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LIMITS OF RATIOS OF TAILS OF MEASURES[1]

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Suppose μ is a positive measure on the half-line $[0, \infty)$, of total mass m, Φ is the sum of a power series with nonnegative coefficients which converges at the point m, and λ is the measure on $[0, \infty)$ whose Fourier transform $\hat{\lambda}$ is $\Phi(\hat{\mu})$. The lower limit of the ratios $\lambda([s, \infty))/\mu([s, \infty))$, as $s \to \infty$, is compared to the number $\Phi'(m)$, under a variety of conditions.

1. Introduction. Suppose μ is a finite positive Borel measure on the half-line $[0, \infty)$, of total variation $||\mu||$, and $\Phi(t) = \sum_0^\infty \Phi_n t^n$ converges absolutely when $t = ||\mu||$. Let μ^0 denote the unit mass at 0. Define $\mu^n = \mu^{n-1} * \mu$, for $n = 1, 2, 3, \cdots$ where $*$ denotes convolution. Then $\lambda = \sum \Phi_n \mu^n$ is a finite measure on $[0, \infty)$ which may be denoted by $\Phi(\mu)$. The behavior of the ratios

$$\frac{\lambda([s, \infty))}{\mu([s, \infty))} \tag{1.1}$$

as $s \to \infty$ is the topic to which the title of this paper refers.

If μ happens to be concentrated on the nonnegative integers, the problem becomes one about coefficients of power series and has therefore a particularly elementary character.

To begin with this special case, suppose $f(x) = \sum_0^\infty f_n x^n$, $f_n > 0$ for all n, $\Phi(t) = \sum_0^\infty \Phi_n t^n$, and

$$g(x) = \Phi(f(x)) = \sum_0^\infty g_n x^n . \tag{1.2}$$

What can one say about the ratios

$$\frac{g_n}{f_n} \quad \text{and} \quad \frac{g_n + g_{n+1} + g_{n+2} + \cdots}{f_n + f_{n+1} + f_{n+2} + \cdots} \tag{1.3}$$

as $n \to \infty$?

When $\{f_n\}$ is a probability measure, i.e., when $f(1) = 1$, then the coefficients g_n have a number of probabilistic interpretations. The following examples were kindly supplied by Peter Ney. If $0 < m < 1$ and $\Phi_n = m^n$ then g_n is the expected number of visits to n by a subcritical branching random walk on the integers. The analogous model with a probability measure μ on $[0, \infty)$ leads to the mean of an age-dependent branching process. (See Section 6 of [1]). If $\Phi_n = 1$ then $\{g_n\}$ is the classical renewal sequence. (This case, however, is not covered in the present paper since $\sum \Phi_n(f(1))^n = \infty$.) If $\{\Phi_n\}$ is itself a probability measure on the positive integers then $\{g_n\}$ is the probability measure of

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the sum of a random number (with distribution $\{\Phi_n\}$) of identically distributed independent random variables. In all of these situations it is often important to know the asymptotic behavior of $\{g_n\}$.

In [1] it is proved that the following four hypotheses, (a) to (d), imply that *the ratios* (1.3) *converge to* $\Phi'(f(1))$:

(a) $f(1) = \sum_0^\infty f_n < \infty$.
(b) $\lim_{n\to\infty} f_{n+1}/f_n = 1$.
(c) $\lim_{n\to\infty} g_n/f_n$ exists (finitely) for the special case $\Phi(t) = t^2$.
(d) Φ is holomorphic on $f(\bar{U})$, where $\bar{U}$ is the closed unit disc in the complex plane.

Among these hypotheses, (c) is rather strong (see Example 1) and is not at all easy to verify; see [1] for some conditions that imply (c). In the present paper I omit (c), I weaken (b) and (d), I add the assumption that $\Phi_n \geqq 0$ for all n, and I show that one can then establish very close relations between the number $\Phi'(f(1))$ and the *lower* limits of the ratios (1.3). These conclusions are weaker than those that are obtained in [1], but they are derived from different hypotheses and their proofs are quite elementary, whereas rather complicated Banach algebra techniques are used in [1]. However, [1] also deals with measures on $(-\infty, \infty)$, whereas the present paper is confined to the half-line $[0, \infty)$.

Section 4 contains some examples that complement the theorems which are stated in Section 2 and are proved in Section 3.

2. Statement of results. In Theorems 1 to 4, the following standing assumptions will be made.

(A) $f(x) = \sum_0^\infty f_n x^n$, $f_n > 0$ for all n, and the series converges when $x = 1$.
(B) $\Phi(t) = \sum_0^\infty \Phi_n t^n$, $\Phi_n \geqq 0$ for all n, and the series converges when $t = f(1)$.

The notation

$$T_n[f] = \sum_{i=n}^\infty f_i \tag{2.1}$$

will be used for the tails of the series $\sum f_i$.

Furthermore, $D^p f$ denotes the pth derivative of f, $\Phi \circ f$ is the composition defined by $(\Phi \circ f)(x) = \Phi(f(x))$, and

$$\Phi'(f(1)) = \sum_{n=0}^\infty n\Phi_n f(1)^{n-1} \tag{2.2}$$

whether this series converges or not.

THEOREM 1. *If* (A) *and* (B) *hold, then*

$$\Phi'(f(1)) \leqq \liminf_{n\to\infty} \frac{T_n(\Phi \circ f]}{T_n[f]}. \tag{2.3}$$

THEOREM 2. *In addition to* (A) *and* (B), *assume that there is a positive integer p such that*

(i) $D^p f$ *is unbounded on* $[0, 1)$, *but*
(ii) $D^p \Phi$ *is bounded on* $[0, f(1))$.

Then

$$\liminf_{n\to\infty} \frac{T_n[\Phi \circ f]}{T_n[f]} = \Phi'(f(1)) . \tag{2.4}$$

The next two theorems have analogous conclusions, but about ratios of individual coefficients, not of tails.

THEOREM 3. *In addition to* (A) *and* (B), *assume that*

$$\liminf_{n\to\infty} (f_n/f_{n+1}) \geqq 1 . \tag{2.5}$$

Then

$$\Phi'(f(1)) \leqq \liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} . \tag{2.6}$$

THEOREM 4. *In addition to* (A) *and* (B), *assume that*

(i) $\sum_0^\infty f_n x^n = \infty$ *for every* $x > 1$, *but*
(ii) $\sum_0^\infty \Phi_n t^n < \infty$ *for some* $t > f(1)$.

Then

$$\liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} \leqq \Phi'(f(1)) . \tag{2.7}$$

REMARKS. (a) If the hypotheses of Theorems 3 and 4 are both satisfied, then

$$\liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} = \Phi'(f(1)) . \tag{2.8}$$

(b) Assumption (2.5) holds whenever $\{f_n\}$ decreases monotonically.
(c) The trivial inequality

$$\liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} \leqq \liminf_{n\to\infty} \frac{T_n[\Phi \circ f]}{T_n[f]} \tag{2.9}$$

shows that (2.8) holds if the hypotheses of Theorems 2 and 3 are both satisfied.

(d) I have not been able to decide whether the conclusion of Theorem 2 holds under the hypotheses of Theorem 4.

Theorems 1 and 2 refer to tails, and it seems therefore natural to try to extend them so that they apply to measures on $[0, \infty)$. To do this, (A) is replaced by the analogous assumption (A*):

(A*) μ is a positive finite Borel measure on $[0, \infty)$, $\mu([s, \infty)) > 0$ for all $s \geqq 0$, and

$$f(x) = \int_0^\infty x^t \, d\mu(t) \qquad (0 \leqq x \leqq 1) . \tag{2.10}$$

Here and later the symbol $\int_0^\infty$ indicates integration over the *closed* half-line $[0, \infty)$.

The tails are now defined for all real $s \geqq 0$ by

$$T_s[f] = \mu([s, \infty)) \qquad (0 \leqq s < \infty) \tag{2.11}$$

if f and μ are related by (2.10).

Observe that $f(1) = \|\mu\|$. If (A*) and (B) hold, it follows that the series $\sum_0^\infty \Phi_n \mu^n$ converges, in the total variation norm, to a measure λ that satisfies (A*) and that is related to $\Phi \circ f$ by

$$(\Phi \circ f)(x) = \int_0^\infty x^t \, d\lambda(t) \qquad (0 \leqq x \leqq 1). \tag{2.12}$$

Thus $T_s[\Phi \circ f] = \lambda([s, \infty))$.

THEOREM 1*. *If* (A*) *and* (B) *hold, then*

$$\Phi'(f(1)) \leqq \liminf_{s \to \infty} \frac{T_s[\Phi \circ f]}{T_s[f]} . \tag{2.13}$$

THEOREM 2*. *In addition to* (A*) *and* (B) *assume that there is a positive integer* p *such that*

(i) $\int_0^\infty t^p \, d\mu(t) = \infty$, *but*
(ii) $\sum_0^\infty n^p \Phi_n f(1)^n < \infty$.

Then

$$\liminf_{s \to \infty} \frac{T_s[\Phi \circ f]}{T_s[f]} = \Phi'(f(1)) . \tag{2.14}$$

It is clear that Theorems 1 and 2 are special cases of Theorems 1* and 2*, obtained by requiring μ to be concentrated on the nonnegative integers.

3. Proofs.

PROOF OF THEOREM 1*. We are assuming (A*) and (B). Let us write the relation (2.10) in the form $f = \hat{\mu}$, and define

$$M_s[f] = M_s[\hat{\mu}] = \mu([0, s)) \qquad (0 \leqq s < \infty). \tag{3.1}$$

If $g = \hat{\lambda}$ for some positive finite Borel measure λ on $[0, \infty)$, then $fg = (\mu * \lambda)\hat{}$, and our first objective is the inequality

$$M_s[fg] \leqq M_s[f] M_s[g] \qquad (0 \leqq s < \infty). \tag{3.2}$$

Define

$$\Delta(s) = \{(x, y) : 0 \leqq x, 0 \leqq y, x + y < s\} ,$$
$$Q(s) = \{(x, y) : 0 \leqq x < s, 0 \leqq y < s\} .$$

If $\mu \times \lambda$ denotes the product measure, then the inclusion $\Delta(s) \subset Q(s)$ gives

$$(\mu * \lambda)([0, s)) = (\mu \times \lambda)(\Delta(s))$$
$$\leqq (\mu \times \lambda)(Q(s)) = \mu([0, s))\lambda([0, s)) ,$$

which is (3.2). It now follows by induction on n that

$$M_s[f^n] \leqq (M_s[f])^n \qquad (n = 0, 1, 2, \cdots) . \tag{3.3}$$

Multiply (3.3) by Φ_n and add. This yields

$$M_s[\Phi \circ f] \leqq \Phi(M_s[f]) \qquad (0 \leqq s < \infty), \tag{3.4}$$

first for all polynomials Φ with nonnegative coefficients, and then, by an obvious passage to the limit, for any Φ that satisfies (B).

Note that $M_s[f] + T_s[f] = f(1)$. The same is true with $\Phi \circ f$ in place of f. Hence (3.4) becomes

$$T_s[\Phi \circ f] \geqq \Phi(f(1)) - \Phi(f(1) - T_s[f]) . \tag{3.5}$$

Divide both sides of (3.5) by $T_s[f]$, which is positive, by (A*), and let $s \to \infty$. Then $T_s[f] \to 0$, and (2.13) is proved.

PROOF OF THEOREM 2*. Let p be the smallest positive integer for which the hypotheses of Theorem 2* hold. Put $g = \Phi \circ f$. We shall prove that

$$\int_0^\infty s^{p-1} x^s T_s[f]\, ds \to \infty \qquad \text{as } x \to 1 \tag{3.6}$$

and that

$$\lim_{x \to 1} \frac{\int_0^\infty s^p x^s T_s[g]\, ds}{\int_0^\infty s^p x^s T_s[f]\, ds} = \Phi'(f(1)) . \tag{3.7}$$

Let us see how these imply the theorem. Let c and s_0 be constants such that $T_s[g] \geqq c T_s[f]$ for all $s > s_0$. By (3.6), the limit in (3.7) is not changed if the integrals are taken over $[s_0, \infty)$ instead of $[0, \infty)$. This limit is therefore $\geqq c$. Thus $c \leqq \Phi'(f(1))$. Consequently,

$$\liminf_{s \to \infty} \frac{T_s[g]}{T_s[f]} \leqq \Phi'(f(1)) . \tag{3.8}$$

In view of Theorem 1*, (3.8) gives the desired conclusion (2.14).

It is therefore enough to prove (3.6) and (3.7).

For $k = 0, 1, 2, \cdots$, define

$$f^{[k]}(x) = \int_0^\infty t^k x^t\, d\mu(t) \qquad (0 \leqq x \leqq 1), \tag{3.9}$$

and

$$m_k = f^{[k]}(1) = \int_0^\infty t^k\, d\mu(t) . \tag{3.10}$$

Note that $f^{[0]} = f$, that $f^{[k]} \leqq m_k$, that $m_p = \infty$, and that $m_k < \infty$ if $0 \leqq k \leqq p - 1$.

A simple computation gives

$$\begin{aligned}\int_0^\infty s^{p-1} x^s T_s[f]\, ds &= \int_0^\infty s^{p-1} x^s\, ds \int_s^\infty d\mu(t) \\ &= \int_0^\infty d\mu(t) \int_0^t x^s s^{p-1}\, ds = \int_0^\infty t^p\, d\mu(t) \int_0^1 x^{tu} u^{p-1}\, du\end{aligned}$$

or

$$\int_0^\infty s^{p-1} x^s T_s[f]\, ds = \int_0^1 f^{[p]}(x^u) u^{p-1}\, du . \tag{3.11}$$

This computation depended only on the fact that $f = \hat{\mu}$. Since $g = \Phi(\hat{\mu})$, (3.11) holds also with g in place of f.

Since $f^{[p]}$ is an increasing function, and since $x \leqq x^u$, the right side of (3.11) is larger than $p^{-1}f^{[p]}(x)$, which tends to $p^{-1}m_p = \infty$ as $x \to 1$. This proves (3.6).

To prove (3.7) we will first show that

$$g^{[p]}(x) - \Phi'(f(x))f^{[p]}(x) \tag{3.12}$$

is a bounded function on $[0, 1)$, and we will begin with the special case $\Phi(t) = t^n$.

For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, in which each α_i is a nonnegative integer, such that $\alpha_1 + \cdots + \alpha_n = p$, positive constants $c(\alpha)$ are determined by

$$(t_1 + \cdots + t_n)^p = \sum_\alpha c(\alpha) t_1^{\alpha_1} \cdots t_n^{\alpha_n} . \tag{3.13}$$

Using this notation, we obtain

$$\begin{aligned}(f^n)^{[p]}(x) &= \int_0^\infty t^p x^t \, d\mu^n(t) \\ &= \int_0^\infty \cdots \int_0^\infty (t_1 + \cdots + t_n)^p x^{t_1 + \cdots + t_n} \, d\mu(t_1) \cdots d\mu(t_n) \\ &= \sum_\alpha c(\alpha) f^{[\alpha_1]}(x) \cdots f^{[\alpha_n]}(x) .\end{aligned}$$

There are n multi-indices α in which one component is p and all others are 0. For these α, $c(\alpha) = 1$. Using (3.10), it follows that

$$0 \leqq (f^n)^{[p]}(x) - nf^{n-1}(x) f^{[p]}(x) \leqq \sum{}' c(\alpha) m_{\alpha_1} \cdots m_{\alpha_n} \tag{3.14}$$

where $\sum'$ indicates that the summation extends only over those α in which $\alpha_i \leqq p - 1$ for all i.

If $0 \leqq k \leqq p - 1$, Hölder's inequality gives

$$m_k^{p-1} \leqq m_{p-1}^k m_0^{p-1-k}$$

so that $m_k \leqq \gamma^{k/p-1} m_0$, where $\gamma = m_{p-1}/m_0$. For any α that occurs in $\sum'$, it follows that

$$m_{\alpha_1} \cdots m_{\alpha_n} \leqq \gamma^{p/p-1} m_0^n = \gamma_1 m_0^n . \tag{3.15}$$

Since $\sum c(\alpha) = n^p$, (3.14) and (3.15) give

$$0 \leqq (f^n)^{[p]} - nf^{n-1}(x) f^{[p]}(x) \leqq \gamma_1 n^p f(1)^n , \tag{3.16}$$

for $n = 1, 2, 3, \cdots$. For $n = 0$, (3.16) holds trivially.

Now multiply (3.16) by Φ_n and add the resulting inequalities. Since $g = \Phi \circ f = \sum \Phi_n f^n$ and $\sum n^p \Phi_n f(1)^n < \infty$, it follows that

$$0 \leqq g^{[p]}(x) - \Phi'(f(x)) f^{[p]}(x) \leqq C \qquad (0 \leqq x < 1) \tag{3.17}$$

for some constant C.

If $0 \leqq x \leqq 1$ and $0 \leqq u \leqq 1$, then $x^u \leqq x \leqq 1$. Since f and Φ' are increasing functions, (3.17) gives

$$\Phi'(f(x)) f^{[p]}(x^u) \leqq g^{[p]}(x^u) \leqq \Phi'(f(1)) f^{[p]}(x^u) + C .$$

Multiply this by $u^{p-1}\,du$ and integrate, to obtain, for $0 < x < 1$,

$$\Phi'(f(x)) \leqq \frac{\int_0^1 g^{[p]}(x^u) u^{p-1}\,du}{\int_0^1 f^{[p]}(x^u) u^{p-1}\,du} \leqq \Phi'(f(1)) + \varepsilon(x) , \tag{3.18}$$

where $\varepsilon(x)$ is C/p divided by the denominator in (3.18); by (3.6) and (3.11), $\varepsilon(x) \to 0$ as $x \to 1$. We have thus proved that the quotient in (3.18) tends to $\Phi'(f(1))$ as $x \to 1$. By (3.11) this gives (3.7), and the proof is complete.

It has already been pointed out that Theorems 1 and 2 are special cases of 1* and 2*, respectively.

PROOF OF THEOREM 3. The hypotheses are now (A), (B), and

$$\liminf_{n\to\infty} (f_n/f_{n+1}) \geqq 1 \,. \tag{3.19}$$

The desired conclusion is

$$\liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} \geqq \Phi'(f(1)) \,. \tag{3.20}$$

We shall first prove this under the assumption that Φ is a polynomial P (with coefficients $\geqq 0$, of course). (3.20) is trivial if $\deg P \leqq 1$. Assume $\deg P > 1$, and assume that (3.20) holds for all polynomials Q with $\deg Q < \deg P$. We may also assume, without loss of generality, that the constant term of P is 0. Then $P(t) = tQ(t)$. Put $g = Q \circ f$. Then

$$(P \circ f)_n = (fg)_n = \sum_{i=0}^{n} f_i g_{n-i} \,. \tag{3.21}$$

Choose λ, $0 < \lambda < 1$. Then there exist integers M and N, with $N > 2M$, such that

$$\sum_{i=0}^{M} f_i \geqq \lambda f(1) \,, \qquad \sum_{i=0}^{M} g_i \geqq \lambda g(1) \,, \tag{3.22}$$

$$f_{n-1} \geqq \lambda f_n \qquad \text{if} \quad 0 \leqq i \leqq M \quad \text{and} \quad n > N \,, \tag{3.23}$$

and

$$g_k \geqq \lambda Q'(f(1)) f_k \qquad \text{if} \quad k \geqq N - M \,. \tag{3.24}$$

Of these, (3.22) is obvious, (3.23) follows from (3.19), and (3.24) uses our induction hypothesis.

Since $N > 2M$, (3.21) gives, for $n > N$,

$$(P \circ f)_n \geqq \sum_{i=0}^{M} f_{n-1} g_i + \sum_{i=0}^{M} f_i g_{n-i} = \mathrm{I} + \mathrm{II} \,,$$

where

$$\mathrm{I} \geqq \lambda f_n \sum_{i=0}^{M} g_i \geqq \lambda^2 f_n g(1)$$

and

$$\mathrm{II} \geqq \sum_{i=0}^{M} f_i \lambda Q'(f(1)) f_{n-i} \geqq \lambda^2 Q'(f(1)) f_n \sum_{i=0}^{M} f_i$$
$$\geqq \lambda^3 Q'(f(1)) f_n f(1) \,.$$

Since $g(1) = Q(f(1))$, it follows that

$$\frac{(P \circ f)_n}{f_n} \geqq \lambda^3 [Q(f(1)) + f(1) Q'(f(1))] = \lambda^3 P'(f(1)) \,.$$

Letting $\lambda \to 1$, we now see that (3.20) holds with P in place of Φ.

To do the general case, fix $\alpha < \Phi'(f(1))$ and let P be a partial sum of the series defining Φ, such that $P'(f(1)) > \alpha$. Then $(\Phi \circ f)_n \geqq (P \circ f)_n$, so that

$$\liminf_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} \geqq \liminf_{n\to\infty} \frac{(P \circ f)_n}{f_n} \geqq P'(f(1)) > \alpha .$$

This is true for every $\alpha < \Phi'(f(1))$. Hence (3.20) is proved.

PROOF OF THEOREM 4. This will be deduced from the case $p = 1$ of Theorem 2, by means of the following lemma:

LEMMA. *If f satisfies* (A), *if $\delta > 0$, and if*

$$\sum_{n=0}^{\infty} f_n x^n = \infty \qquad \text{for every} \quad x > 1 , \tag{3.25}$$

then there exists a sequence $\{\gamma_n\}$, $\gamma_n \geqq 1$, such that

$$\gamma_i \gamma_{n-i} \geqq \gamma_n \qquad \text{if} \quad 0 \leqq i \leqq n , \tag{3.26}$$

$$\sum_{n=0}^{\infty} f_n \gamma_n < f(1) + \delta , \tag{3.27}$$

and

$$\sum_{n=0}^{\infty} n f_n \gamma_n = \infty . \tag{3.28}$$

We shall first prove the lemma.

Choose $k_1 \geqq 2$ so that $T_{k_1}[f] < \delta/2$; see (2.1) for notation. Then choose $\varepsilon_1 > 0$ so that

$$\sum_{n=0}^{k_1} f_n \exp(n\varepsilon_1) < f(1) . \tag{3.29}$$

Assume $p \geqq 1$, and make the following induction hypotheses (which hold when $p = 1$): k_p and ε_p are chosen, $k_p \geqq 2^p$, and

$$T_{k_p}[f] < \delta \cdot 2^{-p} . \tag{3.30}$$

Now choose $k_{p+1} \geqq 2^{p+1}$, and so large that (3.30) holds with $p + 1$ in place of p, and that

$$\sum_{n=1+k_p}^{k_{p+1}} f_n \exp(n\varepsilon_p) > \delta . \tag{3.31}$$

Note that (3.31) can be achieved, because $\varepsilon_p > 0$ and (3.25) holds.

By (3 30) and (3.31) there exists ε_{p+1}, $0 < \varepsilon_{p+1} < \varepsilon_p$, such that

$$\sum_{n=1+k_p}^{k_{p+1}} f_n \exp(n\varepsilon_{p+1}) = \delta \cdot 2^{-p} . \tag{3.32}$$

Our induction hypotheses hold now with $p + 1$ in place of p, and the construction of $\{k_p\}$ and $\{\varepsilon_p\}$ can proceed. We define

$$\begin{aligned} \gamma_n &= \exp(n\varepsilon_1) && \text{if} \quad 0 \leqq n \leqq k_1 \\ &= \exp(n\varepsilon_{p+1}) && \text{if} \quad k_p < n \leqq k_{p+1} , \qquad p = 1, 2, 3, \cdots \end{aligned} \tag{3.33}$$

Then (3.26) holds because $(\log \gamma_n)/n$ decreases as n increases, so that

$$\log \gamma_i + \log \gamma_{n-i} \geqq \frac{i}{n} \log \gamma_n + \frac{n-i}{n} \log \gamma_n = \log \gamma_n .$$

Next, (3.27) holds because (3.29) and (3.32) give

$$\sum_{n=0}^{\infty} f_n \gamma_n = \sum_0^{k_1} f_n \exp(n\varepsilon_1) + \sum_{p=1}^{\infty} \sum_{1+k_p}^{k_{p+1}} f_n \exp(n\varepsilon_{p+1})$$
$$< f(1) + \delta \sum_{p=1}^{\infty} 2^{-p} = f(1) + \delta .$$

Finally, (3.28) holds because

$$\sum_{1+k_p}^{k_{p+1}} n f_n \gamma_n > k_p \sum_{1+k_p}^{k_{p+1}} f_n \exp(n\varepsilon_{p+1}) = k_p \delta 2^{-p} \geqq \delta$$

for $p = 1, 2, 3, \cdots$.

The lemma is thus proved, and we turn to the proof of Theorem 4.

Choose $\delta > 0$ so that $\sum_0^\infty \Phi_n t^n < \infty$ for some $t > f(1) + \delta$, choose $\{\gamma_n\}$ as in the lemma, and define

$$g(x) = \sum_{n=0}^{\infty} f_n \gamma_n x^n \qquad (0 \leqq x \leqq 1) . \tag{3.34}$$

By (3.28), $g'(x) \to \infty$ as $x \to 1$. Our choice of δ ensures that Φ' is bounded on $[0, g(1)]$ since $g(1) < f(1) + \delta$. Theorem 2 implies therefore (via (2.9)) that

$$\liminf_{n\to\infty} \frac{(\Phi \circ g)_n}{g_n} \leqq \Phi'(g(1)) . \tag{3.35}$$

For $k = 1, 2, 3, \cdots$, (3.26) implies that

$$(g^k)_n = \sum g_{i_1} \cdots g_{i_k} \quad (i_1 + \cdots + i_k = n)$$
$$= \sum f_{i_1}\gamma_{i_1} \cdots f_{i_k}\gamma_{i_k}$$
$$\geqq \gamma_n \sum f_{i_1} \cdots f_{i_k} = \gamma_n (f^k)_n .$$

If we multiply this by Φ_k and add, we obtain

$$(\Phi \circ g)_n \geqq \gamma_n (\Phi \circ f)_n \qquad (n = 0, 1, 2, \cdots) . \tag{3.36}$$

Hence

$$\frac{(\Phi \circ f)_n}{f_n} \leqq \frac{(\Phi \circ g)_n}{f_n \gamma_n} = \frac{(\Phi \circ g)_n}{g_n} . \tag{3.37}$$

By (3.35) and (3.37)

$$\liminf \frac{(\Phi \circ f)_n}{f_n} \leqq \Phi'(g(1)) \leqq \Phi'(f(1) + \delta) .$$

If we now let $\delta \to 0$, we obtain the desired inequality (2.7).

4. Examples.

EXAMPLE 1. This is an example of an f that satisfies the standing assumption (A), and also

(a) $f_n^2 \leqq f_{n-1} f_{n+1}$ for $n = 1, 2, 3, \cdots$,
(b) $\lim_{n\to\infty} (f_{n+1}/f_n) = 1$,
(c) $\sum_0^\infty n f_n = \infty$,

although

(d) $\limsup_{n\to\infty} \frac{(f^2)_n}{f_n} = \infty$.

In the conclusion of Theorem 4, lim inf can therefore not be replaced by lim, even when $\Phi(t) = t^2$.

By Theorems 2, 3, and 4, this f does satisfy

$$\liminf_{n\to\infty} \frac{(f^2)_n}{f_n} = \liminf_{n\to\infty} \frac{T_n[f^2]}{T_n[f]} = 2f(1) < \infty .$$

The example shows also that the convexity of $\{\log f_n\}$ (which is another way of stating (a)) does not guarantee the existence of $\lim [(f^2)_n/f_n]$. Conceivably, the convexity of $\{\log f_n\}$ might imply the existence of $\lim (T_n[f^2]/T_n[f])$, I have no counter-example.

The inequality

$$(4.1) \qquad 1 < \varepsilon \sum_{n=0}^{\infty} \exp(-n\varepsilon) < 2 \qquad (0 < \varepsilon \leqq 1)$$

will be used below.

We now begin the construction of f.

Put $n_0 = 0$, $\varepsilon_1 = 1$, $\alpha_1 = 1$, $n_1 = 4$. For $p \geqq 1$, make the induction hypothesis (satisfied when $p = 1$) that ε_p, α_p, n_p are chosen and that

$$(4.2) \qquad \alpha_p n_p \exp(-n_p \varepsilon_p) < \tfrac{1}{2}\varepsilon_p .$$

Define ε_{p+1} to be the left side of (4.2), put

$$(4.3) \qquad \alpha_{p+1} = \alpha_p \exp\{-n_p(\varepsilon_p - \varepsilon_{p+1})\}$$

and let n_{p+1} be an even integer, so large that (4.2) holds with $p + 1$ in place of p, that $n_{p+1} > 2n_p$, that $\alpha_{p+1}n_{p+1} > p + 1$, and that

$$(4.4) \qquad \sum_{n=n_p}^{n_{p+1}} \exp(-n\varepsilon_{p+1}) > \frac{\exp(-n_p \varepsilon_{p+1})}{\varepsilon_{p+1}} .$$

Our induction hypothesis holds now with $p + 1$ in place of p, and the construction of the sequences $\{\varepsilon_p\}$, $\{\alpha_p\}$, and $\{n_p\}$ can proceed.

Define

$$(4.5) \qquad f_n = \alpha_p \exp(-n\varepsilon_p) \qquad (n_{p-1} \leqq n \leqq n_p;\ p = 1, 2, 3, \cdots) .$$

Note that f_{n_p} is determined twice by (4.5); however, (4.3) shows that these two determinations agree.

The choice of ε_{p+1} and α_{p+1} shows that

$$\begin{aligned}\sum_{n_p}^{n_{p+1}} f_n &< \alpha_{p+1} \sum_{n_p}^{\infty} \exp(-n\varepsilon_{p+1}) \\ &< \frac{2\alpha_{p+1}\exp(-n_p\varepsilon_{p+1})}{\varepsilon_{p+1}} \\ &= \frac{2\alpha_p \exp(-n_p\varepsilon_p)}{\varepsilon_{p+1}} = \frac{2}{n_p} .\end{aligned}$$

Thus $\sum_0^\infty f_n < \infty$, and (A) holds. Similarly,

$$\begin{aligned}\sum_{n_p}^{n_{p+1}} nf_n &> n_p\alpha_{p+1} \sum_{n_p}^{n_{p+1}} \exp(-n\varepsilon_{p+1}) \\ &> \frac{n_p\alpha_{p+1}\exp(-n_p\varepsilon_{p+1})}{\varepsilon_{p+1}} = 1 .\end{aligned}$$

Thus f satisfies (c). Properties (a) and (b) hold because ε_p decreases to 0 as $p \to \infty$.

Finally, the convexity of $\{\log f_n\}$ implies that $(f_n)^2 \leqq f_i f_j$ if $i + j = 2n$. Thus

$$(f^2)_{2n} = \sum_{i+j=2n} f_i f_j \geqq 2n(f_n)^2 . \tag{4.6}$$

If n is taken so that $2n = n_p$, then $n_{p-1} \leqq n \leqq n_p$, and (4.5) and (4.6) imply

$$\begin{aligned}(f^2)_{n_p} = (f^2)_{2n} &\geqq 2n\alpha_p^{\,2}\,[\exp(-n\varepsilon_p)]^2 \\ &= n_p \alpha_p^{\,2} \exp(-n_p \varepsilon_p) \\ &= n_p \alpha_p f_{n_p} > p f_{n_p} .\end{aligned}$$

This shows that f has property (d).

EXAMPLE 2. In this example, (A), (B), and hypotheses (i) of Theorems 2 and 4 hold (with $p = 2$ in Theorem 2), but the hypotheses (ii) fail, as do the conclusions. This will be done by taking an f with $f(1) = 1$, $f'(1) = \lambda < \infty$, but f'' unbounded on $[0, 1)$, and by taking $\Phi = f$. If $g = \Phi \circ f$, then

$$g''(x) = f''(x)f'(f(x)) + f'(x)^2 f''(f(x)) . \tag{4.7}$$

If $f''(f(x))/f''(x) \to 1$ as $x \to 1$, it follows that

$$\lim_{x\to 1} \frac{g''(x)}{f''(x)} = \lambda + \lambda^2 . \tag{4.8}$$

Since $g_n/f_n = (g'')_{n-2}/(f'')_{n-2}$ and since $f''(x) \to \infty$ as $x \to 1$, (4.8) suggests (by an argument similar to one used in the proof of Theorem 2*) that

$$\lim_{n\to\infty} \frac{(\Phi \circ f)_n}{f_n} = \lambda + \lambda^2 > \lambda = \Phi'(f(1)) , \tag{4.9}$$

if the above limit exists.

To see that this heuristic argument is correct for suitably chosen f, define

$$f(x) = x + \tfrac{1}{2}(1 - x)^{\frac{3}{2}} . \tag{4.10}$$

The binomial theorem shows that $f_n > 0$ for all n. This f thus has the above mentioned properties, with $\lambda = 1$.

A rather long and tedious computation, based on (4.7), shows that the derivative of

$$g''(x) - 2f''(x) + \tfrac{15}{64} f'(x) \tag{4.11}$$

is bounded in the unit disc. [Here is one way of doing this: Put $u = (1 - x)^{\frac{1}{2}}$; then $(1 - f(x))^{\frac{1}{2}} = u(1 - u/2)^{\frac{1}{2}}$; differentiate (4.11) with respect to x, using (4.7), then express everything in terms of u; in the resulting Laurent expansion, all negative powers of u cancel.] Since $(f'')_n \sim c \cdot n^{-\frac{1}{2}}$, we have $(f')_n \sim cn^{-\frac{3}{2}}$. Hence the boundedness of the derivative of (4.11) implies that

$$(g'')_n = 2(f'')_n + O(1/n) . \tag{4.12}$$

Thus $g_n/f_n \to 2$ as $n \to \infty$, as predicted by (4.9).

This example shows that (ii) cannot be omitted from the hypotheses of Theorems 2 and 4, even if $\lim (\Phi \circ f)_n/f_n$ is assumed to exist as a finite number.

EXAMPLE 3. Let f again satisfy (A), with $f(1) = 1$, for simplicity, and assume that every derivative of f is bounded on $[0, 1)$. (Equivalently, assume that f is infinitely differentiable on the closed unit disc $\bar{U}$.) Note that this can happen if f has radius of convergence 1, i.e., if hypothesis (i) of Theorem 4 holds. Hypothesis (ii) of Theorem 4 says that Φ is holomorphic on an open set containing $\bar{U}$. Let us see whether this can be weakened to the assumption that Φ is infinitely differentiable on $\bar{U}$; of course, (B) is still assumed to hold.

Under these circumstances, I claim that

$$D^p(\Phi \circ f)(1) \geqq D^p\Phi(1) \cdot f'(1)^p \qquad (p = 1, 2, 3, \cdots)\,. \tag{4.13}$$

Since

$$(\Phi \circ f)' = (\Phi' \circ f) \cdot f' \tag{4.14}$$

(4.13) holds when $p = 1$. Assume (4.13) is proved for same p and all such Φ. By (4.14),

$$\begin{aligned} D^{p+1}(\Phi \circ f)(1) &= D^p((\Phi' \circ f) \cdot f')(1) \\ &= \textstyle\sum_{i=0}^{p} \binom{p}{i} D^{p-i}(\Phi' \circ f) \cdot D^i(f') \\ &\geqq D^p(\Phi' \circ f)(1) \cdot f'(1) \\ &\geqq (D^p\Phi')(1) \cdot f'(1)^p \cdot f'(1) \\ &= (D^{p+1}\Phi)(1) \cdot f'(1)^{p+1}\,. \end{aligned}$$

The first of these inequalities is obtained by discarding all terms of the sum with $i > 0$; the second inequality uses the induction hypothesis. Thus (4.13) is proved for all p.

In particular, if $f'(1) > 1$, we apply (4.13) with $\Phi = f$, and obtain (with $g = f \circ f$)

$$\frac{(D^p g)(1)}{(D^p f)(1)} \geqq f'(1)^p \to \infty \qquad \text{as } p \to \infty\,. \tag{4.15}$$

If there were a constant $C < \infty$ such that $g_n \leqq Cf_n$ for all n, then obviously $D^p g \leqq CD^p f$. Hence (4.15) shows that g_n/f_n cannot be bounded.

The following conclusion has thus been reached:

If $f_n > 0$ for all n, if $f(1) = 1, f'(1) > 1$, and f is infinitely differentiable on the closed unit disc, then

$$\limsup_{n\to\infty} \frac{(f \circ f)_n}{f_n} = \infty\,. \tag{4.16}$$

In particular, it is not true that these ratios tend to $f'(f(1)) = f'(1)$.

To see an example of this, in which the radius of convergence is 1, choose $\alpha > 0$ so that $\exp(4^\alpha) < 3$, and define

$$f_n = c \cdot \exp(-n^\alpha) \qquad (n = 0, 1, 2, \cdots)\,, \tag{4.17}$$

where c is picked so that $\sum_0^\infty f_n = 1$; the choice of α is made so that $f'(1) > 1$.

For this example (4.17), it seems very plausible that lim sup can actually be replaced by lim inf in (4.16).

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