

## ON THE CONVERGENCE OF POISSON BINOMIAL TO POISSON DISTRIBUTIONS

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It is well known that under certain conditions, binomial distributions converge to Poisson distributions. Simons and Johnson (1971) showed that the convergence is actually much stronger than in the usual sense. In this note the author shows that the result of Simons and Johnson is also true for Poisson binomial distributions which include binomial distributions as special cases.

**1. Introduction and assertion.** Let  $X_{1n}, X_{2n}, \dots, X_{nn}$ ,  $n = 1, 2, 3, \dots$ , be a triangular array of independent Bernoulli random variables with  $P(X_{in} = 1) = 1 - P(X_{in} = 0) = p_{in}$  and let  $\bar{p}_n = \max_{1 \leq i \leq n} p_{in} \rightarrow 0$  and  $\sum_{i=1}^n p_{in} = \lambda$  (fixed) as  $n \rightarrow \infty$ . It is well known that for  $r = 0, 1, 2, \dots$ ,

$$(1) \quad \lim_{n \rightarrow \infty} P(W_n = r) = p(r; \lambda),$$

where  $W_n = \sum_{i=1}^n X_{in}$  and  $p(r; \lambda) = e^{-\lambda} \lambda^r / r!$ . Stronger versions of (1) which are also known are

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{r=0}^{\infty} |P(W_n = r) - p(r; \lambda)| = 0$$

and

$$(3) \quad \sum_{r=0}^{\infty} |P(W_n = r) - p(r; \lambda)| \leq (16/\lambda) \sum_{i=1}^n p_{in}^2 \quad (\bar{p}_n \leq \frac{1}{4})$$

(Le Cam (1960)).

Simons and Johnson (1971) showed that when the  $p_{in}$ 's are equal, (2) can be strengthened to

$$(4) \quad \lim_{n \rightarrow \infty} \sum_{r=0}^{\infty} h(r) |P(W_n = r) - p(r; \lambda)| = 0$$

where  $h(r) \geq 0$  for  $r = 0, 1, 2, \dots$ , and

$$(5) \quad \sum_{r=0}^{\infty} h(r) p(r; \lambda) < \infty.$$

The aim of this note is to show that (4) is indeed true even without assuming the equality of the  $p_{in}$ 's. The main idea of the proof here is similar to that in [3]. However, because the distribution of  $W_n$  (which is sometimes referred to as the Poisson binomial distribution) does not in general assume a form which is easy to work with, in order to show that it exhibits a certain monotone property in relation to the Poisson distribution, the author uses an identity. This identity was first used by Charles Stein in lectures at Stanford University. Generalizations of it are contained in [1].

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**2. Proof.** Let  $W_n^{(i)} = \sum_{j=1, j \neq i}^n X_{jn}$ . By using the fact that each  $X_{in}$  takes on values 0 and 1 and the independence of the  $X_{in}$ 's, we have, for every bounded function  $f$ ,

$$(6) \quad EW_n f(W_n) = \sum_{i=1}^n p_{in} Ef(W_n^{(i)} + 1).$$

By choosing  $f(w)$  to be 1 if  $w = r + 1$  and 0 if  $w \neq r + 1$ , where  $0 \leq r \leq n$ , (6) yields

$$(7) \quad (r + 1)P(W_n = r + 1) = \sum_{i=1}^n p_{in} P(W_n^{(i)} = r).$$

Now  $1 \geq P(W_n^{(i)} = r | W_n = r) = (1 - p_{in})P(W_n^{(i)} = r)/P(W_n = r)$ . This together with (7) implies

$$(8) \quad P(W_n = r + 1)/P(W_n = r) \leq \lambda/(r + 1)(1 - \bar{p}_n).$$

Using again the fact that each  $X_{in}$  takes on values 0 and 1 and the independence of the  $X_{in}$ 's, we obtain from (6),

$$(9) \quad \begin{aligned} EW_n f(W_n) &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in} E[f(W_n^{(i)} + 1) - f(W_n + 1)] \\ &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in} E\{X_{in}[f(W_n^{(i)} + 1) - f(W_n^{(i)} + 2)]\} \\ &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in}^2 E[f(W_n^{(i)} + 1) - f(W_n^{(i)} + 2)]. \end{aligned}$$

This yields

$$(10) \quad \begin{aligned} &\frac{P(W_n = r + 1)/p(r + 1; \lambda)}{P(W_n = r)/p(r; \lambda)} \\ &= \frac{(r + 1)P(W_n = r + 1)}{\lambda P(W_n = r)} \\ &= 1 + \frac{1}{\lambda P(W_n = r)} \sum_{i=1}^n p_{in}^2 \\ &\quad \times [P(W_n^{(i)} = r) - P(W_n^{(i)} = r - 1)]. \end{aligned}$$

By (8),  $P(W_n^{(i)} = r)/P(W_n^{(i)} = r - 1) \leq (\lambda - p_{in})/r(1 - \max_{j \neq i} p_{jn}) \leq \lambda/r(1 - \bar{p}_n)$  which is  $\leq 1$  for  $n \geq r \geq [\lambda] + 1$  and  $n \geq n_0$  where  $[\lambda]$  is the greatest integer  $\leq \lambda$  and  $n_0$  is independent of  $r$ . Thus for  $n \geq r \geq [\lambda] + 1$  and  $n \geq n_0$ ,  $P(W_n^{(i)} = r) - P(W_n^{(i)} = r - 1) \leq 0$  for  $i = 1, 2, \dots, n$ , which together with (10) in turn implies

$$\frac{P(W_n = r + 1)/p(r + 1; \lambda)}{P(W_n = r)/p(r; \lambda)} \leq 1.$$

Hence noting also that  $P(W_n = r) = 0$  if  $r \geq n + 1$ , we have for  $r \geq [\lambda] + 1$  and  $n \geq n_0$ ,

$$(11) \quad P(W_n = r)/p(r; \lambda) \leq p(W_n = [\lambda] + 1)/p([\lambda] + 1; \lambda) \leq S < \infty$$

where  $S$  is some constant by virtue of (1). Finally, (4) follows, from (1), (5), (11) and the dominated convergence theorem, as in Simons and Johnson (1971). The assertion is proved.

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