The Annals of Probability
1973, Vol. 1, No. 6, 1071–1073

AN ESTIMATE FOR $E(|S_n|)$ FOR VARIABLES IN THE DOMAIN OF NORMAL ATTRACTION OF A STABLE LAW OF INDEX α, $1 < \alpha < 2$

BY WILLIS L. OWEN

University of Oklahoma Health Sciences Center

An estimate of the expected value of the sum of independent identically distributed random variables in the domain of normal attraction of a stable law of index α is obtained. This estimate is then used to obtain a generalization of the Helly-Bray lemma.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables defined on a probability space (Ω, F, P). Let $E(X_1) = 0$, and let $S_n = X_1 + \cdots + X_n$. Under these assumptions, Von Bahr and Esséen [4] have proved that for $1 \leqq r < s \leqq 2$

$$E(|S_n|^r) \leqq 2n^{r/s}(E|X_1|^s)^{r/s} .$$

In particular, if $X_1, X_2, \cdots$ are in the domain of normal attraction of a stable law of index α, $1 < \alpha < 2$, this result becomes

$$E(|S_n|) \leqq K(\beta)n^{1/\beta} \qquad \text{for} \quad 1 < \beta < \alpha ,$$

where $K(\beta)$ is a constant for fixed β. In the investigation of certain optimal stopping problems, one wants an estimate of this form for the case $\alpha = \beta$. This cannot be obtained from the Von Bahr–Esséen result. It is the purpose of this note to establish such a result and use it to obtain a generalization of the Helly–Bray lemma.

2. The estimate. Let Y be a stable random variable of index α, $1 < \alpha < 2$, and let V be the distribution of Y. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables which are in the domain of normal attraction of V. We also assume that $E(X_1) = 0$. Let F denote the common distribution of the X_i. As usual, we let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. Our assumption about the X_i implies that there is a constant $a > 0$, such that S_n/an^r converges in distribution to Y, where $r = 1/\alpha$. Since none of our arguments depends on the value of the constant a, we will take $a = 1$ for ease of notation.

Much is known about the distribution V and F (see, for example, [1]). In particular, we know the form of F, from which it is clear that there must be some constant $C > 0$ such that $F(x) \leqq C|x|^{-\alpha}$ if $x < 0$ and $1 - F(x) \leqq Cx^{-\alpha}$ if $x > 0$. Throughout this note, C will denote this constant.

We will use r to denote $1/\alpha$.

We may now prove our first lemma.

Received December 7, 1971; revised March 5, 1973.

AMS 1970 *subject classifications.* Primary 60G50; Secondary 60F99.

Key words and phrases. Stable laws, domains of attraction, convergence of moments.

LEMMA 1. *There exist positive constants K and s, both independent of n such that, for $t > s$,*

$$P(|S_n/n^r| > t) \leqq K/t^\alpha . \tag{1}$$

PROOF. The proof is by truncation. For fixed n and $t > 1$, define $Y_i = X_i I(|X_i| < tn^r)$, where I denotes the indicator function. Let $Z_i = X_i - Y_i$, $S_{1n} = Y_1 + \cdots + Y_n$, and $S_{2n} = Z_1 + \cdots + Z_n$. Note that

$$P(|S_n/n^r| > t) \leqq P(|S_{1n}/n^r| > t) + P(S_{2n} \neq 0) . \tag{2}$$

Since $P(S_{2n} \neq 0) \leqq nP(Z_1 \neq 0)$ and $P(Z_1 \neq 0) = P(|X_1| \geqq tn^r)$, we get

$$P(S_{2n} \neq 0) \leqq 2C/t^\alpha . \tag{3}$$

Note that since $E(X_1) = 0$,

$$E(Y_1) = \int_{|x| \geqq tn^r} x\, dF .$$

So, integrating by parts and using our remark about F, we get

$$E(Y_1) \leqq C/t^{\alpha-1}n^{1-r} + C/(\alpha - 1)t^{\alpha-1}n^{1-r} .$$

So, $E(S_{1n}/n^r) \leqq \alpha C t^{1-\alpha}/(\alpha - 1)$. Since $\alpha > 1$ and $t > 1$, we must have $|E(S_{1n} n^r)| \leqq \alpha C/(\alpha - 1)$. Let $B = \alpha C/(\alpha - 1)$ and let $s = 1 + B$. Henceforth, we take $t > s$. Then

$$P(|S_{1n}/n^r| > t) \leqq P(|S_{1n}/n^r - n^{1-r}E(Y_1)| > t - B) . \tag{4}$$

From Chebyshev's inequality, we get

$$P(|S_{1n}/n^r - n^{1-r}E(Y_1)| > t - B) \leqq (t - B)^{-2} \operatorname{Var}(S_{1n}/n^r) . \tag{5}$$

Now, $\operatorname{Var}(Y_1) \leqq E(Y_1^2)$ and $\operatorname{Var}(S_{1n}/n^r) = n^{1-2r} \operatorname{Var}(Y_1)$. Moreover, $\int_{-1}^{1} x^2\, dF \leqq 1 \leqq (tn^r)^{2-\alpha}$, and integrating by parts, we can get

$$\int_{-tn^r}^{-1} x^2\, dF \leqq [1 + 2C/(2 - \alpha)](tn^r)^{2-\alpha}$$

and

$$\int_{1}^{tn^r} x^2\, dF \leqq [1 + 2C/(2 - \alpha)](tn^r)^{2-\alpha} .$$

Hence, there is a constant K', independent of n, such that $\operatorname{Var}(Y_1) \leqq K'(tn^r)^{2-\alpha}$. So, from (4) and (5), we get

$$P(|S_{1n}/n^r| > t) \leqq K'(1 - B)^{-2}t^{-\alpha} . \tag{6}$$

If we let $K = K'(1 - B)^{-2} + 2C$, then (2), (3), and (6) give the result. □

The next theorem gives our estimate for $E(|S_n|)$.

THEOREM 1. *For each real number q with $0 \leqq q < \alpha$, there exists a finite positive real number Q, depending on q but independent of n, such that*

$$E(|S_n/n^r|^q) \leqq Q . \tag{7}$$

In particular, there exists a constant M, independent of n, such that

$$E(|S_n|) \leqq Mn^r . \tag{8}$$

PROOF. For the proof we integrate by parts and use the estimate of the previous lemma.

Since the result is clear for $q = 0$, choose q such that $0 < q < \alpha$. Let K and s be as in Lemma 1. Then

$$E(|S_n/n^r|^q) = \int_0^s x^q \, dP(|S_n/n^r| \leqq x) + \int_s^\infty x^q \, dP(|S_n/n^r| \leqq x) .$$

Now $\int_0^s x^q \, dP(|S_n/n^r| \leqq x) \leqq s^q$, and we can integrate by parts to get

$$\int_s^\infty x^q \, dP(|S_n/n^r| \leqq x) \leqq s^q + Ks^{q-\alpha}q/(\alpha - q) .$$

Letting $Q = 2s^q + Ks^{q-\alpha}q/(\alpha - q)$ gives (7). Equation (8) follows from (7) with $q = 1$. □

Although convergence in distribution does not imply convergence in p-mean, it is possible to make this conclusion in the particular case studied here.

Meyer ([3] Theorem 22, page 19) proves that if H is a set of integrable random variables and if there exists a function $G: [0, \infty) \to 0, \infty)$ which is increasing and satisfies $G(t)/t \to \infty$ as $t \to \infty$ and $\sup\{E[G(|f|)] : f \in H\} < \infty$, then H is uniformly integrable. We use this fact to prove a moment-convergence result for S_n/n^r.

THEOREM 2. *As* $n \to \infty$

$$E(S_n/n^r) \to E(Y) . \tag{9}$$

Moreover, for all q *with* $0 \leqq q < \alpha$, *we have*

$$E(|S_n/n^r|^q) \to E(|Y|^q) . \tag{10}$$

PROOF. It is known [see, e.g., Loève, page 183] that the result will follow if we can show $\{S_n/n^r\}$ is uniformly integrable. Since $\{S_n/n^r\}$ is uniformly integrable if $\{|S_n/n^r|\}$ is, it is only necessary to show $\{|S_n/n^r|^q\}$ is uniformly integrable.

The result is clear if $q = 0$; so let $q > 0$ and choose s, such that $0 < q < s < \alpha$. Let $u = s/q$ and let $G(t) = |t|^u$. Then $G(t)/t \to \infty$ as $t \to \infty$.

Moreover, $\sup E[G(|S_n/n^r|^q)] = \sup E[|S_n/n^r|^s]$ where the supremum is taken over all positive integers n. So, from (7), we get

$$\sup E[G(|S_n/n^r|^q)] < \infty .$$

Hence $\{|S_n/n^r|^q\}$ is uniformly integrable, and we are done. □

REFERENCES

[1] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading.

[2] LOÈVE, MICHEL (1955). *Probability Theory.* Van Nostrand, Princeton.

[3] MEYER, PAUL A. (1966). *Probability and Potentials.* Blaisdell, Waltham.

[4] VON BAHR, B. and ESSÉEN, C. G. (1965). Inequalities for the rth absolute moment of a sum of random variables, $1 \leqq r \leqq 2$. *Ann. Math. Statist.* **36** 299–303.

DEPARTMENT OF BIOSTATISTICS
UNIVERSITY OF OKLAHOMA
OKLAHOMA CITY, OKLAHOMA 73190