tfrac{1}{2} \lim_{m\to\infty} P(\textstyle\bigcup_{n=m}^\infty [|X_n - a| \geqq \varepsilon]) .
\end{aligned}$$

LEMMA 3. *Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of random variables such that $(X_1, \cdots, X_n)$ and Y_n are independent for each $n \geqq 1$. Suppose there exist a finite constant C and a random variable Y such that $\lim_{n\to\infty} (X_n + Y_n) = C$* a.e. *and Y_n converges in distribution to Y. Then $X_n \to_{\text{a.e.}} K$, a finite constant, and $Y_n \to_{\text{a.e.}} C - K$ and $P[Y = C - K] = 1$.*

PROOF. Suppose Y is not a degenerate random variable. Then we can choose a real number λ such that $E|e^{i\lambda Y}| < 1$ (cf. [10] page 202). Since $\lim_{n\to\infty} \exp(i\lambda(X_n + Y_n)) = e^{i\lambda C}$ a.e., we obtain (cf. [1] Theorem 2) that

$$\lim_{n\to\infty} E[\exp(i\lambda(X_n + Y_n)) \mid X_1, \cdots, X_n] = e^{i\lambda C} \quad \text{a.e.}$$

By the independence of $(X_1, \cdots, X_n)$ and Y_n, we therefore have

$$\lim_{n\to\infty} e^{i\lambda X_n} Ee^{i\lambda Y_n} = e^{i\lambda C} \quad \text{a.e.} \tag{5}$$

But $\lim_{n\to\infty} |Ee^{i\lambda Y_n}| = |Ee^{i\lambda Y}| < 1$, contradicting (5). Therefore Y must be degenerate, say $P[Y = C - K] = 1$. The desired conclusion then follows from Lemma 2.

PROOF OF THEOREM 1. By the Borel–Cantelli lemma, it is easy to show that (1) and (2) are equivalent. We now assume that there exists a nonvoid sequence $(c_n, n \geqq 0)$ such that $\lim_{n\to\infty} n^{-1/\alpha} \sum_{i=1}^n c_{n-i}X_i = 0$ a.e. Let $m = \inf\{i \geqq 0 : c_i \neq 0\}$. Using Lemma 3, it is easy to see that $\lim_{n\to\infty} n^{-1/\alpha}c_m X_{n-m} = 0$ a.e. Therefore (3) implies (2).

We now prove that (1) implies (3). For $1 \leqq \alpha \leqq 2$, this is a special case of Theorem 9 below. Now assume that $\alpha \geqq 2$. Let $(c_n, n \geqq 0)$ be any nonvoid sequence of real numbers such that $\sum c_n^2 < \infty$. Let $Z_n = \sum_{i=1}^n c_{i-1}X_i$ and $Z = \lim_{n\to\infty} Z_n$. Then for any $\varepsilon > 0$.

$$P[|\sum_{i=1}^n c_{n-i}X_i| > \varepsilon n^{1/\alpha}] = P[|Z_n| > \varepsilon n^{1/\alpha}] \leqq P[\sup_{j\geqq 1} |Z_j| > \varepsilon n^{1/\alpha}] . \tag{6}$$

By Lemma 1, $E|X_1|^\alpha < \infty$ implies that $E|Z|^\alpha < \infty$, and so by Doob's martingale dominated inequality ([3] page 317), $E(\sup_{j\geqq 1} |Z_j|)^\alpha < \infty$. Hence it follows from (6) that $\sum_1^\infty P[|\sum_{i=1}^n c_{n-i}X_i| > \varepsilon n^{1/\alpha}] < \infty$ and the proof is complete.

Using Lemma 1 (ii), we obtain in the following theorem necessary and sufficient conditions for $\sum_{i=1}^n c_{n-i}X_i/\log n$ to converge to 0 with probability one.

THEOREM 2. *Let $X_1, X_2, \cdots$ be* i.i.d. *random variables such that $EX_1 = 0$. Then the following statements are equivalent*:

$$Ee^{t|X_1|} < \infty \quad \forall t > 0 \tag{7}$$

$$\lim_{n\to\infty} X_n/\log n = 0 \quad \text{a.e.} \tag{8}$$

(9) $\lim_{n\to\infty} \sum_{i=1}^n c_{n-i}X_i/\log n = 0$ a.e. *for some (or equivalently for every) nonvoid sequence of real numbers $(c_n, n \geqq 0)$ such that $\sum_{n=0}^\infty c_n^2 < \infty$.*

PROOF. We shall only show that (7) implies (9), as the rest of the proof is similar to that of Theorem 1. Let $Z_n = \sum_{i=1}^n c_{i-1}X_i$. Given $\varepsilon > 0$, choose t such that $t\varepsilon = \gamma > 1$. By Lemma 1, $\sup_{n\geqq 1} Ee^{t|Z_n|} = K < \infty$. Therefore

$$\sum_{n=1}^\infty P[|\sum_{i=1}^n c_{n-i}X_i| > \varepsilon \log n] = \sum_{n=1}^\infty P[|Z_n| > \varepsilon \log n] \leqq \sum_{n=1}^\infty n^{-\gamma}Ee^{t|Z_n|} \leqq K \sum_{n=1}^\infty n^{-\gamma} < \infty .$$

We remark that as is evident from the proceeding proof, (7) implies not only almost everywhere convergence in (8) and (9), but it implies complete convergence as well.

We now extend the results of Theorem 2 to the case where $E\exp(t|X_1|^\alpha) < \infty$ for all $t > 0$ with $\alpha > 1$. One may expect that in this case,

$$\lim_{n\to\infty} (\log n)^{-1/\alpha} \sum_{i=1}^n c_{n-i}X_i = 0 \quad \text{a.e.}$$

However in order that the above limiting behavior holds, one has to narrow down the choice of the weighting sequence (c_n). In Theorem 7, we shall show that if $X_1, X_2, \cdots$ are i.i.d. bounded random variables with $EX_1 = 0$ and $EX_1^2 = 1$, then for any $\gamma \in (\frac{1}{2}, 1)$,

$$\limsup_{n\to\infty} (\log n)^{\gamma-1} \sum_{j=1}^n j^{-\gamma}X_{n-j+1} > 0 \quad \text{a.e.}$$

In the following theorem, we therefore restrict ourselves to weighting sequences (c_n) satisfying $c_n = O(n^{-\beta})$ with $\beta = \alpha/(\alpha+1)$.

THEOREM 3. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For* $\alpha > 1$, *the following statements are equivalent*:

(10) $$E\exp(t|X_1|^\alpha) < \infty \quad \forall t > 0$$

(11) $$\lim_{n\to\infty} (\log n)^{-1/\alpha} X_n = 0 \quad \text{a.e.}$$

(12) $\lim_{n\to\infty} (\log n)^{-1/\alpha} \sum_{i=1}^n c_{n-i} X_i = 0$ a.e. *for some (or equivalently for every) nonvoid sequence of real numbers* $(c_n, n \geqq 0)$ *such that* $c_n = O(n^{-\beta})$, *where* $\beta = \alpha/(\alpha+1)$.

LEMMA 4. *Let* $X_1, X_2, \cdots$ *be independent random variables such that* $EX_n = 0$ *and* $|X_n| \leqq \varepsilon$ *for all* n. *Let* (a_n) *be a nonvoid sequence of real numbers satisfying* $\sum_1^\infty a_n^2 = A < \infty$, *and let* $Z = \sum_1^\infty a_n X_n$. *Then for any* $\zeta > 0$,

$$P[Z \geqq \zeta\varepsilon] \leqq \exp(-\zeta^2(4A)^{-1}) .$$

PROOF. For all $t > 0$ and $n \geqq 1$, $Ee^{tX_n} \leqq \exp(t^2\varepsilon^2)$ (See [2]). Therefore $P[Z \geqq \zeta\varepsilon] \leqq e^{-\zeta t\varepsilon} Ee^{tZ} \leqq \exp\{-\zeta t\varepsilon + t^2\varepsilon^2 A\}$. Setting $t = \zeta(2\varepsilon A)^{-1}$, we obtain the desired inequality.

PROOF OF THEOREM 3. We shall only show that (10) implies (12). Suppose that (10) holds and that $(c_n, n \geqq 0)$ is any sequence of real numbers such that $c_n = O(n^{-\beta})$ where $\beta = \alpha/(\alpha+1)$. Let $X_1^*, X_2^*, \cdots$ be i.i.d. and independent of $(X_1, X_2, \cdots)$ and have the same distribution as X_1. Let $X_i^s = X_i - X_i^*$. By Lemma 3, it follows that

$$(\log n)^{-1/\alpha} \sum_{i=1}^n c_{n-i} X_i \to_{\text{a.e.}} 0 \quad \text{iff} \quad (\log n)^{-1/\alpha} \sum_{i=1}^n c_{n-i} X_i^s \to_{\text{a.e.}} 0 .$$

Therefore without loss of generality, we shall assume that X_1 is symmetric. Let $\eta = (2\beta - 1)^{-1} = (\alpha+1)/(\alpha-1)$, $\gamma = (\alpha(\alpha+1))^{-1}$, and choose $\lambda > 1$ such that for $n = 1, 2, \cdots$,

$$\lambda \geqq n^\beta |c_n| + \{\textstyle\sum_{j=0}^\infty |c_j|^{\alpha/(\alpha-1)}\}^{\alpha-1}$$

Given $0 < \varepsilon < 1$, put $t = 3\lambda/\varepsilon$ and let $E\exp(t|X_1|^\alpha) = e^\xi$. Take $\rho \in (0, 1)$ such that $\xi\rho < 1$ and $2\lambda^2\rho < 2\beta - 1$. Put $n' = [(\rho\log n)^\eta]$, $n'' = [\rho\log n]$ and set

(13)
$$\begin{aligned}
X_j^{(1)} &= X_j I_{[|X_j| \leqq \rho\varepsilon(\log j)^\gamma]} \\
X_j^{(2)} &= X_j I_{[|X_j| > \rho\varepsilon(\log j)^{1/\alpha}]} \\
X_j^{(3)} &= X_j I_{[\rho\varepsilon(\log j)^\gamma < |X_j| \leqq \rho\varepsilon(\log j)^{1/\alpha}]} \\
V_n &= (\log n)^{-1/\alpha} \textstyle\sum_{j=0}^{n''-1} c_j X_{n-j} \\
T_n^{(1)} &= (\log n)^{-1/\alpha} \textstyle\sum_{j=n''}^{n-1} c_j X_{n-j}^{(1)} \\
T_n^{(2)} &= (\log n)^{-1/\alpha} \textstyle\sum_{j=n''}^{n-1} c_j X_{n-j}^{(2)} \\
T_n^{(3)} &= (\log n)^{-1/\alpha} \textstyle\sum_{j=n''}^{n'-1} c_j X_{n-j}^{(3)} \\
U_n^{(3)} &= (\log n)^{-1/\alpha} \textstyle\sum_{j=n'}^{n-1} c_j X_{n-j}^{(3)} .
\end{aligned}$$

By Hölder inequality,

$$|\sum_1^{n''} c_{j-1} X_j|^\alpha \leqq \{\sum_1^{n''} |c_{j-1}|^{\alpha/(\alpha-1)}\}^{\alpha-1} \sum_1^{n''} |X_j|^\alpha \leqq \lambda \sum_1^{n''} |X_j|^\alpha$$

and therefore

$$\begin{aligned} P[|V_n| > \varepsilon^{1/\alpha}] &= P[|\sum_1^{n''} c_{j-1} X_j|^\alpha > \varepsilon \log n] \\ &\leqq P\left[t \sum_1^{n''} |X_j|^\alpha > \frac{t\varepsilon}{\lambda} \log n\right] \\ &\leqq \exp(-3 \log n) E \exp(t \sum_1^{n''} |X_j|^\alpha) \\ &= \exp(-3 \log n + \xi n'') \leqq n^{-2}. \end{aligned}$$

Hence $\sum P[|V_n| > \varepsilon^{1/\alpha}] < \infty$ and so

(14) $$\limsup_{n\to\infty} |V_n| \leqq \varepsilon^{1/\alpha} \quad \text{a.e.}$$

Since for $j = 1, \cdots, n$, $X_j^{(1)}$ are independent, $EX_j^{(1)} = 0$ and $(\log n)^{-\gamma}|X_j^{(1)}|/\rho \leqq \varepsilon$, it follows from Lemma 4 that for any $n \geqq n_0$,

$$\begin{aligned} P[T_n^{(1)} \geqq 2\varepsilon] &= P[(\log n)^{-1/\alpha+\gamma}(\log n)^{-\gamma} \sum_{j=n''}^{n-1} (c_j \rho)(X_{n-j}^{(1)}/\rho) \geqq 2\varepsilon] \\ &\leqq \exp\{-(\sum_{j=n''}^{\infty} c_j^2 \rho^2 (\log n)^{-2/\alpha+2\gamma})^{-1}\} \\ &\leqq \exp\{-2(\log n)^{2\beta-1+2/\alpha-2\gamma}\} = n^{-2}. \end{aligned}$$

Hence $\sum P[T_n^{(1)} \geqq 2\varepsilon] < \infty$ and therefore

(15) $$\limsup_{n\to\infty} T_n^{(1)} \leqq 2\varepsilon \quad \text{a.e.}$$

Now (10) implies (11), and so we have

(16) $$\lim_{n\to\infty} T_n^{(2)} = 0 \quad \text{a.e.}$$

Since $\lambda\rho < 1$, $X_j^{(3)} \leqq \rho\varepsilon(\log j)^{1/\alpha}$ and $|c_j| \leqq \lambda j^{-\beta}$, therefore $T_n^{(3)} \geqq \varepsilon$ implies that there are at least $(n'')^\beta$ nonzero $X_{n-j}^{(3)}$ for $j = n'', \cdots, n' - 1$. Choose $K > 1$ such that $K^{1/(1+\alpha)}\rho^\beta > 2$ and take $\theta > (K/\rho\varepsilon)^{1/(1+\alpha)}$. Letting $n_\beta = (n'')^\beta$, we have for all large n

(17) $$\begin{aligned} P[T_n^{(3)} \geqq \varepsilon] &\leqq \binom{n'}{n_\beta}\{P[|X_1| > (\rho\varepsilon \log (n - n'))^\gamma]\}^{n_\beta} \\ &\leqq \binom{n'}{n_\beta}\{P[\theta|X_1|^\alpha > (K \log n)^{1/(\alpha+1)}]\}^{n_\beta} \\ &\leqq \binom{n'}{n_\beta}[\exp\{-(K \log n)^{1/(\alpha+1)}\} E \exp(\theta|X_1|^\alpha)]^{n_\beta} \end{aligned}$$

Since $n' \sim (\rho \log \eta)^{\eta-\beta} n_\beta$, we have

(18) $$\begin{aligned} \log \binom{n'}{n_\beta} &= (n' + \tfrac{1}{2}) \log n' - (n' - n_\beta + \tfrac{1}{2}) \log (n' - n_\beta) \\ &\quad - (n_\beta + \tfrac{1}{2}) \log n_\beta + O(1) \\ &= n' \log \frac{n'}{n' - n_\beta} + n_\beta \log \frac{n' - n_\beta}{n_\beta} \\ &\quad + \tfrac{1}{2}(\log n' - \log (n' - n_\beta) - \log n_\beta) + O(1) \\ &= n_\beta(1 + o(1)) + n_\beta(\eta - \beta)(\log_2 n)(1 + o(1)) + O(\log n') \\ &= (\eta - \beta)(\rho \log n)^\beta (\log_2 n)(1 + o(1)) \end{aligned}$$

where $\log_2 n$ denotes $\log\log n$. From (17) and (18), we have for all large n

$$\begin{aligned} P[T_n^{(3)} \geqq \varepsilon] &\leqq \exp\{(\eta - \beta)(\rho \log n)^\beta (\log_2 n)(1 + o(1)) \\ &\qquad - (K \log n)^{1/(1+\alpha)}(\rho \log n)^\beta (1 + o(1))\} \\ &\leqq n^{-2} . \end{aligned}$$

The last inequality above follows from the fact that $\beta + (1 + \alpha)^{-1} = 1$ and $K^{1/(1+\alpha)}\rho^\beta > 2$. Therefore $\sum P[T_n^{(3)} \geqq \varepsilon] < \infty$ and so

(19) $$\limsup_{n\to\infty} T_n^{(3)} \leqq \varepsilon \quad \text{a.e.}$$

Since for $j = 1, \cdots, n$, $X_j^{(3)}$ are independent, $EX_j^{(3)} = 0$ and $(\log n)^{-1/\alpha}|X_j^{(3)}|/\rho \leqq \varepsilon$, we obtain from Lemma 4 that

$$\begin{aligned} P[U_n^{(3)} \geqq 2\varepsilon] &= P[(\log n)^{-1/\alpha} \textstyle\sum_{j=n'}^{n-1} (c_j \rho)(X_{n-j}^{(3)}/\rho) \geqq 2\varepsilon] \\ &\leqq \exp\{-(\textstyle\sum_{j=n'}^{\infty} c_j^2 \rho^2)^{-1}\} \leqq n^{-2} . \end{aligned}$$

Therefore $\sum P[U_n^{(3)} \geqq 2\varepsilon] < \infty$ and so

(20) $$\limsup_{n\to\infty} U_n^{(3)} \leqq 2\varepsilon \quad \text{a.e.}$$

From (14), (15), (16), (19), (20), we have

$$\limsup_{n\to\infty} (\log n)^{-1/\alpha} \textstyle\sum_{j=0}^{n-1} c_j X_{n-j} \leqq \varepsilon^{1/\alpha} + 5\varepsilon \quad \text{a.e.}$$

Since ε is an arbitrary positive number, the above upper limit is $\leqq 0$ a.e. Replacing X_n by $-X_n$, it is easy to see that (12) holds.

Let us now consider the case where $E\exp(t|X_1|^\alpha) < \infty$ for all $t > 0$ with $\alpha < 1$. A modification of the proof of Theorem 3 gives the following theorem.

THEOREM 4. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For* $0 < \alpha < 1$, *the following statements are equivalent*:

(21) $$E\exp(t|X_1|^\alpha) < \infty \quad \forall t > 0$$

(22) $$\lim_{n\to\infty} (\log n)^{-1/\alpha} X_n = 0 \quad \text{a.e.}$$

(23) $\lim_{n\to\infty} (\log n)^{-1/\alpha} \sum_{i=1}^{n} c_{n-i} X_i = 0$ a.e. *for some* (*or equivalently for every*) *nonvoid sequence of real numbers* $(c_n, n \geqq 0)$ *such that there exists* $\nu > \frac{1}{2}$ *for which* $c_n = O(n^{-\nu})$.

PROOF. To prove that (21) implies (23), we can again assume that X_1 is symmetric. Let $(c_n, n \geqq 0)$ be a sequence of real numbers such that $c_n = O(n^{-\nu})$ with $\frac{1}{2} < \nu < 1$. Let $\eta = (2\nu - 1)^{-1}$, $\gamma = (1 - \nu)/\alpha$ and choose $\lambda > 1$ such that $\lambda \geqq n^\nu |c_{n-1}|$ for all $n \geqq 1$. Given $0 < \varepsilon < 1$, put $t = 3\lambda/\varepsilon$ and let $E\exp(t|X_1|^\alpha) = e^\xi$. Take $\rho \in (0, 1)$ such that $(2\lambda^2 + \xi)\rho < 2\nu - 1$. Put $n' = [(\rho \log n)^\eta]$, $n'' = [\rho \log n]$ and define $X_j^{(1)}$, $X_j^{(2)}$, $X_j^{(3)}$, V_n, $T_n^{(1)}$, $T_n^{(2)}$, $T_n^{(3)}$, $U_n^{(3)}$ by (13). We note that since $\alpha < 1$, we have

$$|\textstyle\sum_{j=1}^{n''} c_{j-1} X_j|^\alpha \leqq \sum_{j=1}^{n''} |c_{j-1} X_j|^\alpha \leqq \lambda \sum_{j=1}^{n''} j^{-\alpha\nu} |X_j|^\alpha .$$

Therefore

$$
\begin{aligned}
P[|V_n| > \varepsilon^{1/\alpha}] &= P[|\textstyle\sum_1^{n''} c_{j-1}X_j|^\alpha > \varepsilon \log n] \\
&\leqq P\left[t \textstyle\sum_1^{n''} j^{-\alpha\nu}|X_j|^\alpha > \dfrac{t\varepsilon}{\lambda} \log n\right] \\
&\leqq \exp\left(-\dfrac{t\varepsilon}{\lambda}\log n\right) E\exp(t \textstyle\sum_1^{n''} j^{-\alpha\nu}|X_j|^\alpha) \\
&\leqq \exp(-3\log n)\{E\exp(t|X_1|^\alpha)\}^\sigma \qquad \text{where} \quad \sigma = \textstyle\sum_1^{n''} j^{-\alpha\nu} \\
&\leqq \exp(-3\log n + \xi \textstyle\sum_1^{n''} j^{-\alpha\nu}) \leqq n^{-2}.
\end{aligned}
$$

Furthermore, since $2\nu - 1 + 2/\alpha - 2\gamma > 1$,

$$
\begin{aligned}
\textstyle\sum_1^\infty P[T_n^{(1)} \geqq 2\varepsilon] &= \textstyle\sum_1^\infty P[(\log n)^{-1/\alpha+\gamma}(\log n)^{-\gamma} \sum_{j=n''}^{n-1} (c_j\rho)(X_{n-j}^{(1)}/\rho) \geqq 2\varepsilon] \\
&\leqq \textstyle\sum_1^\infty \exp\{-2(\log n)^{2\nu-1+2/\alpha-2\gamma}\} < \infty .
\end{aligned}
$$

Choose $K > 1$ such that $K^{\alpha\gamma}\rho^\nu > 2$ and take $\theta > (K/\rho\varepsilon)^{\alpha\gamma}$. Letting $n_\nu = (n'')^\nu$, we have for all large n

$$
\begin{aligned}
P[T_n^{(3)} \geqq \varepsilon] &\leqq \tbinom{n'}{n_\nu}\{P[\theta|X_1|^\alpha > (K\log n)^{\alpha\gamma}]\}^{n_\nu} \\
&\leqq \exp\{(\eta - \nu)(\rho\log n)^\nu(\log_2 n)(1 + o(1)) \\
&\qquad - (K\log n)^{\alpha\gamma}(\rho\log n)^\nu(1 + o(1))\} \\
&\leqq n^{-2}, \qquad \text{since} \quad \alpha\gamma + \nu = 1 \quad \text{and} \quad K^{\alpha\gamma}\rho^\nu > 2 .
\end{aligned}
$$

The rest of the proof is similar to that of Theorem 3.

We shall use the following notations below. Let $e_1(x) = e^x$, $e_2(x) = e_1(e^x)$, etc., and let $\log_2 x = \log\log x$, $\log_3 x = \log(\log_2 x)$, etc. We shall also write $\log_1 x = \log x$, $\log_0 x = e_0(x) = 1$ and $e_k = e_k(1)$. Suppose $Ee_k(t|X_1|^\alpha) < \infty$ for all $t > 0$ with $k \geqq 2$, $\alpha > 0$. In order that the limiting relation

$$\lim_{n\to\infty} (\log_k n)^{-1/\alpha} \textstyle\sum_{i=1}^n c_{n-i}X_i = 0 \quad \text{a.e.}$$

holds, we have to further narrow down the choice of the weighting sequence (c_n). In fact, in Section 3, it will be shown that if $X_1, X_2, \cdots$ and i.i.d. coin-tossing random variables, then for $k = 2, 3, \cdots$,

$$\limsup_{n\to\infty} (\log_k n)^{-1} \textstyle\sum_{j \geqq e_{k-2}}^n X_{n-j+1}/j(\log_0 j) \cdots (\log_{k-2} j) = 1 \quad \text{a.e.}$$

THEOREM 5. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For any* $\alpha > 0$ *and* $k = 1, 2, \cdots$, *the following statements are equivalent*:

(24) $$Ee_k(t|X_1|^\alpha) < \infty \quad \forall t > 0$$

(25) $$\lim_{n\to\infty} (\log_k n)^{-1/\alpha} X_n = 0 \quad \text{a.e.}$$

(26) $\lim_{n\to\infty} (\log_k n)^{-1/\alpha} \sum_{i=1}^n c_{n-i}X_i = 0$ a.e. *for some* (*or equivalently for every*) *nonvoid sequence of real numbers* $(c_n, n \geqq 0)$ *such that* $\sum_{n=0}^\infty |c_n| < \infty$.

PROOF. Suppose that (24) holds. To prove (26), given $\varepsilon > 0$, define

$$
\begin{aligned}
X_j' &= X_j I_{[|X_j| \leqq \varepsilon(\log_k j)^{1/\alpha}]} \\
X_j'' &= X_j I_{[|X_j| > \varepsilon(\log_k j)^{1/\alpha}]} .
\end{aligned}
$$

Since (25) implies (25), we have

$$\lim_{n\to\infty} (\log_k n)^{-1/\alpha} \sum_{i=1}^n c_{n-i} X_i'' = 0 \quad \text{a.e.}$$

The desired conclusion is now obvious since

$$|\sum_{i=1}^n c_{n-i} X_i'| \leqq \varepsilon(\log_k n)^{1/\alpha} \sum_{i=0}^\infty |c_i| \,.$$

3. Some analogs of the law of the iterated logarithm. In this section, we shall consider the fluctuation behavior of weighted sums of normal and coin-tossing random variables. We first consider the normal case. Suppose $X_1, X_2, \cdots$ are i.i.d. $N(0, 1)$ random variables. Then $E\exp(t|X|^2) < (=) \infty$ according as $t < (\geqq) \frac{1}{2}$. The following theorem shows that in this case, in place of the usual iterated logarithm $(2n \log_2 n)^{\frac{1}{2}}$ for sample sums, we have the single logarithm $(2 \log n)^{\frac{1}{2}}$ for the fluctuation behavior of weighted sums $\sum_{i=1}^n c_{n-i} X_i$.

THEOREM 6. *Let $X_1, X_2, \cdots$ be* i.i.d. $N(0, 1)$ *random variables, and let $(c_n, n \geqq 0)$ be a nonvoid sequence of real numbers such that $\sum_{n=0}^\infty c_n^2 = \sigma^2 < \infty$. Then*

$$\limsup_{n\to\infty} (2 \log n)^{-\frac{1}{2}} \sum_{i=1}^n c_{n-i} X_i = \sigma \quad \text{a.e.} \tag{27}$$

PROOF. Take any $\gamma > 2$ and let $b_n = \sigma(\gamma \log n)^{\frac{1}{2}}$. Then for $n \geqq 2$,

$$P[\sum_{i=1}^n c_{n-i} X_i \geqq b_n] \leqq \frac{K}{b_n} \exp\left(-\frac{b_n^2}{2\sigma^2}\right)$$

where K is a positive constant. Therefore $\sum_{n=1}^\infty P[\sum_{i=1}^n c_{n-i} X_i \geqq b_n] < \infty$ and so

$$\limsup_{n\to\infty} (2 \log n)^{-\frac{1}{2}} \sum_{i=1}^n c_{n-i} X_i \leqq \sigma \quad \text{a.e.}$$

To prove the reverse inequality, let $0 < \delta < 1$ and $1 < \zeta < (1 - \zeta)^{-1}$. For $k = 1, 2, \cdots$, define $n_k = [k^\zeta]$,

$$U_{k+1} = \sum_{i=n_k+1}^{n_{k+1}} c_{n_{k+1}-i} X_i$$
$$T_{k+1} = \sum_{i=1}^{n_k} c_{n_{k+1}-i} X_i \,.$$

Since $U_k \sim N(0, \sigma_k^2)$ where

$$\sigma_k^2 = \sum_{i=0}^{n_k - n_{k-1} - 1} c_i^2 \to \sigma^2 \qquad \text{as } k \to \infty \,,$$

it follows that $\sum_{k=2}^\infty P[U_k > \{2(1-\delta)\sigma^2 \log n_k\}^{\frac{1}{2}}] = \infty$. But the U_i's are independent, and so by the Borel–Cantelli lemma, $P[U_k > \{2(1-\delta)\sigma^2 \log n_k\}^{\frac{1}{2}} \text{ i.o.}] = 1$. It then suffices to show

$$\lim_{k\to\infty} (\log n_k)^{-\frac{1}{2}} T_k = 0 \quad \text{a.e.} \tag{28}$$

This follows by an easy application of the Borel–Cantelli lemma, noting that T_k is normal with $ET_k = 0$ and $\lim_{k\to\infty} ET_k^2 = 0$.

The fluctuation behavior (27) holds for all nonvoid sequences (c_n) such that $\sum c_n^2 < \infty$. In contrast to this, the corresponding behavior of weighted sums $\sum_{i=1}^n c_{n-i} X_i$ of coin-tossing random variables and more generally, of bounded random variables with 0 mean, depends on the particular weighting sequence (c_n).

THEOREM 7. *Let $X_1, X_2, \cdots$ be* i.i.d. *bounded random variables such that $EX_1 = 0$ and $E|X_1| \neq 0$.*

(i) *For any sequence* $(c_n, n \geqq 0)$ *of real numbers such that* $\sum c_n^2 < \infty$,

(29) $$\lim_{n\to\infty} (\log n)^{-\frac{1}{2}} \sum_{i=1}^n c_{n-i} X_i = 0 .$$

(ii) *For any* $\gamma \in (\frac{1}{2}, 1)$, *there exists* $K_\gamma \in (0, \infty)$ *such that*

(30) $$\limsup_{n\to\infty} (\log n)^{\gamma-1} \sum_{j=1}^n j^{-\gamma} X_{n-j+1} = K_\gamma \quad \text{a.e.}$$

LEMMA 5. *Suppose that* $X_1, X_2, \cdots$ *are* i.i.d. *bounded symmetric random variables. Let* (a_n) *be a sequence of real numbers such that* $\sum a_n^2 < \infty$ *and let* $Z_n = \sum_{i=1}^n a_i X_i$. *Then* $\sup_{n\geqq 1} E \exp(tZ_n^2) < \infty$ *for any* $t > 0$.

PROOF. When $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$ (i.e., X_1 is a coin-tossing random variable), the lemma has been proved by Zygmund ([17] page 214) using the Khintchine inequality. In general, when the X_i's are i.i.d. symmetric random variables with $|X_1| \leqq C$, X_i and $T_i X_i$ have the same distribution, where $T_1, T_2, \cdots$ are i.i.d. coin-tossing random variables and are independent of $(X_1, X_2, \cdots)$. Therefore by the Khintchine inequality, we have

$$E|Z_n|^{2k} = E|\textstyle\sum_1^n a_i X_i T_i|^{2k} \leqq Ek^k(\sum_1^n a_i^2 X_i^2)^k \leqq k^k(C^2 \sum_1^n a_i^2)^k .$$

The rest of the proof is similar to that of Zygmund.

LEMMA 6. *Let* $X_1, X_2, \cdots$ *be independent random variables. Suppose that there exist* $\zeta > 0$ *and* $T > 0$ *for which*

(31) $$Ee^{tX_j} \leqq \zeta \exp(\zeta^2 t^2) \qquad \forall t \in (-T, 0) \ \text{and} \ j = 1, 2, \cdots$$

and that there exist $a > 0$ *and* $p > 0$ *such that*

(32) $$P[X_j \geqq a] \geqq p \qquad \text{for all sufficiently large } j .$$

Then given any $\gamma \in (\frac{1}{2}, 1)$, *there exists* K_γ *such that* $0 < K_\gamma \leqq \infty$ *and* (30) *holds.*

PROOF. We can assume that $p < 1$. Given $\varepsilon > 0$, choose $\rho > |\log p|^{-1}$ such that

(33) $$\varepsilon\rho^\gamma T > 3 \qquad \text{and} \qquad (2\gamma - 1)\varepsilon^2\rho^{2\gamma-1} > 9\zeta^2$$

Now take any $\eta \in (0, |\log p|^{-1})$ and let $n' = [\rho \log n]$, $n'' = [\eta \log n]$. Define

$$A_n = [X_n \geqq a, X_{n-1} \geqq a, \cdots, X_{n-n''+1} \geqq a]$$
$$B_n = [\textstyle\sum_1^{n'} j^{-\gamma} X_{n-j+1} \geqq a \sum_{j=1}^{n''} j^{-\gamma} - \varepsilon] .$$

For $k > p^{-1}$, let $n_k = [2\rho k \log k]$. Then

$$PA_{n_k} \geqq p^{[\eta \log n_k]} \geqq (2\rho k \log k)^{-\eta|\log p|}$$

Since $\eta|\log p| < 1$, $\sum PA_{n_k} = \infty$. We note that

$$\begin{aligned} &P[\textstyle\sum_{n''+1}^{n'} j^{-\gamma} X_{n-j+1} \leqq -\varepsilon] \\ &\quad = P[\textstyle\sum_{n''+1}^{n'} (-tj^{-\gamma}) X_{n-j+1} \geqq t\varepsilon] , \qquad \text{where } t = (\log n)^{\gamma-\frac{1}{2}} \\ &\quad \leqq e^{-t\varepsilon} E \exp(-\textstyle\sum_{n''+1}^{n'} tj^{-\gamma} X_{n-j+1}) \\ &\quad \leqq \zeta \exp\{-t\varepsilon + \zeta^2 t^2 \textstyle\sum_{n''+1}^{n'} j^{-2\gamma}\} , \qquad \text{by (31), if } n \text{ is large} \\ &\quad \to 0 \qquad \text{as } n \to \infty . \end{aligned}$$

Therefore for all n sufficiently large,

(34) $$P[\sum_{n''+1}^{n'} j^{-\gamma} X_{n-j+1} > -\varepsilon] > \tfrac{1}{2}\,.$$

Obviously, on the event A_n,

(35) $$\sum_{j=1}^{n''} j^{-\gamma} X_{n-j+1} \geqq a \sum_{j=1}^{n''} j^{-\gamma}\,.$$

It then follows from (34) and (35) that $PB_n \geqq \frac{1}{2}PA_n$, and so $\sum PB_{n_k} = \infty$. We observe that $B_{n_k}, B_{n_{k+1}}, \cdots$ are independent for $k \geqq k_0$. Hence $P[B_n \text{ i.o.}] = 1$ and

(36) $$\limsup_{n\to\infty} (\log n)^{\gamma-1} \sum_{j=1}^{n'} j^{-\gamma} X_{n-j+1} \geqq a\eta^{1-\gamma}/(1-\gamma) \quad \text{a.e.}$$

Let $\tau = 3\varepsilon^{-1}(\log n)^\gamma$. Since $\varepsilon\rho^\gamma T > 3$, it follows that $\tau j^{-\gamma} < T$ if $j > n'$. Hence using (31), we have

$$\begin{aligned} P[\textstyle\sum_{n'+1}^{n} j^{-\gamma} X_{n-j+1} &\leqq -\varepsilon(\log n)^{1-\gamma}] \\ &= P[\textstyle\sum_{n'+1}^{n} (-\tau j^{-\gamma}) X_{n-j+1} \geqq \tau\varepsilon(\log n)^{1-\gamma}] \\ &\leqq \exp(-3\log n) E \exp(-\textstyle\sum_{n'+1}^{n} \tau j^{-\gamma} X_{n-j+1}) \\ &\leqq \zeta \exp\{-3\log n + \zeta^2\tau^2 \textstyle\sum_{n'+1}^{n} j^{-2\gamma}\} \\ &\leqq \zeta n^{-2}\,. \end{aligned}$$

Therefore by the Borel–Cantelli lemma

(37) $$\liminf_{n\to\infty} (\log n)^{\gamma-1} \sum_{n'+1}^{n} j^{-\gamma} X_{n-j+1} \geqq -\varepsilon \quad \text{a.e.}$$

From (36) and (37), we obtain that with probability 1,

(38) $$\limsup_{n\to\infty} (\log n)^{\gamma-1} \sum_{j=1}^{n} j^{-\gamma} X_{n-j+1} \geqq a(1-\gamma)^{-1}\eta^{1-\gamma} - \varepsilon\,.$$

By the zero-one law, the upper limit on the left-hand side of (38) must be equal a.e. to a constant, say K_γ, and (38) implies that $\infty \geqq K_\gamma \geqq a(1-\gamma)^{-1}\eta^{1-\gamma} - \varepsilon$ for any $\varepsilon > 0$ and $\eta \in (0, |\log p|^{-1})$. Therefore $K_\gamma \geqq a(1-\gamma)^{-1}|\log p|^{\gamma-1} > 0$.

PROOF OF THEOREM 7. To prove (i), we note that using Lemma 3 and symmetrization as in the proof of Theorem 3, we can without loss of generality assume that X_1 is symmetric and we shall make such an assumption. For any sequence $(c_n, n \geqq 0)$ such that $\sum c_n^2 < \infty$, let $Z_n = \sum_{i=1}^n c_{i-1}X_i$. Given $\varepsilon > 0$, choose t such that $t\varepsilon^2 = \alpha > 1$. Then using Lemma 5, we have

$$\begin{aligned} \textstyle\sum_{n=1}^\infty P[|\sum_{i=1}^n c_{n-i}X_i| > \varepsilon(\log n)^{\frac{1}{2}}] &= \textstyle\sum_{n=1}^\infty P[Z_n^2 > \varepsilon^2 \log n] \\ &\leqq \textstyle\sum_{n=1}^\infty n^{-\alpha} E \exp(tZ_n^2) \\ &< \infty\,. \end{aligned}$$

To prove (ii), let $|X_1| \leqq C$. Then (31) holds with $\zeta = C$ (which we can assume to be $\geqq 1$). Also since $E|X_1| \neq 0$, we can choose $a > 0$ such that $P[X_1 \geqq a] = p > 0$. Therefore Lemma 6 implies that $K_\gamma > 0$. It remains to show that $K_\gamma < \infty$. Choose $\delta > 0$ such that

(39) $$(2\gamma - 1)\delta^{2\gamma-1} > 9C^2\,.$$

Letting $n' = [\delta \log n]$, we have

$$\begin{aligned} P[\textstyle\sum_{n'+1}^{n} j^{-\gamma} X_{n-j+1} \geqq (\log n)^{1-\gamma}] &\leqq \exp\{-t(\log n)^{1-\gamma}\} E \exp(t \textstyle\sum_{n'+1}^{n} j^{-\gamma} X_j) \qquad \text{where} \quad t = 3(\log n)^{\gamma} \\ &\leqq \exp\{-3 \log n + C^2 t^2 \textstyle\sum_{n'+1}^{n} j^{-2\gamma}\} \\ &\leqq n^{-2} . \end{aligned}$$

The last inequality above follows from (39). Therefore using the Borel–Cantelli lemma, we obtain

$$\limsup_{n\to\infty} (\log n)^{\gamma-1} \textstyle\sum_{n'+1}^{n} j^{-\gamma} X_{n-j+1} \leqq 1 \quad \text{a.e.} \tag{40}$$

Furthermore, it is easy to see that

$$|\textstyle\sum_{j=1}^{n'} j^{-\gamma} X_{n-j+1}| \leqq C(1-\gamma)^{-1}(\delta \log n)^{1-\gamma}(1+o(1)) \tag{41}$$

Therefore it follow from (40) and (41) that $K_\gamma < \infty$.

Suppose $X_1, X_2, \cdots$ are i.i.d. coin-tossing random variables, i.e., $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$. Then it is well known ([4] page 210) that

$$\limsup_{n\to\infty} N_n \log 2/\log n = 1 \quad \text{a.e.}$$

where N_n denotes the length of the success run beginning at the nth trial. From this it follows that for $k = 2, 3, \cdots$,

$$\limsup_{n\to\infty} (\log_k n)^{-1} \textstyle\sum_{j \geqq e_{k-2}}^{[\log n]} X_{n-j+1}/j(\log_0 j) \cdots (\log_{k-2} j) = 1 \quad \text{a.e.}$$

Using Lemma 4, it can be proved that

$$\limsup_{n\to\infty} (\log_k n)^{-1} \textstyle\sum_{j=[\log n]+1}^{n} X_{n-j+1}/j(\log_0 j) \cdots (\log_{k-2} j) = 0 \quad \text{a.e.}$$

Therefore we obtain

$$\limsup_{n\to\infty} (\log_k n)^{-1} \textstyle\sum_{j \geqq e_{k-2}}^{n} X_{n-j+1}/j(\log_0 j) \cdots (\log_{k-2} j) = 1 \quad \text{a.e.} \tag{42}$$

If instead of the weighting sequence $j(\log_0 j) \cdots (\log_{k-2} j)$, we use the weighting sequence $j(\log_0 j) \cdots (\log_{k-1} j)$, then the corresponding weighted sum of i.i.d. random variables with $Ee_k(|X_1| \log |X_1|) < \infty$ and $EX_1 = 0$ will, with probability 1, be of smaller order than $\log_k n$, as we shall show in the theorem below,

THEOREM 8. *Suppose $k \in \{2, 3, \cdots\}$ and $\alpha \geqq 1$. Let $X_1, X_2, \cdots$ be* i.i.d. *random variables such that $EX_1 = 0$ and $Ee_k(t|X_1|^\alpha \log^+ |X_1|) < \infty$ for all $t > 0$. Let $(c_n, n \geqq 0)$ be a sequence of real numbers such that*

$$\limsup_{n\to\infty} |c_n| n(\log n) \cdots (\log_{k-1} n)(\log_k n)^{\beta(\alpha-1)} < \infty \qquad \textit{for some} \quad \beta > 1/\alpha .$$

Then

$$\lim_{n\to\infty} (\log_k n)^{-1/\alpha} \textstyle\sum_{i=1}^{n} c_{n-i} X_i = 0 \quad \text{a.e.}$$

PROOF. Choose $\lambda > 0$ such that for $n \geqq e_k$,

$$|c_n| n(\log n) \cdots (\log_{k-1} n)(\log_k n)^{\beta(\alpha-1)} \leqq \lambda .$$

Let $n' = [\log n]$, and let

$$\pi_\alpha = \left(\sum_{n \geqq e_k}^{\infty} \frac{1}{n(\log n) \cdots (\log_{k-1} n)(\log_k n)^{\beta/\alpha}}\right)^{\alpha-1}$$

Without loss of generality, we shall assume that X_1 is symmetric. Given $\varepsilon > 0$, define for $n \geqq e_k$,

$$X_n^{(1)} = X_n I_{[|X_n| \leqq \varepsilon(\log_k n/\log_{k+1} n)^{1/\alpha}]}$$
$$X_n^{(2)} = X_n I_{[|X_n| > \varepsilon(\log_k n/\log_{k+1} n)^{1/\alpha}]}$$
$$X_n' = X_n I_{[|X_n| \leqq \varepsilon(\log_k n)^{1/\alpha}]}$$
$$X_n'' = X_n I_{[|X_n| > \varepsilon(\log_k n)^{1/\alpha}]} .$$

It is easy to see that with probability 1,

$$\text{(43)} \qquad \limsup_{n\to\infty} (\log_k n)^{-1/\alpha} |\textstyle\sum_{j=0}^{n'} c_j X_{n-j}^{(1)}| \leqq \lambda\varepsilon , \quad \text{if} \quad \alpha = 1$$
$$= 0 , \quad \text{if} \quad \alpha > 1 .$$

Using Hölder inequality in the case $\alpha > 1$, we obtain that

$$|\textstyle\sum_{j\geqq e_k}^{n'} c_j X_{n-j}^{(2)}|^\alpha \leqq \lambda^\alpha \pi_\alpha \sum_{j\geqq e_k}^{n'} d_j |X_{n-j}^{(2)}|^\alpha$$

where $d_j^{-1} = j(\log j) \cdots (\log_{k-1} j)$. Letting $\omega_n = \sum_{j\geqq e_k}^{n'} d_j$, it then follows that

$$P[|\textstyle\sum_{j\geqq e_k}^{n'} c_j X_{n-j}^{(2)}| > \varepsilon^{1/\alpha}(\log_k n)^{1/\alpha}]$$
$$\text{(44)} \qquad \leqq P[\lambda^\alpha \pi_\alpha \textstyle\sum_{j\geqq e_k}^{n'} d_j |X_{n-j}^{(2)}|^\alpha > \varepsilon \log_k n]$$
$$= P\left[e_{k-1}\left(\frac{t}{\omega_n} \textstyle\sum_{j\geqq e_k}^{n'} d_j |X_{n-j}^{(2)}|^\alpha\right) > e_{k-1}\left(\frac{\varepsilon t \lambda^{-\alpha}}{\pi_\alpha \omega_n} \log_k n\right)\right]$$
$$\leqq P\left[\frac{1}{\omega_n} \textstyle\sum_{j\geqq e_k}^{n'} d_j e_{k-1}(t|X_{n-j}^{(2)}|^\alpha) > e_{k-1}\left(\frac{\varepsilon t \lambda^{-\alpha}}{\pi_\alpha \omega_n} \log_k n\right)\right].$$

The last inequality in (44) follows from Jensen's inequality, since e_{k-1} is a convex function. Set $\varepsilon t = 2\lambda^\alpha \pi_\alpha \log_{k+1} n$. Then $e_{k-1}((\varepsilon t \lambda^{-\alpha}/\pi_\alpha \omega_n) \log_k n) \geqq 2 \log n$ for $n \geqq n_0$. Furthermore, if $e_k \leqq j \leqq n'$, then

$$t|X_{n-j}^{(2)}|^\alpha = |X_{n-j}^{(2)}|^\alpha 2\lambda^\alpha \pi_\alpha \varepsilon^{-1} \log_{k+1} n \leqq \delta |X_{n-j}^{(2)}|^\alpha \log^+ |X_{n-j}^{(2)}|$$

for $n \geqq n_1 \geqq n_0$, where $\delta = 4\alpha\lambda^\alpha \pi_\alpha \varepsilon^{-1}$. Hence it follows from (44) that for $n \geqq n_1$,

$$P[|\textstyle\sum_{j\geqq e_k}^{n'} c_j X_{n-j}^{(2)}| > \varepsilon^{1/\alpha}(\log_k n)^{1/\alpha}]$$
$$\leqq P\left[\frac{1}{\omega_n} \textstyle\sum_{j\geqq e_k}^{n'} d_j e_{k-1}(\delta |X_{n-j}^{(2)}|^\alpha \log^+ |X_{n-j}^{(2)}|) > 2 \log n\right]$$
$$\leqq n^{-2} E \exp\{\textstyle\sum_{j\geqq e_k}^{n'} \omega_n^{-1} d_j e_{k-1}(\delta |X_{n-j}^{(2)}|^\alpha \log^+ |X_{n-j}^{(2)}|)\}$$
$$\leqq n^{-2} \textstyle\prod_{j\geqq e_k}^{n'} \{E e_k(\delta |X_1|^\alpha \log^+ |X_1|)\}^{\omega_n^{-1} d_j}$$
$$= n^{-2} E e_k(\delta |X_1|^\alpha \log^+ |X_1|) .$$

Hence $\sum_n P[|\sum_{j\geqq e_k}^{n'} c_j X_{n-j}^{(2)}| > \varepsilon^{1/\alpha}(\log_k n)^{1/\alpha}] < \infty$ and therefore

$$\text{(45)} \qquad \limsup_{n\to\infty} (\log_k n)^{-1/\alpha} |\textstyle\sum_{j\geqq e_k}^{n'} c_j X_{n-j}^{(2)}| \leqq \varepsilon^{1/\alpha} \quad \text{a.e.}$$

Since $\lim_{n\to\infty} (\log_k n)^{-1/\alpha} X_n = 0$ a.e., it follows that

$$\text{(46)} \qquad \lim_{n\to\infty} (\log_k n)^{-1/\alpha} \textstyle\sum_{j=n'+1}^{n-1} c_j X_{n-j}'' = 0 \quad \text{a.e.}$$

Using Lemma 4, it can be proved that

$$\text{(47)} \qquad \limsup_{n\to\infty} (\log_k n)^{-1/\alpha} |\textstyle\sum_{j=n'+1}^{n-1} c_j X_{n-j}'| \leqq 2\varepsilon \quad \text{a.e.}$$

The desired conclusion then follows from (43), (45), (46) and (47).

4. Extension to double arrays of weights. The following theorem is an extension of a theorem of [2] concerning double arrays of weights.

THEOREM 9. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For* $1 \leqq \alpha \leqq 2$, $E|X_1|^\alpha < \infty$ *if and only if for every array* a_{nk} *of real numbers such that* $\limsup_{n\to\infty} \sum_1^n a_{nk}^2 < \infty$, *we have*

$$\lim_{n\to\infty} n^{-1/\alpha} \sum_{k=1}^n a_{nk} X_k = 0 \quad \text{a.e.} \tag{48}$$

PROOF. Suppose $E|X_1|^\alpha < \infty$ and a_{nk} is a double array of real numbers such that $\limsup_{n\to\infty} A_n = A < \infty$, where $A_n = \sum_1^n a_{nk}^2$. For $\varepsilon > 0$, choose $M = M(\varepsilon) > 1$ such that $E|X_1|^\alpha I_{[|X_1|>M]} \leqq \varepsilon^\alpha$ and define

$$X_k' = X_k I_{[|X_k|\leqq M]} - EX_1 I_{[|X_1|\leqq M]}$$
$$X_k'' = X_k I_{[|X_k|>M]} - EX_1 I_{[|X_1|>M]}$$
$$T_n' = n^{-1/\alpha} \sum_{k=1}^n a_{nk} X_k'$$
$$T_n'' = n^{-1/\alpha} \sum_{k=1}^n a_{nk} X_k'' .$$

By the strong law of large numbers,

$$\begin{aligned} |T_n''|^\alpha &= n^{-1}\{|\textstyle\sum_1^n a_{nk} X_k''|^2\}^{\alpha/2} \leqq n^{-1}\{(\sum_1^n a_{nk}^2)(\sum_1^n X_k''^2)\}^{\alpha/2} \\ &\leqq n^{-1} A_n^{\alpha/2} \textstyle\sum_1^n |X_k''|^\alpha \\ &\leqq A^{\alpha/2} E|X_1''|^\alpha (1 + o(1)) \leqq 2^\alpha A^{\alpha/2} \varepsilon^\alpha (1 + o(1)) \quad \text{a.e.} \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} |T_n''| \leqq 2A^{\frac{1}{2}}\varepsilon \quad \text{a.e.} \tag{49}$$

From Lemma 4,

$$P[T_n' \geqq 4\varepsilon] \leqq \exp\left(-\frac{\varepsilon^2}{A_n M^2}\, n^{2/\alpha}\right).$$

Therefore $\sum P[T_n' \geqq 4\varepsilon] < \infty$ and so

$$\limsup_{n\to\infty} T_n' \leqq 4\varepsilon \quad \text{a.e.} \tag{50}$$

From (49) and (50), we obtain

$$\limsup_{n\to\infty} n^{-1/\alpha} \sum_{k=1}^n a_{nk} X_k \leqq (4 + 2A^{\frac{1}{2}})\varepsilon \quad \text{a.e.}$$

Replacing X_n by $-X_n$, we obtain (48) since ε is arbitrary. To prove the converse, let $a_{nk} = 0$ for $k \neq n$ and $a_{nn} = 1$. Then (48) implies that $\lim_{n\to\infty} n^{-1/\alpha} X_n = 0$ a.e. and so $E|X_1|^\alpha < \infty$.

Theorems 2 and 5 also can be easily extended to double arrays of weights. We state the results below.

THEOREM 10. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *Then* $Ee^{t|X_1|} < \infty$ *for all* $t > 0$ *if and only if for every array* a_{nk} *of real numbers such that* $\limsup_{n\to\infty} \sum_1^n a_{nk}^2 < \infty$, *we have*

$$\lim_{n\to\infty} \sum_{k=1}^n a_{nk} X_k / \log n = 0 \quad \text{a.e.} \tag{51}$$

THEOREM 11. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables such that* $EX_1 = 0$. *For*

$\alpha > 0$ and $k = 1, 2, \cdots$, $Ee_k(t|X_1|^\alpha) < \infty$ for all $t > 0$ if and only if for every array a_{nk} of real numbers such that $\limsup_{n\to\infty} \sum_{k=1}^n |a_{nk}| < \infty$, *we have*

$$\lim_{n\to\infty} (\log_k n)^{-1/\alpha} \sum_{k=1}^n a_{nk} X_k = 0 \quad \text{a.e.} \tag{52}$$

REFERENCES

[1] BLACKWELL, D. and DUBINS, L. (1962). Merging of opinions with increasing information. *Ann. Math. Statist.* **33** 882-886.

[2] CHOW, Y. S. (1966). Some convergence theorems for independent random variables. *Ann. Math. Statist.* **37** 1482-1493.

[3] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[4] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **1** Wiley, New York.

[5] GAL, I. S. (1951). Sur la majoration des suites des fonctions. *Proc. Kon. Ned. Akad. Wet Ser. A* **54** 243-351.

[6] GAPOSHKIN, V. F. (1965). The law of the iterated logarithm for Cesaro's and Abel's methods of summation. *Theor. Probability Appl.* **10** 411-420.

[7] HANSON, D. L. and KOOPMAN, L. H. (1965). On the convergence rate of the law of large numbers for linear combinations of independent random variables. *Ann Math. Statist.* **36** 559-564.

[8] HILL, J. D. (1951). The Borel property of summability methods. *Pacific J. Math.* **1** 399-409.

[9] LAI, T. L. (1972). Control charts based on weighted sums. *Ann. Statist.* **2**.

[10] LOÈVE, M. (1955). *Probability Theory.* Van Nostrand, Princeton.

[11] MARCINKIEWICZ, J. et ZYGMUND, A. (1937). Sur les fonctions independantes. *Fund. Math.* **29** 60-90.

[12] PRUITT, W. E. (1966). Summability of independent random variables. *J. Math. Mech.* **15** 769-776.

[13] STACKELBERG, O. (1964). On the law of the iterated logarithm. *Proc. Kon Ned. Akad. Wet. Ser. A* **67** 48-67.

[14] STOUT, W. F. (1968). Some results on the complete and almost sure convergence of linear combinations of independent random variables and martingale differences. *Ann. Math. Statist.* **39** 1549-1562.

[15] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **3** 211-226.

[16] TOMKINS, R. J. (1971). An iterated logarithm theorem for some weighted averages of independent random variables. *Ann. Math. Statist.* **42** 760-763.

[17] ZYGMUND, A. (1959). *Trigonometric Series* **1**. Cambridge Univ. Press.

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