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ON THE MOMENTS AND LIMIT DISTRIBUTIONS OF SOME FIRST PASSAGE TIMES

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Let S_n, $n = 1, 2, 3, \cdots$, denote the partial sums of i.i.d. random variables with positive, finite mean. The first passage times $\min\{n; S_n > c\}$ and $\min\{n; S_n > c \cdot a(n)\}$, where $c \geqq 0$ and $a(y)$ is a positive, continuous function on $[0, \infty)$, such that $a(y) = o(y)$ as $y \uparrow \infty$, are investigated. Necessary and sufficient conditions for finiteness of their moments and moment generating functions are given. Under some further assumptions on $a(y)$, asymptotic expressions for the moments and the excess over the boundary are obtained when $c \to \infty$. Convergence to the normal and stable distributions is established when $c \to \infty$. Finally, some of the results are generalized to a class of random processes.

1. Introduction. Let $(\Omega, \mathscr{A}, P)$ be a probability space, let $X_1, X_2, \cdots$ be a sequence of random variables on Ω, let $\mathscr{F}_n = \sigma\{X_1, X_2, \cdots, X_n\}$, $n = 1, 2, \cdots$, and set $\mathscr{F}_0 = \{\phi, \Omega\}$. A random variable N, defined on Ω, with positive integer values, is called a stopping time (a stopping variable) of the sequence $X_1, X_2, \cdots$, if the event $\{N = n\} \in \mathscr{F}_n$ for every $n \geqq 1$.

The only case considered here is when $X_1, X_2, \cdots$ are independent, identically distributed (i.i.d.) random variables with expectation $\theta > 0$. Let X be a random variable on Ω, independent of, and with the same distribution as, $X_1, X_2, \cdots$. Set $S_0 = 0$, $S_n = \sum_{\nu=1}^{n} X_\nu$, $X^- = \min\{0, X\}$ and $X^+ = \max\{0, X\}$.

Let c be a nonnegative constant and let $a(y)$, where $y \in [0, \infty)$, be a positive, continuous function, such that $a(y)/y \to 0$ as $y \to \infty$. The stopping times studied are

$$\text{(I)} \qquad \min\{n; S_n > c\}$$

and

$$\text{(II)} \qquad \min\{n; S_n > c \cdot a(n)\}.$$

According to the strong law of large numbers the stopping times are finite with probability 1 in both cases, that is, they are proper random variables.

The purpose of this paper is to find necessary and sufficient conditions for finiteness of the moments and moment generating functions of the stopping times, to find asymptotic expressions for the moments and the excess over the boundary as $c \to \infty$, and to prove that, under general conditions, a stopping time, suitably normed, has a limit distribution as $c \to \infty$. When treating Case II as $c \to \infty$

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some further conditions on the functions $a(y)$ are imposed. For sufficiently large y, $a(y)$ is then supposed to be a non-decreasing, concave, differentiable function, that varies regularly at infinity with exponent α, where $0 \leqq \alpha < 1$, i.e. $a(y) = y^\alpha \cdot L(y)$, where $L(y)$ varies slowly at infinity. (About regular and slow variation, see Feller [14] pages 268 ff.) To simplify, this restricted situation is called Case III.

Case I has been treated by Heyde [19], [20], [21], [22], and Chow [8], who has also studied Case III with $L(y) \equiv 1$, i.e. $a(y) = y^\alpha$, $0 \leqq \alpha < 1$, but with considerably weaker conditions on $X_1, X_2, \cdots$. Heyde [21] and Siegmund [31] have studied the asymptotic normality in Case I and Case III respectively (the latter with $L(y) \equiv 1$), and in Heyde [22] a theorem about attraction to stable laws is proved in Case I. When $P(X \geqq 0) = 1$ Case I reduces to classical renewal theory.

In the last chapter some of the results are generalized to separable random processes with independent, stationary increments and without positive jumps.

2. First passage times across horizontal barriers.

2.1. Throughout this chapter only Case I is considered. Let $X_1, X_2, \cdots$ be i.i.d. random variables such that $0 < EX = \theta < \infty$. Define $N = N(c) = \min\{n; S_n > c\}$.

THEOREM 2.1. *Let* $r \geqq 1$. *Then*

(a) $E|X^-|^r < \infty \Rightarrow EN^r < \infty$

(b) $EN^r < \infty \Rightarrow E|X^-|^r < \infty$

(c) $E(X^+)^r < \infty \Leftrightarrow ES_N{}^r < \infty$.

PROOF. If $X_1, X_2, \cdots$ are nonnegative random variables then $X_\nu{}^- \equiv 0$, $\nu = 1, 2, \cdots$, and (a) reduces to a well-known result from renewal theory (see e.g. Prabhu [26] Theorem 2.1, page 155). If $P(X < 0) > 0$ and $c = 0$, then $N = N(0)$ equals the first (strong ascending) ladder index, and then (a) and (b) are known from Heyde [19] Theorem 3, page 221, if $r \geqq 2$ is an integer, and from Feller [14] page 396, if $r = 1$. ($EX^- < \infty$, since $EX = \theta < \infty$.)

Set $M_n = \max_{1 \leqq \nu \leqq n} S_\nu$. Then $\{M_n \leqq c\} = \{N > n\}$ and thus $EN^r < \infty \Leftrightarrow \sum_{n=1}^\infty n^{r-1} P(M_n \leqq c) < \infty$, which is seen after a simple rearrangement of the sum. (a) then follows from Heyde [20] Theorem 3, page 705, if $r \geqq 1$ is an integer.

From $n^{-1}P(S_n \leqq 0) \leqq P(M_n \leqq 0) \leqq P(M_n \leqq c) \leqq P(S_n \leqq c)$ (see Heyde [19] page 220) it follows that $EN^r < \infty \Rightarrow \sum_{n=1}^\infty n^{r-1} P(M_n \leqq c) < \infty \Rightarrow \sum_{n=1}^\infty n^{r-2} P(S_n \leqq 0) < \infty \Rightarrow E|X^-|^r < \infty$, according to Smith [34] Theorem 7, page 274, if $r \geqq 1$, and so (b) is proved.

It remains to prove (a) when $r \geqq 1$ is not an integer, and (c). The proofs are based on some lemmas.

LEMMA 2.1. *Let* $r \geqq 1$. *If* $E|X|^r < \infty$, *then*

$$E|X_N| \leqq [E|X_N|^r]^{1/r} \leqq [EN \cdot E|X|^r]^{1/r} .$$

Furthermore, since $P(X_N > 0) = 1$, $|X_N|$ *and* $|X|$ *above may be replaced by* $X_N{}^+$ *and* X^+, *respectively.*

PROOF. The first inequality of this well-known lemma is true because $(E|X|^r)^{1/r}$ is a non-decreasing function of r (see e.g. Loève [23] page 156). The second inequality follows from $|X_N|^r \leqq |X_1|^r + |X_2|^r + \cdots + |X_N|^r$ and Wald's lemma. The rest of the proof is immediate.

PROOF OF (c). Since $S_N \geqq X_1^+$, $E|X^-|^r < \infty$ does not imply that $ES_N^r < \infty$. Suppose that $E(X^+)^r < \infty$. From $S_N \leqq c + X_N$ and $P(X_N > 0) = 1$ it follows that $X_N = X_N^+$, and hence

$$ES_N^r \leqq E(c + X_N^+)^r \leqq 2^{r-1}(c^r + E(X_N^+)^r) \leqq 2^{r-1}(c^r + EN \cdot E(X^+)^r) < \infty ,$$

where the c_r-inequalities (Loève [23] page 155), Lemma 2.1, and (a) with $r = 1$, have been used. The converse is immediate, since $S_N \geqq X_1^+$.

PROOF OF (a). The following truncation argument shows that it suffices to prove (a) when $E|X|^r < \infty$. Define

$$\begin{aligned} X_n' &= X_n \quad \text{if} \quad X_n \leqq A \\ &= 0 \quad \text{if} \quad X_n > A , \qquad n = 1, 2, \cdots, \ S_n' = \textstyle\sum_{\nu=1}^n X_\nu', \ n = 1, 2, \cdots , \end{aligned}$$

and $N' = \min\{n; S_n' > c\}$, and choose A such that $0 < EX_n' = \theta_A < \infty$. Obviously $E|X^-|^r < \infty \Rightarrow E|X'|^r < \infty$. Since $X_n' \leqq X_n$, $S_n' \leqq S_n$ it follows that $N \leqq N'$, and hence $EN^r \leqq E(N')^r$. Thus it suffices to prove (a), when $E|X|^r < \infty$.

Assume that $E|X|^r < \infty$, where $r > 1$.

$$\theta^r EN^r \leqq 2^{r-1}(E|S_N - N\theta|^r + ES_N^r)$$

and

$$E|S_N - N\theta|^r \leqq 2^{r-1}(ES_N^r + \theta^r EN^r) ,$$

where the c_r-inequalities have been used. Thus it follows from (c) that

$$EN^r < \infty \Leftrightarrow E|S_N - N\theta|^r < \infty .$$

(If $r = 2$ then $E|S_N - N\theta|^2 = \operatorname{Var} X \cdot EN$ according to Chow, Robbins and Teicher [7] Theorem 2, page 791).

Set $Y_\nu = X_\nu - \theta$, $\nu = 1, 2, \cdots$. Then $EY_\nu = 0$, and thus (a) is proved if $E|X|^r < \infty \Rightarrow E|\sum_{\nu=1}^N Y_\nu|^r < \infty$, $r > 1$.

The following lemma is Theorem 9, page 1502, of Burkholder [5].

LEMMA 2.2. *Let $\{Z_n\}$ be a martingale and suppose that $E|Z_n|^p < \infty$, $p > 1$. Set $Y_1 = Z_1$, $Y_n = Z_n - Z_{n-1}$, $n = 2, 3, \cdots$. Then*

$$c_p E|\textstyle\sum_{\nu=1}^n Y_\nu^2|^{p/2} \leqq E|Z_n|^p \leqq C_p E|\sum_{\nu=1}^n Y_\nu^2|^{p/2} ,$$

where c_p and C_p are constants depending only on p.

When Y_ν, $\nu = 1, 2, \cdots$ are i.i.d. random variables such that $EY_\nu = 0$ and $E|Y_\nu|^p < \infty$, then the lemma reduces to a well-known inequality, originally due to Marcinkiewicz and Zygmund (see [25] Theorem 13, page 87).

If $Y_\nu = X_\nu - \theta$, $\nu = 1, 2, \cdots$, then $\{\sum_{\nu=1}^n Y_\nu\}_{n=1}^\infty$ is a martingale and, since $E|Y|^r < \infty$, Lemma 2.2 applies. Define $N_n = \min\{N, n\}$ and define $U_k = \sum_{\nu=1}^k Y_\nu \cdot I\{N_n \geqq \nu\}$, $k = 1, 2, \cdots, n$, where $I\{\cdot\}$ denotes the indicator function of the set in braces. Then also $\{U_k\}_{k=1}^n$ is a martingale, $U_n = \sum_{\nu=1}^{N_n} Y_\nu = S_{N_n} - N_n\theta$, and $E|U_k|^r \leqq E|U_n|^r \leqq E|\sum_{\nu=1}^{n} Y_\nu|^r < \infty$, since $\{|U_k|^r\}_{k=1}^n$ is a submartingale. (See Doob [11] Theorem 2.1, page 300). Thus $\{U_k\}_{k=1}^n$ satisfies Lemma 2.2, and hence

$$c_r \cdot E|\textstyle\sum_{\nu=1}^{N_n} Y_\nu^2|^{r/2} \leqq E|U_n|^r = E|\sum_{\nu=1}^{N_n} Y_\nu|^r \leqq C_r \cdot E|\sum_{\nu=1}^{N_n} Y_\nu^2|^{r/2}.$$

The following lemma completes the proof of (a).

LEMMA 2.3. *Set* $\mu_s = E|Y|^s$, $s \leqq r$. *If* $\mu_r < \infty$, *then*

(a) $E|\sum_{\nu=1}^{N} Y_\nu|^r \leqq C_r \cdot \mu_r \cdot EN < \infty$ *if* $1 < r \leqq 2$

(b) $E|\sum_{\nu=1}^{N} Y_\nu|^r \leqq C(r, \mu_r) \cdot EN^{r/2} < \infty$ *if* $r > 2$

where $C(r, \mu_r)$ *is a constant depending only on* r *and* μ_r.

PROOF OF (a). If $1 < r \leqq 2$ then $EN < \infty$ and thus only the first inequality has to be proved. Furthermore, $\frac{1}{2} < r/2 \leqq 1$ and so the c_r-inequalities (Loève [23] page 155) and Wald's lemma on the right-hand inequality above give

$$E|U_n|^r \leqq C_r \cdot E|\textstyle\sum_{\nu=1}^{N_n} Y_\nu^2|^{r/2} \leqq C_r \cdot E \sum_{\nu=1}^{N_n} |Y_\nu|^r = C_r \cdot EN_n \cdot \mu_r \leqq C_r \mu_r EN.$$

Now Fatou's lemma gives

$$E|\textstyle\sum_{\nu=1}^{N} Y_\nu|^r \leqq \liminf_{n\to\infty} E|U_n|^r \leqq C_r \mu_r EN,$$

and thus (a) is proved.

PROOF OF (b). The proof consists of a step-wise reduction to (a). If $r > 2$ then $EN^{[r]} < \infty$ and thus also $EN^{r/2} < \infty$. Hence it remains to prove the first inequality.

Assume that $2^k < r \leqq 2^{k+1}$, where $k \geqq 1$ is a integer. Then

$$\begin{aligned} E|\textstyle\sum_{\nu=1}^{N_n} Y_\nu|^r &\leqq C_r \cdot E|\textstyle\sum_{\nu=1}^{N_n} Y_\nu^2|^{r/2} \leqq 2^{r/2-1} \cdot C_r \cdot E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)|^{r/2} \\ &\quad + 2^{r/2-1} \cdot C_r \cdot \mu_2^{r/2} EN_n^{r/2} \leqq 2^{r/2-1} \cdot C_r \cdot E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)|^{r/2} \\ &\quad + 2^{r/2-1} C_r \mu_r EN^{r/2}. \end{aligned}$$

Since $\{\sum_{\nu=1}^{n} (Y_\nu^2 - \mu_2)\}_{n=1}^\infty$ is a martingale and

$$E|\textstyle\sum_{\nu=1}^{n} (Y_\nu^2 - \mu_2)|^{r/2} \leqq C_{r/2} n^{r/4} E|Y_1^2 - \mu_2|^{r/2} < \infty,$$

Lemma 2.2 is applicable. Define the martingale transform $\{U_{1k}\}_{k=1}^n$, where

$$U_{1k} = \textstyle\sum_{\nu=1}^{k} (Y_\nu^2 - \mu_2) \cdot I\{N_n \geqq \nu\}, \qquad k = 1, 2, \cdots, n.$$

$U_{1n} = \sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)$. With the same arguments as before it follows that

$$E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)|^{r/2} \leqq C_{r/2} \cdot E|\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)^2|^{r/4}.$$

If $k = 1$, i.e. $1 < r/2 \leqq 2$, then it follows as in the proof of (a) that

$$E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)|^{r/2} \leqq C_{r/2} \cdot E|Y_1^2 - \mu_2|^{r/2} \cdot EN.$$

If $r/2 > 2$ then

$$\begin{aligned} E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)|^{r/2} &\leqq C_{r/2} \cdot E|\textstyle\sum_{\nu=1}^{N_n} (Y_\nu^2 - \mu_2)^2|^{r/4} \\ &\leqq 2^{r/4-1} \cdot C_{r/2} \cdot E|\textstyle\sum_{\nu=1}^{N_n} ((Y_\nu^2 - \mu_2)^2 - E(Y_\nu^2 - \mu_2)^2)^{r/4} \\ &\quad + 2^{r/4-1} \cdot C_{r/2} \cdot E|Y_1^2 - \mu_2|^2 \cdot EN^{r/4} \end{aligned}$$

and the process continues. It terminates, however, after k steps, (where $2^k < r \leqq 2^{k+1}$). If all the terms are collected into one inequality it follows that

$$E|\sum_{\nu=1}^{N_n} Y_\nu|^r \leqq b_0 \cdot EN + \sum_{\nu=1}^{k} b_\nu \cdot EN^{r/2^\nu},$$

where $b_0, b_1, \cdots, b_k$ are constants depending only on r and μ_s, $s \leqq r$. Since $(E|Z|^r)^{1/r}$ is a non-decreasing function of r it follows that $\mu_s \leqq \mu_r^{s/r}$, $s \leqq r$, and thus $b_0, b_1, \cdots, b_k$ may be increased to $b_0', b_1', \cdots, b_k'$, which depend only on r and μ_r. Furthermore, $EN^{r2^{-\nu}} \leqq EN^{r/2}$, $\nu = 1, 2, \cdots, k$, and thus

$$E|\sum_{\nu=1}^{N_n} Y_\nu|^r \leqq C(r, \mu_r) \cdot EN^{r/2},$$

where $C(r, \mu_r)$ is a constant depending only on r and μ_r.

Finally Fatou's lemma gives

$$E|\sum_{\nu=1}^{N} Y_\nu|^r \leqq \liminf_{n\to\infty} E|\sum_{\nu=1}^{N_n} Y_\nu|^r \leqq C(r, \mu_r) \cdot EN^{r/2}$$

and thus the lemma is proved, and hence also Theorem 2.1 a.

The following result on moment generating functions is Theorem 1, page 220 of Heyde [19].

THEOREM 2.2. *There exists a* $t_0 > 0$ *such that* $Ee^{tN} < \infty$, $|t| < t_0$ *if and only if there exists a* $t_1 > 0$ *such that* $Ee^{tX^-} < \infty$, $|t| < t_1$.

2.2. In the remaining part of this chapter asymptotic properties of $N = N(c)$ will be studied.

THEOREM 2.3. *Let* $r \geqq 1$. *Then*

(a) $N/c \rightarrow_{\text{a.s.}} \theta^{-1}$, *as* $c \to \infty$.
(b) $E|X^-|^r < \infty \Leftrightarrow E(N/c)^r \to \theta^{-r}$, *as* $c \to \infty$.
(c) $E|X^-|^r < \infty \Rightarrow E(N/c)^s \to \theta^{-s}$, *as* $c \to \infty$ *for all* s, $0 \leqq s \leqq r$.
(d) $E(X^+)^r < \infty \Leftrightarrow E(S_N/c)^r \to 1$, *as* $c \to \infty$.

(a) is proved in Heyde [20], Theorem 7, page 710. If $P(X \geqq 0) = 1$ the result of Doob [10], Theorem 1, pages 423–424, is obtained. Chow [8] has proved (b) and (c) when $r = 2$, pages 385–386, and Chow and Robbins [6] have proved (b) when $r = 1$. Heyde [20] has proved that $E|X^-|^{r+1} < \infty \Rightarrow E(N/c)^r \to \theta^{-r}$ as $c \to \infty$ if $r \geqq 1$ is an integer, see Theorem 6, page 709. If $P(X \geqq 0) = 1$ then $X^- \equiv 0$ and the results follow from Hatori [18] Theorem 1, page 141.

PROOF OF (b). Let $r > 1$. The necessity follows from Theorem 2.1 b since $EN^r < \infty$ if $c < \infty$.

Now assume that $E|X^-|^r < \infty$. (a) and Fatou's lemma give $\liminf_{c\to\infty} E(N/c)^r \geqq \theta^{-r}$.

It remains to prove $\limsup_{c\to\infty} E(N/c)^r \leqq \theta^{-r}$. This is first proved under the assumption that also $E|X|^r < \infty$. Let $0 < \delta < \theta$ and define $N_*(c) = \min\{n; S_n > c, n > c/\delta\}$. For simplicity, assume that $c_0 = c/\delta$ is an integer.

Define $A_0 = \{S_{c_0} > c\}$ and $A_\nu = \{c - \nu < S_{c_0} \leqq c - (\nu - 1)\}$, $\nu = 1, 2, \cdots$

$$\begin{aligned} EN^r = EN^r(c) &\leqq EN_*{}^r(c) = E(N_*{}^r(c) \mid A_0) \cdot P(A_0) \\ &+ \textstyle\sum_{\nu=1}^\infty E(N_*{}^r(c) \mid A_\nu) \cdot P(A_\nu) \leqq E(N_*{}^r(c) \mid S_{c_0} = c) \cdot P(A_0) \\ &+ \textstyle\sum_{\nu=1}^\infty E(N_*{}^r(c) \mid S_{c_0} = c - \nu) \cdot P(A_\nu) = E(c_0 + N(0))^r \cdot P(A_0) \\ &+ \textstyle\sum_{\nu=1}^\infty E(c_0 + N(\nu))^r \cdot P(A_\nu) . \end{aligned}$$

Let $0 \leqq t \leqq 1$ and let $0 \leqq \eta \leqq 1$. Then $(1 + t)^r = 1 + r(1 + \eta t)^{r-1} \cdot t \leqq 1 + r \cdot 2^{r-1} t$ and thus

$$(a + b)^r \leqq a^r + r \cdot 2^{r-1} ab^{r-1} + r \cdot 2^{r-1} a^{r-1} b + b^r .$$

From this inequality and the c_r-inequalities it follows that

$$\begin{aligned} E\left(\frac{N(c)}{c}\right)^r &\leqq E\left(\frac{1}{\delta} + \frac{N(0)}{c}\right)^r \cdot P(A_0) + \textstyle\sum_{\nu=1}^\infty E\left(\frac{1}{\delta} + \frac{N(\nu)}{c}\right)^r P(A_\nu) \\ &\leqq \frac{1}{\delta^r} + r \cdot 2^{r-1}\left(\frac{1}{\delta^{r-1}} \cdot \frac{EN(0)}{c} + \frac{1}{\delta} \cdot \frac{EN^{r-1}(0)}{c^{r-1}}\right) + \frac{EN^r(0)}{c^r} \\ &\quad + 2^{r-1} \cdot \textstyle\sum_{\nu=1}^\infty \left(\frac{1}{\delta^r} + \frac{EN^r(\nu)}{c^r}\right) P(A_\nu) \\ &= \frac{1}{\delta^r} + r \cdot 2^{r-1}\left(\frac{EN(0)}{c \cdot \delta^{r-1}} + \frac{EN^{r-1}(0)}{\delta \cdot c^{r-1}}\right) + \frac{EN^r(0)}{c^r} \\ &\quad + \frac{2^{r-1}}{\delta^r}(1 - P(A_0)) + 2^{r-1} \textstyle\sum_{\nu=1}^\infty \frac{EN^r(\nu)}{c^r} \cdot P(A_\nu) . \end{aligned}$$

By Theorem 2.1 a $EN^s(0) < \infty$, $s \leqq r$ and by the strong law of large numbers $1 - P(A_0) = P(S_{c_0} \leqq c_0 \delta) \to 0$ as $c \to \infty$. Thus

$$\limsup_{c\to\infty} E\left(\frac{N(c)}{c}\right)^r \leqq \frac{1}{\delta^r} + 2^{r-1} \limsup_{c\to\infty} \textstyle\sum_{\nu=1}^\infty \frac{EN^r(\nu)}{c^r} \cdot P(A_\nu) .$$

The next step is to prove that $\lim_{c\to\infty} \sum_{\nu=1}^\infty E(N^r(\nu)/c^r) \cdot P(A_\nu) = 0$. If $\nu = 1, 2, \cdots$, then

$$\begin{aligned} P(A_\nu) &= P(c - \nu < S_{c_0} \leqq c - (\nu - 1)) \\ &= P(c_0(\delta - \theta) - \nu < S_{c_0} - c_0\theta \leqq c_0(\delta - \theta) - \nu + 1) \\ &\leqq P(c_0(\theta - \delta) + \nu - 1 \leqq |S_{c_0} - c_0\theta| < c_0(\theta - \delta) + \nu) \end{aligned}$$

if c is so large that $c_0(\theta - \delta) - 1 > 0$. This can always be achieved since δ is chosen before c.

$N(\nu) \leqq N(\nu - 1) + \min\{n; S_n - S_{N(\nu-1)} > 1 \mid S_{N(\nu-1)} = \nu - 1\} = N(\nu - 1) + N_1(1)$ and so $N(\nu) \leqq N_1(1) + N_2(1) + \cdots + N_\nu(1)$, where $N_k(1)$, $k = 1, 2, \cdots, \nu$ are distributed as $N(1)$. Minkowski's inequality implies that

$$(EN^r(\nu))^{1/r} \leqq \nu \cdot (EN_1{}^r(1))^{1/r} .$$

By Theorem 2.1 a

$$E\left(\frac{N(\nu)}{\nu}\right)^r \leqq EN_1{}^r(1) < \infty , \qquad \nu = 1, 2, \cdots$$

Hence

$$\sum_{\nu=1}^{\infty} E\left(\frac{N(\nu)}{c}\right)^r \cdot P(A_\nu)$$
$$\leqq \frac{EN_1{}^r(1)}{c^r} \cdot \sum_{\nu=1}^{\infty} \nu^r P(A_\nu)$$
$$\leqq \frac{EN_1{}^r(1)}{c^r} \cdot \sum_{\nu=1}^{\infty} \nu^r \cdot P(c_0(\theta-\delta)+\nu-1 \leqq |S_{c_0}-c_0\theta| < c_0(\theta-\delta)+\nu)$$
$$\leqq \frac{EN_1{}^r(1)}{c^r} \cdot \sum_{\nu=1}^{\infty} (\nu + c_0(\theta-\delta)-1)^r$$
$$\times P(c_0(\theta-\delta)+\nu-1 \leqq |S_{c_0}-c_0\theta| < c_0(\theta-\delta)+\nu)$$
$$\leqq \frac{EN_1{}^r(1)}{c^r} \cdot E|S_{c_0}-c_0\theta|^r .$$

If $r > 2$ then, by the inequalities of Marcinkiewicz and Zygmund (see [25] Theorem 13, page 87) and Minkowski

$$E|S_{c_0}-c_0\theta|^r \leqq C(r) \cdot E|\sum_{\nu=1}^{c_0} (X_\nu-\theta)^2|^{r/2}$$
$$\leqq C(r) \cdot c_0{}^{r/2} E|X-\theta|^r = C(r) \cdot \left(\frac{c}{\delta}\right)^{r/2} \cdot E|X-\theta|^r ,$$

where $C(r)$ is a constant depending only on r. Hence

$$0 \leqq \limsup_{c\to\infty} \sum_{\nu=1}^{\infty} E\left(\frac{N(\nu)}{c}\right)^r \cdot P(A_\nu)$$
$$\leqq \limsup_{c\to\infty} \frac{EN_1{}^r(1)}{c^r} \cdot C(r) \cdot \left(\frac{c}{\delta}\right)^{r/2} \cdot E|X-\theta|^r = 0 .$$

If $1 < r \leqq 2$ then, by the inequality of Marcinkiewicz and Zygmund and the c_r-inequalities,

$$E|S_{c_0}-c_0\theta|^r \leqq C(r) E|\sum_{\nu=1}^{c_0} (X_\nu-\theta)^2|^{r/2}$$
$$\leqq C(r) \sum_{\nu=1}^{c_0} E|X_\nu-\theta|^r = C(r) \frac{c}{\delta} E|X-\theta|^r ,$$

where $C(r)$ is a constant depending only on r. Hence

$$0 \leqq \limsup_{c\to\infty} \sum_{\nu=1}^{\infty} E\left(\frac{N(\nu)}{c}\right)^r \cdot P(A_\nu)$$
$$\leqq \limsup_{c\to\infty} \frac{EN_1{}^r(1)}{c^r} \cdot C(r) \cdot \frac{c}{\delta} \cdot E|X-\theta|^r = 0 .$$

Thus $\limsup_{c\to\infty} E(N(c)/c)^r \leqq \delta^{-r}$ for all δ, $0 < \delta < \theta$. It follows that $\limsup_{c\to\infty} E(N(c)/c)^r \leqq \theta^{-r}$.

To complete the proof of (b) it remains to remove the assumption that $E|X|^r < \infty$. Thus, suppose only that $E|X^-|^r < \infty$ and define X_n', S_n' and $N' = N'(c)$ as in the proof of Theorem 2.1 a. Choose A such that $EX_n' = \theta_A > 0$. Since

$X_n' \leqq X_n$, $S_n' \leqq S_n$, $N(c) \leqq N'(c)$ and $E|X'|^r < \infty$ the above result implies that

$$\limsup_{c\to\infty} E\left(\frac{N(c)}{c}\right)^r \leqq \limsup_{c\to\infty} E\left(\frac{N'(c)}{c}\right)^r \leqq \frac{1}{\theta_A{}^r}\,.$$

However, A may be chosen such that θ_A is arbitrarily close to θ and thus $\limsup_{c\to\infty} E(N(c)/c)^r \leqq \theta^{-r}$.

PROOF OF (c). (c) follows from (a), (b) and Corollary 1, page 164, of Loève [23].

PROOF OF (d). The necessity follows from Theorem 2.1 c, since $ES_N{}^r < \infty$ if $c < \infty$. Now let $r = 1$. Then by Wald's lemma and (b)

$$E\,\frac{S_N}{c} = \frac{ES_N}{c} = \frac{\theta \cdot EN}{c} \to \theta \cdot \frac{1}{\theta} = 1 \qquad \text{as } c \to \infty\,.$$

Finally, let $r > 1$. From $c < S_N \leqq c + X_N{}^+$, Minkowski's inequality and Lemma 2.1 it follows that

$$c \leqq [ES_N{}^r]^{1/r} \leqq [E(c + X_N{}^+)]^{1/r} \leqq c + (E(X_N{}^+)^r)^{1/r} \leqq c + (EN \cdot E(X^+)^r)^{1/r}\,.$$

Hence

$$1 \leqq \left(E\left(\frac{S_N}{c}\right)^r\right)^{1/r} \leqq 1 + c^{-1} \cdot (EN)^{1/r} \cdot (E(X^+)^r)^{1/r}$$
$$= 1 + c^{1/r-1} \cdot \left(\frac{EN}{c}\right)^{1/r} \cdot (E(X^+)^r)^{1/r}\,.$$

By (b) the last term tends to 0 as $c \to \infty$ and so (d) is proved.

REMARK. Allowing $\theta = +\infty$ the following generalization of Corollary 3, page 386, of Chow [8] can be proved.

COROLLARY 2.3.1. *Let* $r \geqq 1$. *If* $E|X^-|^r < \infty$ *and* $0 < EX = \theta \leqq \infty$, *then* $E(N/c)^r \to \theta^{-r}$ *as* $c \to \infty$, *where* θ^{-r} *is interpreted as* 0 *if* $\theta = +\infty$.

PROOF. If $0 < \theta < \infty$ there is nothing more to prove. Let $\theta = +\infty$. The method of Prabhu [26] Theorem 3.1, page 161, is used. Define X_n', S_n' and N' as in the proof of Theorem 2.1 a. Again $X_n' \leqq X_n$, $S_n' \leqq S_n$ and it follows that $N \leqq N'$. Furthermore, A is chosen such that $0 < EX_n' = \theta_A < \infty$. Theorem 2.3 b applied to the sequence $X_1', X_2', \cdots$ gives

$$0 \leqq \limsup_{c\to\infty} E\left(\frac{N}{c}\right)^r \leqq \limsup_{c\to\infty} E\left(\frac{N'}{c}\right)^r = \frac{1}{\theta_A{}^r}\,.$$

Since A may be chosen arbitrarily large and since $1/\theta_A \to 0$ as $A \to \infty$ it follows that $\limsup_{c\to\infty} E(N/c)^r = 0$ and hence $\lim_{c\to\infty} E(N/c)^r = 0$.

THEOREM 2.4. *If* $E(X^+)^r < \infty$, *where* $r \geqq 1$, *then*

(a) $[E(S_N - c)^r]/c \to 0$, *as* $c \to \infty$,

(b) $[ES_N - c]/c^{1/r} \to 0$, *as* $c \to \infty$,

(c) $[EN - c/\theta]/c^{1/r} \to 0$, *as* $c \to \infty$.

$S_N - c$ is called the excess over the boundary. See also Lorden [24].

To prove the theorem the following lemma is used.

LEMMA 2.4. *Let* $E(X^+)^r < \infty$, *where* $r \geqq 1$. *Then*

(a) $EX_N{}^r/c \to 0$, *as* $c \to \infty$,

(b) $EX_N/c^{1/r} \to 0$, *as* $c \to \infty$.

PROOF. The technique is similar to the one used in Gundy and Siegmund [16] page 1916. For a slightly different situation where $r = 2$, see Siegmund [32] page 1075, Lemma 2.

Let $\varepsilon > 0$ be an arbitrarily small given number and choose n_0 so large that $E((X^+)^r \cdot I\{(X^+) > (\varepsilon^2 n)^{1/r}\}) < \varepsilon$ if $n \geqq n_0$.

$$\begin{aligned} EX_N{}^r = E(X_N{}^+)^r &= E((X_N{}^+)^r \cdot I\{(X_N{}^+)^r \leqq \varepsilon N\}) + E((X_N{}^+)^r \cdot I\{(X_N{}^+)^r > \varepsilon N\}) \\ &\leqq \varepsilon EN + E(\textstyle\sum_{k=1}^N (X_k{}^+)^r \cdot I\{(X_k{}^+)^r > \varepsilon k\}) \\ &= \varepsilon EN + E((\textstyle\sum_{k=1}^N (X_k{}^+)^r \cdot I\{(X_k{}^+)^r > \varepsilon k) \cdot I\{N \leqq \varepsilon n_0\}) \\ &\qquad + E((\textstyle\sum_{k=1}^N (X_k{}^+)^r \cdot I\{(X_k{}^+)^r > \varepsilon k) \cdot I\{N > \varepsilon n_0\}) \\ &\leqq \varepsilon EN + E((\textstyle\sum_{k=1}^{[\varepsilon n_0]} (X_k{}^+)^r) \cdot I\{N \leqq \varepsilon n_0\}) \\ &\qquad + E((\textstyle\sum_{k=1}^{[\varepsilon n_0]} (X_k{}^+)^r) \cdot I\{N > \varepsilon n_0\}) \\ &\qquad + E((\textstyle\sum_{k=[\varepsilon n_0]+1}^N (X_k{}^+)^r \cdot I\{(X_k{}^+)^r > \varepsilon^2 n_0\}) \cdot I\{N > \varepsilon n_0\}) \\ &\leqq \varepsilon EN + \varepsilon n_0 E(X^+)^r + E(\textstyle\sum_{k=1}^N (X_k{}^+)^r \cdot I\{(X_k{}^+)^r > \varepsilon^2 n_0\}) \\ &= \varepsilon EN + \varepsilon n_0 E(X^+)^r + EN \cdot E((X^+)^r \cdot I\{(X^+)^r > \varepsilon^2 n_0\}) \\ &\leqq \varepsilon EN + \varepsilon n_0 E(X^+)^r + EN \cdot \varepsilon = \varepsilon(2 \cdot EN + n_0 E(X^+)^r)\,. \end{aligned}$$

The last inequality holds because of the way ε and n_0 are chosen and the preceding equality is a consequence of Wald's lemma.

Thus $0 \leqq EX_N{}^r/EN \leqq 2\varepsilon + \varepsilon(n_0 E(X^+)^r/EN)$, from which it follows that $0 \leqq \limsup_{c\to\infty} EX_N{}^r/EN \leqq 2\varepsilon$. Since ε was arbitrary $0 \leqq \limsup_{c\to\infty} EX_N{}^r/EN \leqq 0$ and thus $\lim_{c\to\infty} EX_N{}^r/EN = 0$. By Theorem 2.3 b $EN/c \to \theta^{-1}$ as $c \to \infty$, from which (a) follows.

(b) is an easy consequence of (a) and the lemma is proved.

PROOF OF THEOREM 2.4. From $c < S_N \leqq c + X_N$ and Lemma 2.4 a it follows that

$$0 \leqq \frac{E(S_N - c)^r}{c} \leqq \frac{EX_N{}^r}{c} \to 0 \qquad \text{as } c \to \infty\,.$$

(b) is an easy consequence of (a), and (c) follows from (b) and Wald's lemma.

The next problem is to establish the asymptotic normality of $N = N(c)$ if $\operatorname{Var} X = \sigma^2 < \infty$.

THEOREM 2.5. *Let* $0 < EX = \theta < \infty$ *and* $\operatorname{Var} X = \sigma^2 < \infty$. *Then*

$$\mathscr{L}\left(\frac{N - c/\theta}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \textit{as } c \to \infty\,.$$

The proofs in the case when $P(X \geqq 0) = 1$ are due to Feller [13] if X has a lattice distribution, and due to Takacs [36] if X has a non-lattice distribution. The present proof, which is quite different from that of Heyde (see [21] Theorem 4, page 148), is a combination of the following lemmas, which are Theorem 1, page 193, and Lemma 3, page 197, of Rényi [27], and which are similarly applied there.

LEMMA 2.5. *Let $Y_1, Y_2, \cdots$ be* i.i.d. *random variables with expectation* 0 *and variance* 1, *and let $Z_n = \sum_{\nu=1}^{n} Y_\nu$. Let $\nu(t)$ be a positive integer valued random variable for any $t > 0$, and assume that $\nu(t)/t \to_P k$ as $t \to \infty$, where k is a positive constant. Then*

$$\mathscr{L}\left(\frac{Z_{\nu(t)}}{(\nu(t))^{\frac{1}{2}}}\right) \Rightarrow N(0, 1), \qquad \text{as } t \to \infty .$$

Lemma 2.5 is a special case of a theorem originally due to Anscombe (see [1]).

LEMMA 2.6. *Let $Y_1, Y_2, \cdots$ be* i.i.d. *random variables and suppose that the second moment exists. If $\nu(t)$ is as in the previous lemma, then*

$$\frac{Y_{\nu(t)}}{(\nu(t))^{\frac{1}{2}}} \to_P 0 , \qquad \text{as } t \to \infty .$$

PROOF OF THEOREM 2.5. By Lemma 2.5, with $Y_m = (X_m - \theta)/\sigma$, $t = c$, $\nu(t) = N$ $(= N(c))$, and $k = \theta^{-1}$, it is found that

$$\mathscr{L}\left(\frac{S_N - N\theta}{\sigma N^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty .$$

From $c < S_N \leqq c + X_N$ it follows that

$$\frac{c - N\theta}{\sigma N^{\frac{1}{2}}} < \frac{S_N - N\theta}{\sigma N^{\frac{1}{2}}} \leqq \frac{c + X_N - N\theta}{\sigma N^{\frac{1}{2}}} .$$

Lemma 2.6 implies that $X_N/\sigma N^{\frac{1}{2}} \to_P 0$ as $c \to \infty$, and hence

$$\mathscr{L}\left(\frac{c - N\theta}{\sigma N^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty .$$

Theorem 2.3a and Cramér's theorem (see Cramér [9] page 254) imply that

$$\mathscr{L}\left(\frac{N\theta - c}{\sigma(c/\theta)^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty$$

from which the conclusion follows.

Theorem 2.5 does not imply that $EN \sim c/\theta$ or $\operatorname{Var} N \sim \sigma^2 c/\theta^3$ as $c \to \infty$. The following theorems state, however, that these results are true.

Let $X_1, X_2, \cdots$ be i.i.d. random variables and let $N_1, N_1 + N_2, N_1 + N_2 + N_3, \cdots$ be the successive strong ascending ladder indices, i.e. $N_1 = \min\{n; S_n > 0\}$, $N_1 + N_2 = \min\{n > N_1; S_n > S_{N_1}\}, \cdots$ and let $Y_1 = S_{N_1}$, $Y_1 + Y_2 = S_{N_1+N_2}, \cdots$ be the corresponding ladder heights. Then $N_1, N_2, \cdots$ and $Y_1, Y_2, \cdots$ are two sequences of i.i.d. positive random variables. Furthermore, $Y_1, Y_2, \cdots$ constitute a renewal process (see e.g. Prabhu [26] Chapter 6.2, page 208 ff.) and hence the results from renewal theory are applicable.

With a slightly different notation, and with the assumption that $\operatorname{Var} X < \infty$ also in (a), the following theorem is stated and proved in Heyde [21] page 147.

THEOREM 2.6. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables with a non-lattice distribution and suppose that* $0 < EX = \theta < \infty$ *and* $E(X^+)^2 < \infty$. *Then*

(a) $EN = c/\theta + EN_1 \cdot EY_1^2/2(EY_1)^2 + o(1)$, *as* $c \to \infty$;

if also $\operatorname{Var} X = \sigma^2 < \infty$, *then*

(b) $\operatorname{Var} N = \sigma^2 c/\theta^3 + o(c)$, *as* $c \to \infty$.

Siegmund [32] has proved $EN \sim c/\theta$ and $\operatorname{Var} N \sim \sigma^2 c/\theta^3$ as $c \to \infty$ for more general sequences $X_1, X_2, \cdots$. If $P(X \geqq 0) = 1$ then $N_1 = N_2 = \cdots = 1$ and $Y_i = X_i$, $i = 1, 2, \cdots$ and the theorem reduces to the classical results (see e.g. Smith [33]).

The present proof of (a) makes use of the ladder variables defined above, a method introduced by Blackwell [3] in a similar context.

PROOF OF (a). The crossing of the boundary c must occur at a ladder point. Therefore, let $M = \min\{m; \sum_{\nu=1}^{m} Y_\nu > c\}$. By Theorem 2.1 a EN_1 is finite and by Theorem 2.1 c $EY_1 = ES_{N_1}$ and $EY_1^2 = ES_{N_1}^2$ are finite. Hence the relation (a) for M, (which is already known to be true), gives

$$EM = \frac{c}{EY_1} + \frac{EY_1^2}{2(EY_1)^2} + o(1) \qquad \text{as } c \to \infty .$$

Since the crossing occurs at a ladder point it follows that

$$N = N_1 + N_2 + \cdots + N_M .$$

All expectations involved are finite, and thus Wald's lemma implies that

$$EY_1 = \theta \cdot EN_1 \qquad \text{and} \qquad EN = EN_1 \cdot EM .$$

Finally $EN = EN_1 \cdot EM = EN_1 \cdot (c/\theta EN_1 + EY_1^2/2(EY_1)^2 + o(1))$ as $c \to \infty$ and (a) is proved.

Now let $X_1, X_2, \cdots$ have a lattice distribution, i.e. a discrete distribution with masses at the points md, where $m = 0, \pm 1, \pm 2, \cdots$ and where $d > 0$ is the span. Assume that $d = 1$ for notational convenience. Then $N = \min\{n; S_n > c\} = \min\{n; S_n > [c]\}$. Therefore, define $N^{(n)} = \min\{k; S_k > n\}$.

With this notation the corresponding result is

THEOREM 2.7. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables with a lattice distribution with span* $d = 1$ *and suppose that* $0 < EX = \theta < \infty$ *and* $E(X^+)^2 < \infty$. *Then*

(a) $EN^{(n)} = n/\theta + EN_1 \cdot EY_1^2/2(EY_1)^2 + 1/2\theta + o(1)$, *as* $n \to \infty$;

If also $\operatorname{Var} X = \sigma^2 < \infty$, *then*

(b) $\operatorname{Var} N^{(n)} = \sigma^2 n/\theta^3 + o(n)$, *as* $n \to \infty$.

With the original notation the result is

THEOREM 2.7′. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables with a lattice distribution with span* $d = 1$ *and suppose that* $0 < EX = \theta < \infty$ *and* $E(X^+)^2 < \infty$. *Then*

(a) $EN = c/\theta + EN_1 \cdot EY_1^2/2(EY_1)^2 + 1/2\theta - (c - [c])/\theta + o(1)$, *as* $c \to \infty$;

If also $\operatorname{Var} X = \sigma^2 > \infty$, *then*

(b) $\operatorname{Var} N = \sigma^2 c/\theta^3 + o(c)$, *as* $c \to \infty$.

Note that the statements about EN in the last three theorems are sharper than Theorem 2.4 c with $r \geqq 2$.

If $P(X \geqq 0) = 1$, then again $N_1 = N_2 = \cdots = 1$, and $Y_i = X_i$, $i = 1, 2, \cdots$, and formulas similar to formulas (6.7) and (6.10) of Feller [13] page 111, are obtained. The proof of Theorem 2.7 a is carried through in the same way as the proof given above of Theorem 2.6 a with the only change that M is replaced by $M^{(n)} = \min\{k; \sum_{\nu=1}^k Y_\nu > n\}$. Hence

$$EM^{(n)} = \frac{n}{EY_1} + \frac{EY_1^2}{2(EY_1)^2} + \frac{1}{2EY_1} + o(1) \qquad \text{as } n \to \infty ,$$

according to Feller [13] page 111.

As mentioned above, a proof of (b) is found in Siegmund [32].

Since $0 < Y_1 = S_{N_1} \leqq X_{N_1}^+$, and $EY_1 = \theta \cdot EN_1$, Lemma 2.1 implies that

$$0 \leqq EY_1^2 \leqq E(X_{N_1}^+)^2 \leqq EN_1 \cdot E(X^+)^2 ,$$

and hence

$$0 \leqq \frac{EN_1 \cdot EY_1^2}{2(EY_1)^2} \leqq \frac{(EN_1)^2 \cdot E(X^+)^2}{2(EN_1)^2 \cdot \theta^2} = \frac{E(X^+)^2}{2\theta^2} .$$

Thus, the following corollaries are proved. (See also Lorden [24] Theorem 1, page 521, for a related result).

COROLLARY 2.6.1. *If* X *has a non-lattice distribution with* $0 < EX = \theta < \infty$ *and* $E(X^+)^2 < \infty$, *then*

$$\frac{c}{\theta} \leqq EN \leqq \frac{c}{\theta} + \frac{E(X^+)^2}{2\theta^2} + o(1) , \qquad \textit{as } c \to \infty .$$

COROLLARY 2.7.1. *If* X *has a lattice distribution with* $d = 1$ *and* $0 < EX = \theta < \infty$ *and* $E(X^+)^2 < \infty$, *then*

$$\frac{n}{\theta} \leqq EN^{(n)} \leqq \frac{n}{\theta} + \frac{E(X^+)^2}{2\theta^2} + \frac{1}{2\theta} + o(1) , \qquad \textit{as } n \to \infty$$

$$\frac{c}{\theta} \leqq EN \leqq \frac{c}{\theta} + \frac{E(X^+)^2}{2\theta^2} + \frac{1}{2\theta} + o(1) , \qquad \textit{as } c \to \infty .$$

2.3. Let $X_1, X_2, \cdots$ be i.i.d. random variables such that $0 < EX = \theta < \infty$. In the first of the following theorems it is assumed that $E|X|^r < \infty$ or $E(X^+)^r < \infty$, where $1 \leqq r < 2$, and in the second theorem it is assumed that the sequence $X_1, X_2, \cdots$ belongs to the domain of attraction of a stable law with exponent β, $1 < \beta \leqq 2$. (See e.g. Feller [14] Chapters IX. 8 and XVII. 5, and Gnedenko-Kolmogorov [15] Chapter 7.)

THEOREM 2.8. *If* $E|X|^r < \infty$, *where* $1 \leqq r < 2$, *then*
(a) $(N - c/\theta)/c^{1/r} \to_{\text{a.s.}} 0$, *as* $c \to \infty$;
If $E(X^+)^r < \infty$, *where* $1 \leqq r < 2$, *then*
(b) $(EN - c/\theta)/c^{2-r} \to 0$, *as* $c \to \infty$.
(c) $(ES_N - c)/c^{2-r} \to 0$, *as* $c \to \infty$.

Note that because of Theorems 2.5—2.7 the theorem is false when $r = 2$. When $r = 1$ the theorem is a part of Theorem 2.3. Since $2 - r < r^{-1}$ when $1 < r < 2$, (b) and (c) are sharper results than (c) and (b) of Theorem 2.4.

To prove the theorem two lemmas are needed. The first lemma is Theorem 1 of Richter [28], where a proof is found.

LEMMA 2.7. *Let* $Y, Y_1, Y_2, \cdots$ *be random variables such that* $Y_n \to_{\text{a.s.}} Y$, *as* $n \to \infty$. *Further, let* $\nu(t)$ *be a positive integer valued random variable for any* $t > 0$, *and assume that* $\nu(t) \to_{\text{a.s.}} \infty$, *as* $t \to \infty$. *Then*

$$Y_{\nu(t)} \to_{\text{a.s.}} Y\,, \qquad \textit{as } t \to \infty\,.$$

LEMMA 2.8. *Let* $E|X|^r < \infty$, *where* $1 \leqq r < 2$. *Then*

$$\frac{X_N}{c^{1/r}} \to_{\text{a.s.}} 0\,, \qquad \textit{as } c \to \infty\,.$$

PROOF. According to Loève [23] page 243,

$$\frac{1}{n^{1/r}} \sum_{\nu=1}^{n} (X_\nu - \theta) \to_{\text{a.s.}} 0\,, \qquad \text{as } n \to \infty\,.$$

Thus, also

$$\frac{1}{(n-1)^{1/r}} \sum_{\nu=1}^{n-1} (X_\nu - \theta) \to_{\text{a.s.}} 0$$

and

$$n^{-(1/r)} \sum_{\nu=1}^{n-1} (X_\nu - \theta) \to_{\text{a.s.}} 0\,, \qquad \text{as } n \to \infty\,.$$

Since $X_n/n^{1/r} = n^{-(1/r)} \sum_{\nu=1}^{n} (X_\nu - \theta) - n^{-(1/r)} \sum_{\nu=1}^{n-1} (X_\nu - \theta) + \theta/n^{1/r}$ it follows that

$$\frac{X_n}{n^{1/r}} \to_{\text{a.s.}} 0\,, \qquad \text{as } n \to \infty\,.$$

Since $N/c \to_{\text{a.s.}} \theta^{-1}$, as $c \to \infty$, Lemma 2.7, applied to $Y_n = X_n/n^{1/r}$, $n = 1, 2, \cdots$, and $Y \equiv 0$, gives that

$$\frac{X_N}{N^{1/r}} \to_{\text{a.s.}} 0\,, \qquad \text{as } c \to \infty\,.$$

Now $X_N/c^{1/r} = X_N/N^{1/r} \cdot (N\theta/c)^{1/r} \cdot \theta^{-(1/r)}$ and hence the result follows.

Next the theorem will be proved.

PROOF OF (a). Lemma 2.7, applied to the sequence $Y_n = (S_n - n\theta)/n^{1/r}$, $n = 1, 2, \cdots$, with $Y \equiv 0$, gives $(S_N - N\theta)/N^{1/r} \to_{\text{a.s.}} 0$, as $c \to \infty$. From $N/c \to_{\text{a.s.}} \theta^{-1}$, as $c \to \infty$ it also follows that

$$\frac{S_N - N\theta}{c^{1/r}} \to_{\text{a.s.}} 0\,, \qquad \text{as } c \to \infty\,.$$

Since $c < S_N \leqq c + X_N$ it follows that

$$\frac{c - N\theta}{c^{1/r}} < \frac{S_N - N\theta}{c^{1/r}} \leqq \frac{c - N\theta + X_N}{c^{1/r}} .$$

By Lemma 2.8 $(c - N\theta)/c^{1/r} \rightarrow_{\text{a.s.}} 0$, as $c \rightarrow \infty$, from which the result follows.

PROOF OF (b). If $P(X \geqq 0) = 1$ the result is already proved by Täcklind [37] page 9. If $P(X < 0) > 0$ the ladder indices $N_1, N_1 + N_2, \cdots$ and ladder heights $Y_1, Y_1 + Y_2, \cdots$ and $M = \min\{n; \sum_{\nu=1}^{n} Y_\nu > c\}$ are defined as in Theorems 2.6 and 2.7. From Theorem 2.1 c it follows that

$$EY_1^r = ES_{N_1}^r < \infty ,$$

and thus, according to Täcklind's theorem,

$$EM = \frac{c}{EY_1} + o(c^{2-r}) \qquad \text{as } c \rightarrow \infty ,$$

where $1 \leqq r < 2$. Since $EN = EM \cdot EN_1$, and $EY_1 = \theta \cdot EN_1$ it follows that

$$EN = EN_1 \cdot \left(\frac{c}{\theta \cdot EN_1} + o(c^{2-r})\right) \qquad \text{as } c \rightarrow \infty$$

and so (b) is proved.

PROOF OF (c).

$$\frac{ES_N - c}{c^{2-r}} = \theta \cdot \frac{EN - c/\theta}{c^{2-r}} \rightarrow 0 \qquad \text{as } c \rightarrow \infty$$

according to Wald's lemma and (b).

If the sequence $X_1, X_2, \cdots$ belongs to the domain of attraction of a stable law with exponent β, then X possesses absolute moments of every order $< \beta$. (See Feller [14] Lemma 2, page 545, or Gnedenko and Kolmogorov [15] Theorem 3, page 179).

THEOREM 2.9. *Let $B_1, B_2, \cdots$ be positive normalizing coefficients such that*

$$P\left(\frac{S_n - n\theta}{B_n} \leqq x\right) \rightarrow G_\beta(x) , \qquad \textit{as } n \rightarrow \infty ,$$

where $G_\beta(x)$ is the distribution function of a stable law with exponent β and $1 < \beta \leqq 2$. Then

$$P\left(\frac{N - c/\theta}{B(c/\theta)/\theta} \geqq -x\right) \rightarrow G_\beta(x) , \qquad \textit{as } c \rightarrow \infty ,$$

where $B(y)$ is such that $B(n) = B_n$, $n = 1, 2, 3, \cdots$ and $B(y) \sim B([y])$ elsewhere. Moreover,

(a) *If $\beta = 2$ then $U(x) = \int_{-x}^{x} u^2 \, dF_X(u)$ varies slowly at infinity and*

$$\frac{n \cdot U(B_n)}{B_n^2} \rightarrow k_1 , \qquad \textit{as } n \rightarrow \infty ,$$

where $k_1 > 0$ is a constant;

(b) *If* $1 < \beta < 2$ *then*

$$P(|X-\theta| > x) \sim \frac{L(x)}{x^\beta}\,, \qquad \text{as } x \to \infty\,,$$

where $L(x)$ *varies slowly at infinity and*

$$\frac{n \cdot L(B_n)}{B_n{}^\beta} \to \frac{2-\beta}{\beta} \cdot k_2\,, \qquad \text{as } n \to \infty\,,$$

where $k_2 > 0$ *is a constant.*

The theorem is stated and proved in Heyde [22] Theorem 2, where $B(c)$ corresponds to $B(c/\theta)$ above. It is possible, however, to prove it using the same technique as in the proof of Theorem 2.5. To do so, a result corresponding to Lemma 2.6 has to be proved.

LEMMA 2.9. *If the assumptions of Theorem* 2.9 *are satisfied, then*

(a) $B_N/B(c/\theta) \to_P 1$, *as* $c \to \infty$;
(b) $X_N/B(c/\theta) \to_P 0$, *as* $c \to \infty$.

PROOF OF (a). Since $N/c \to_{\text{a.s.}} \theta^{-1}$, as $c \to \infty$, there is for every $\varepsilon > 0$ a c_0, such that if $c > c_0$, then $P(A) = P(|N - c/\theta| > (c/\theta)\varepsilon) < \varepsilon$. Then

$$\begin{aligned} P\left(\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right) &= P\left(\left\{\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right\} \cap A\right) \\ &\quad + P\left(\left\{\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right\} \cap A^c\right) \\ &\leqq P(A) + P\left(\left\{\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right\} \cap A^c\right). \end{aligned}$$

Now

$$\frac{B((c/\theta)(1+\varepsilon))}{B(c/\theta)} \to (1+\varepsilon)^\beta \qquad \text{as } c \to \infty$$

and

$$\frac{B((c/\theta)(1-\varepsilon))}{B(c/\theta)} \to (1-\varepsilon)^\beta\,, \qquad \text{as } c \to \infty$$

(see e.g. Feller [14] page 305). If $\delta > 0$ is given, it is possible to choose $\varepsilon > 0$ so small that $\max\{(1+\varepsilon)^\beta - 1, 1 - (1-\varepsilon)^\beta\} = (1+\varepsilon)^\beta - 1 < \delta$. Then

$$P\left(\left\{\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right\} \cap A^c\right) = 0 \qquad \text{if } c > c_1\,.$$

Thus

$$P\left(\left|\frac{B_N}{B(c/\theta)} - 1\right| > \delta\right) \leqq P(A) < \varepsilon \qquad \text{if } c > \max\{c_0, c_1\}\,,$$

which proves (a).

PROOF OF (b). As mentioned before, $E|X|^s < \infty$ if $s < \beta$. Since $1 < \beta \leqq 2$, s may be chosen such that $1 < s < 2$. According to Theorem 2.8 a $(N - c/\theta)/c^{1/s} \to_{\text{a.s.}}$

0, as $c \to \infty$, and hence also $(N - c/\theta)/(c/\theta)^{1/s} \to_{\text{a.s.}} 0$, as $c \to \infty$. Given $\varepsilon > 0$, c_0 is chosen such that if $c > c_0$, then $P(A) = P(|N - c/\theta| > (c/\theta)^{1/s} \cdot \varepsilon) < \varepsilon$. Define $n_1 = [c/\theta - (c/\theta)^{1/s} \cdot \varepsilon]$ and $n_2 = [c/\theta + (c/\theta)^{1/s} \cdot \varepsilon]$.

$$\begin{aligned} P\left(\frac{|X_N|}{B(c/\theta)} > \delta\right) &= P\left(\left\{\frac{|X_N|}{B(c/\theta)} > \delta\right\} \cap A\right) + P\left(\left\{\frac{|X_N|}{B(c/\theta)} > \delta\right\} \cap A^c\right) \\ &\leqq P(A) + P\left(\left\{\frac{|X_N|}{B(c/\theta)} > \delta\right\} \cap A^c\right) \\ &\leqq \varepsilon + P\left(\left\{\max_{n_1 < n \leqq n_2} |X_n| > \delta \cdot B\left(\frac{c}{\theta}\right)\right\} \cap A^c\right) \\ &\leqq \varepsilon + P\left(\max_{n_1 < n \leqq n_2} |X_n| > \delta \cdot B\left(\frac{c}{\theta}\right)\right) \\ &\leqq \varepsilon + \sum_{n=n_1+1}^{n_2} P\left(|X_n| > \delta \cdot B\left(\frac{c}{\theta}\right)\right) \\ &= \varepsilon + (n_2 - n_1) \cdot P\left(|X| > \delta \cdot B\left(\frac{c}{\theta}\right)\right). \end{aligned}$$

First assume that $\beta = 2$. Then, according to Feller [14] page 303, $[x^2 \cdot P(|X| > x)]/U(x) \to 0$ as $x \to \infty$. Since $n_2 - n_1 < 4 \cdot (c/\theta)^{1/s} \cdot \varepsilon$ it follows from (a) of the theorem, that

$$\begin{aligned} P\left(\frac{|X_N|}{B(c/\theta)} > \delta\right) &\leqq \varepsilon + 4 \cdot \left(\frac{c}{\theta}\right)^{1/s} \cdot \varepsilon \cdot P\left(|X| > \delta \cdot B\left(\frac{c}{\theta}\right)\right) \\ &= \varepsilon + \frac{4\varepsilon\theta}{\delta^2\theta^{1/s}} \cdot \frac{1}{c^{1-1/s}} \cdot \frac{c/\theta \cdot U(B(c/\theta))}{(B(c/\theta))^2} \cdot \frac{U(B(c/\theta) \cdot \delta)}{U(B(c/\theta))} \\ &\quad \times \frac{(B(c/\theta) \cdot \delta)^2 \cdot P(|X| > \delta \cdot B(c/\theta))}{U(B(c/\theta) \cdot \delta)} \\ &\to \varepsilon + \frac{4\varepsilon\theta}{\delta^2\theta^{1/s}} \cdot 0 \cdot k_1 \cdot 1 \cdot 0 = \varepsilon \qquad \text{as } c \to \infty. \end{aligned}$$

Since ε is arbitrary the conclusion follows.

Now suppose that $1 < \beta < 2$. Then $P(|X| > \delta \cdot B(c/\theta)) \leqq P(|X - \theta| > \delta/2 \cdot B(c/\theta))$ if $c > c_1$ and hence if $c > \max\{c_0, c_1\}$ it follows from (b) of the theorem that

$$\begin{aligned} P\left(\frac{|X_N|}{B(c/\theta)} > \delta\right) &\leqq \varepsilon + 4\left(\frac{c}{\theta}\right)^{1/s} \cdot \varepsilon \cdot P\left(|X - \theta| > \frac{\delta}{2} \cdot B\left(\frac{c}{\theta}\right)\right) \\ &\sim \varepsilon + \frac{4\varepsilon\theta}{(\delta/2)^\beta \cdot \theta^{1/s}} \cdot \frac{1}{c^{1-1/s}} \cdot \frac{c/\theta \cdot L(B(c/\theta))}{(B(c/\theta))^\beta} \cdot \frac{L(B(c/\theta) \cdot \delta/2)}{L(B(c/\theta))} \\ &\to \varepsilon + \frac{4\varepsilon\theta}{(\delta/2)^\beta \cdot \theta^{1/s}} \cdot 0 \cdot \frac{2-\beta}{\beta} k_2 \cdot 1 = \varepsilon, \qquad \text{as } c \to \infty, \end{aligned}$$

which completes the proof.

PROOF OF THEOREM 2.9. Since $\lim_{\delta \to 0} \lim_{n \to \infty} B_n/B_{[n(1+\delta)]} = \lim_{\delta \to 0} (1 + \delta)^{-(1/\beta)} = 1$, Theorem 3 page 77–78, of Richter [28] applies. Let $Y_n = (S_n - n\theta)/B_n$, $n =$

1, 2, $\cdots$, and let Y have a stable distribution with exponent β. Then $P(Y_N \leqq x) \to P(Y \leqq x)$ as $c \to \infty$, that is $P((S_N - N\theta)/B_N \leqq x) \to G_\beta(x)$, as $c \to \infty$. (Another way to obtain this would be to prove a generalization of Lemma 2.5.)

From $(c - N\theta)/B(c/\theta) < (S_N - N\theta)/B(c/\theta) \leqq (c - N\theta + X_N)/B(c/\theta)$, Lemma 2.9 and Cramér's theorem ([9] page 254) it follows that $P((N\theta - c)/B(c/\theta) \geqq -x) \to G_\beta(x)$, as $c \to \infty$, from which the conclusion follows.

If the sequence $X_1, X_2, \cdots$ belongs to the domain of normal attraction of a stable law with exponent β, $1 < \beta \leqq 2$, then $B_n = A \cdot n^{1/\beta}$, where A is some constant, and the theorem then states that $P((N - c/\theta)/(A/\theta) \cdot (c/\theta)^{1/\beta} \geqq -x) \to G_\beta(x)$, as $c \to \infty$, where thus $B(y) = A \cdot y^{1/\beta}$, $y > 0$. If $\beta = 2$, then $\operatorname{Var} X = \sigma^2 < \infty$ and $A = \sigma$, (see Gnedenko and Kolmogorov [15], Theorem 4 page 181), and the statement reduces to Theorem 2.5.

3. First passage times across general continuous barriers.

3.1. In this section Case II is considered. Let $X_1, X_2, \cdots$ be i.i.d. random variables and let $0 < EX = \theta < \infty$. Define

$$N = N(c) = \min\{n; S_n > c \cdot a(n)\},$$

where c is a nonnegative constant, and $a(y)$, $y \in [0, \infty)$, is a positive, continuous function such that $a(y)/y \to 0$ as $y \to \infty$.

THEOREM 3.1. *Let $r \geqq 1$. Then*

(a) $E|X^-|^r < \infty \Rightarrow EN^r < \infty$

(b) $EN^r < \infty \Rightarrow E|X^-|^r < \infty$

(c) $E|X|^r < \infty \Rightarrow ES_N{}^r < \infty$.

PROOF OF (a). Assume that $E|X^-|^r < \infty$ and $c > 0$. By the assumptions there is for every $\delta > 0$ an a_0, such that $c \cdot a(y) \leqq a_0 + \delta y$ for all $y \geqq 0$. Choose $\delta < \theta$, and introduce the stopping time $N_* = \min\{n; S_n > a_0 + \delta n\}$. Then $N \leqq N_*$ and hence it suffices to prove that $E|X^-|^r < \infty \Rightarrow EN_*{}^r < \infty$. But

$$N_* = \min\{n; S_n > a_0 + \delta n\} = \min\{n; S_n - \delta n > a_0\} = \min\{n; \textstyle\sum_1^n Y_\nu > a_0\},$$

where $Y_\nu = X_\nu - \delta$, $\nu = 1, 2, \cdots$. Since $EY_\nu = \theta - \delta > 0$ it follows from Theorem 2.1 a that $E|Y_\nu{}^-|^r < \infty \Rightarrow EN_*{}^r < \infty$, and since $E|X^-|^r < \infty \Leftrightarrow E|Y^-|^r < \infty$, the result follows.

PROOF OF (b). Define $N_1 = \min\{n; S_n > 0\}$, i.e. N_1 is the first ladder index. Then $N_1 \leqq N$ and thus $EN^r < \infty \Rightarrow EN_1{}^r < \infty$ and according to Theorem 2.1 b $EN_1{}^r < \infty \Rightarrow E|X^-|^r < \infty$.

PROOF OF (c). If $r = 1$ Wald's lemma and (a) imply that $ES_N = \theta EN < \infty$. Let $r > 1$. From (a) it is known that $c \cdot a(y) \leqq a_0 + \delta y$ and thus $S_N \leqq c \cdot a(N-1) + X_N \leqq a_0 + \delta(N-1) + X_N \leqq a_0 + \delta N + X_N$. Therefore it follows from Minkowski's inequality, Lemma 2.1, and (a), that

$$\begin{aligned}(ES_N{}^r)^{1/r} &\leqq a_0 + \delta(EN^r)^{1/r} + (E|X_N|^r)^{1/r} \\ &\leqq a_0 + \delta(EN^r)^{1/r} + (EN \cdot E|X|^r)^{1/r} < \infty,\end{aligned}$$

if $E|X|^r < \infty$.

REMARK. Since the class of admissible barriers is rather large, there does not exist a general converse to (c). There are, however, several cases where more precise results can be given. One example is Case I, (see Theorem 2.1 c). If $a(y)$ is assumed to be non-decreasing, then $S_N \geqq X_1^+$ and thus $E(X^+)^r < \infty$ is necessary. Finally, let $a(y) = y^\alpha$, $y \geqq 0$, $0 < \alpha < 1$, and let r be such that $\alpha r > 1$. By the usual arguments it follows that

$$c \cdot N^\alpha < S_N \leqq c \cdot N^\alpha + X_N^+ , \qquad \text{and}$$
$$c^r \cdot EN^{\alpha r} < ES_N{}^r \leqq 2^{r-1} \cdot c^r \cdot EN^{\alpha r} + 2^{r-1} \cdot EN \cdot E(X^+)^r$$

Therefore, $ES_N{}^r < \infty$ if and only if $E(X^+)^r < \infty$ and $E|X^-|^{\alpha r} < \infty$, since $a(y)$ is non-decreasing and because of (a) and (b).

THEOREM 3.2. *There exists a* $t_0 > 0$ *such that* $Ee^{tN} < \infty$, $|t| < t_0$, *if and only if there exists a* $t_1 > 0$ *such that* $Ee^{tX^-} < \infty$, $|t| < t_1$.

PROOF. Define N_1 and N_* as before, i.e. $N_1 = \min\{n; S_n > 0\}$ and $N_* = \min\{n; S_n > a_0 + \delta n\} = \min\{n; \sum_{\nu=1}^n (X - \delta) > a_0\}$. Then $N_1 \leqq N \leqq N_*$. Since N_1 and N_* are stopping times of Case I-type, and since $Ee^{tX^-} < \infty \Leftrightarrow Ee^{t(X-\delta)^-} < \infty$, the theorem follows from Theorem 2.2.

3.2. In the remaining part of this chapter Case III will be studied. Let X_1, $X_2, \cdots$ be i.i.d. random variables such that $0 < EX = \theta < \infty$, and define $N = N(c) = \min\{n; S_n > c \cdot a(n)\}$, where $c \geqq 0$ is a constant and $a(y)$, $y \in [0, \infty)$, is a positive continuous function such that $a(y)/y \to 0$ as $y \to \infty$. Furthermore, assume that $a(y)$, for sufficiently large values of y, is a non-decreasing, concave, differentiable function that varies regularly at infinity with exponent α, $0 \leqq \alpha < 1$, i.e. $a(y) = y^\alpha \cdot L(y)$, where $L(y)$ varies slowly at infinity. It is sufficient to assume that $a(y)$ has these properties for large y, since $N \to_{\text{a.s.}} +\infty$ as $c \to \infty$.

Some examples of functions $a(y)$ of the above kind are

$$a(y) = y^\alpha \cdot \log y , \qquad 0 \leqq \alpha < 1$$

and

$$a(y) = y^\alpha \cdot \arctan y , \qquad 0 \leqq \alpha < 1 .$$

Any function $a(y)$ that has a positive finite limit when $y \to \infty$ is slowly varying (see Feller [14] page 269).

Earlier work has mainly dealt with the case when $L(y) \equiv 1$. (See e.g. Chow [8] and Siegmund [31].) If $\alpha = 0$ and $L(y) \equiv 1$ Case I is obtained.

Let $\lambda = \lambda(c)$ denote the solution of the equation $c \cdot a(y) = \theta y$. If c is sufficiently large, λ is unique. If $a(y) \to 1$ as $y \to \infty$ then

$$c/\theta\lambda = 1/a(\lambda) \to 1 \qquad \text{as } c \to \infty ,$$

since $c \to \infty$ implies that $\lambda \to \infty$. If $\alpha = 0$ and $a(y)$ has a finite limit when $y \to \infty$ it is no restriction to assume that the limit is equal to 1, because a limit other than 1 may be absorbed in the constant c.

The following theorem corresponds to Theorem 2.3 (a)—(c).

THEOREM 3.3. *Let* $r \geqq 1$ *and* $0 \leqq \alpha < 1$. *Then*

(a) $N/\lambda \to_{\text{a.s.}} 1$, *as* $c \to \infty$
(b) $E|X^-|^r < \infty \Leftrightarrow E(N/\lambda)^r \to 1$, *as* $c \to \infty$
(c) $E|X^-|^r < \infty \Rightarrow E(N/\lambda)^s \to 1$, *as* $c \to \infty$ *for all* s, $0 \leqq s \leqq r$.

PROOF OF (a). For $0 < \alpha < 1$ a proof is found in Siegmund [30] Lemma 4, page 1643. For $\alpha = 0$ the same method gives that $(a(N)/a(\lambda)) \cdot \lambda/N \to_{\text{a.s.}} 1$, as $c \to \infty$, from which (a) follows from Sreehari [35] Lemma 2.3, page 259, and from the fact that $N \to_{\text{a.s.}} \infty$, as $c \to \infty$.

When $r = 1$ and $0 < \alpha < 1$ Siegmund [30] has proved that $EN/\lambda \to 1$ as $c \to \infty$, and there $X_1, X_2, \cdots$ are not assumed to be identically distributed.

To prove (b), introduce the auxiliary stopping time N^* of Siegmund [30] page 1643. Define

$$N^* = N^*(c) = \min\{n; S_n > \theta\alpha(\lambda)n + \theta\lambda(1 - \alpha(\lambda))\}$$
$$= \min\left\{n; \sum_{\nu=1}^n \frac{X_\nu - \theta\alpha(\lambda)}{\theta(1 - \alpha(\lambda))} > \lambda\right\},$$

where $\alpha(\lambda)$ is a function which tends to α as $\lambda \to \infty$. The first part of the following lemma shows that the choice $\alpha(\lambda) = (\lambda \cdot a'(\lambda))/a(\lambda)$ is permitted.

LEMMA 3.1. *Let* $0 \leqq \alpha < 1$. *Define* $\alpha(\lambda t) = (\lambda \cdot a'(\lambda t))/a(\lambda)$. *Then*

(a) $\alpha(\lambda t) \to \alpha \cdot t^{\alpha-1}$, *as* $\lambda \to \infty$.
(b) *If* N^* *is as above with* $\alpha(\lambda) = (\lambda \cdot a'(\lambda))/a(\lambda)$, *then*

$$E|X^-|^r < \infty \Leftrightarrow E\left(\frac{N^*}{\lambda}\right)^r \to 1, \qquad as\ \lambda \to \infty.$$

where $r \geqq 1$.

PROOF OF (a). Let $0 < \alpha < 1$. With $u(t) = a'(t)$, $U(t) = a(t)$ and $s = \alpha$ the result is proved in the lemma on page 422 of Feller [4]. Now let $\alpha = 0$. If $0 < t_1 < t_2$, then

$$\frac{a(\lambda t_2) - a(\lambda t_1)}{a(\lambda)} = \int_{t_1}^{t_2} \frac{\lambda \cdot a'(\lambda y)}{a(\lambda)}\, dy \geqq \frac{\lambda \cdot a'(\lambda t_2)}{a(\lambda)} = \alpha(\lambda t_2) \geqq 0,$$

since $a'(y)$ is non-increasing and nonnegative. But

$$\frac{a(\lambda t_2) - a(\lambda t_1)}{a(\lambda)} \to 1 - 1 = 0 \qquad \text{as } \lambda \to \infty$$

because of the slow variation of $a(y)$. Hence

$$\alpha(\lambda t) \to 0 \qquad \text{as } \lambda \to \infty.$$

With $t = 1$ (a) states that $\alpha(\lambda) \to \alpha$ as $\lambda \to \infty$, $0 \leqq \alpha < 1$, and thus N^* is a stopping time of the above kind.

PROOF OF (b). Choose λ_0 so large that $0 \leqq \alpha(\lambda) < 1$ if $\lambda > \lambda_0$. If $E(N^*/\lambda)^r \to 1$ as $\lambda \to \infty$ then $E(N^*)^r$ is finite if λ is finite and, since $E((X_\nu - \theta\alpha(\lambda))/\theta(1 -$

$\alpha(\lambda))) = 1$, it follows from Theorem 2.1 b that $E|((X_\nu - \theta\alpha(\lambda))/\theta(1 - \alpha(\lambda)))^-|^r < \infty$, which in turn is equivalent to $E|X^-|^r < \infty$, and so the necessity is proved.

Now assume that $E|X^-|^r < \infty$. Choose ε, such that $0 < \varepsilon < 1 - \alpha$ and then λ_1, such that $\lambda > \lambda_1$ implies that $|\alpha(\lambda) - \alpha| < \varepsilon$. (When $\alpha = 0$, λ_1 is chosen such that $\lambda > \lambda_1$ implies that $0 \leqq \alpha(\lambda) < \varepsilon$). Set $\lambda_2 = \max\{\lambda_0, \lambda_1\}$. Define

$$\begin{aligned} N^{**} &= \min\{n; S_n > \theta(\alpha + \varepsilon)n + \lambda\theta(1 - \alpha + \varepsilon)\} \\ &= \min\left\{n; \sum_{\nu=1}^n \frac{X_\nu - \theta(\alpha + \varepsilon)}{\theta(1 - \alpha + \varepsilon)} > \lambda\right\} \end{aligned}$$

and

$$\begin{aligned} N_{**} &= \min\{n; S_n > \theta(\alpha - \varepsilon)n + \lambda\theta(1 - \alpha - \varepsilon)\} \\ &= \min\left\{n; \sum_{\nu=1}^n \frac{X_\nu - \theta(\alpha - \varepsilon)}{\theta(1 - \alpha - \varepsilon)} > \lambda\right\} \end{aligned}$$

and let $\lambda > \lambda_2$. Then $N_{**} \leqq N^* \leqq {}^{**}$, and according to Theorem 2.3 b it follows that

$$E\left(\frac{N^{**}}{\lambda}\right)^r \to \left(\frac{1 - \alpha + \varepsilon}{1 - \alpha - \varepsilon}\right)^r \qquad \text{as } \lambda \to \infty ,$$

and

$$E\left(\frac{N_{**}}{\lambda}\right)^r \to \left(\frac{1 - \alpha - \varepsilon}{1 - \alpha + \varepsilon}\right)^r \qquad \text{as } \lambda \to \infty .$$

Hence

$$\left(\frac{1 - \alpha - \varepsilon}{1 - \alpha + \varepsilon}\right)^r \leqq \liminf_{\lambda\to\infty} E\left(\frac{N^*}{\lambda}\right)^r \leqq \limsup_{\lambda\to\infty} E\left(\frac{N^*}{\lambda}\right)^r \leqq \left(\frac{1 - \alpha + \varepsilon}{1 - \alpha - \varepsilon}\right)^r .$$

Since ε is arbitrary (b) follows.

Proof of Theorem 3.3 (b). Let $r \geqq 1$ and assume that $E|X^-|^r < \infty$. If $\alpha(\lambda) = \lambda \cdot a'(\lambda)/a(\lambda)$, Lemma 3.1 b is true, if $\lambda \to \infty$ is replaced by $c \to \infty$, since $c \to \infty$ implies that $\lambda \to \infty$.

Furthermore, $\theta\alpha(\lambda)y + \lambda\theta(1 - \alpha(\lambda))$ is a line support to $c \cdot a(y)$ $(= \theta\lambda a(y)/a(\lambda))$ at the point $(\lambda, \lambda\theta)$. Because of the concavity of $a(y)$ it follows that $N \leqq N^*$. Therefore $\limsup_{c\to\infty} E(N/\lambda)^r \leqq \limsup_{c\to\infty} E(N^*/\lambda)^r = 1$, and from Theorem 3.3 a and Fatou's lemma it follows that $\liminf_{c\to\infty} E(N/\lambda)^r \geqq 1$, which proves the sufficiency. The necessity follows as in Theorem 2.3 b.

Proof of (c). (c) follows from (a), (b) and Corollary 1, page 164, of Loève [23].

Remark. If $\alpha = 0$ and $a(y) \to 1$ then $\lambda/c/\theta = a(\lambda) \to 1$ as $c \to \infty$, and thus Theorem 3.3 is true with λ replaced by c/θ. For example, if $a(y) = 2/\pi \arctan y$ this is the case.

Since $S_N - c \cdot a(N)$ is the excess over the boundary, (a) and (b) of the following theorem correspond to Theorem 2.4 a and b.

Theorem 3.4. *Let $r \geqq 1$ and $0 \leqq \alpha < 1$. If $E(X^+)^r < \infty$, then*

(a) $[E(S_N - c \cdot a(N))^r]/\lambda \to 0$ *as* $c \to \infty$,
(b) $[E(S_N - c \cdot a(N))]/\lambda^{1/r} \to 0$ *as* $c \to \infty$,
(c) $[E(N - c/\theta \cdot a(N))]/\lambda^{1/r} \to 0$ *as* $c \to \infty$,
(d) $\limsup_{c\to\infty} (EN - \lambda)/\lambda^{1/r} \leqq 0$.

The proof is similar to the proof of Theorem 2.4.

LEMMA 3.2. *Let* $r \geqq 1$. *If* $E(X^+)^r < \infty$, *then*

(a) $EX_N{}^r/\lambda \to 0$, *as* $c \to \infty$,
(b) $EX_N/\lambda^{1/r} \to 0$, *as* $c \to \infty$.

PROOF. Exactly as in the proof of Lemma 2.4 a it follows that $\lim_{c\to\infty} EX_N{}^r/EN = 0$. Since $EN/\lambda \to 1$ as $c \to \infty$, (a) follows, and then (b) is immediate.

PROOF OF THEOREM 3.4(a)—(c). $0 \leqq [E(S_N - c \cdot a(N))^r]/\lambda \leqq EX_N{}^r/\lambda \to 0$ as $c \to \infty$ by Lemma 3.2a. Thus (a) is proved. (b) is then immediate, and (c) follows from (b) and Wald's lemma.

PROOF OF (d). Set again

$$N^* = \min\{n; S_n > \theta\alpha(\lambda)n + \lambda\theta(1 - \alpha(\lambda))\} = \min\{n; \textstyle\sum_{\nu=1}^{n} X_\nu' > \lambda\},$$

where $X_\nu' = [X_\nu - \theta\alpha(\lambda)]/\theta(1 - \alpha(\lambda))$. Choose c_0 (and λ_0) such that $c > c_0$ implies $0 \leqq \alpha(\lambda) = \lambda a'(\lambda)/a(\lambda) < 1$. As in the proof of Lemma 2.4a it is found that

$$0 \leqq \frac{E(X'_{N^*})^r}{EN^*} \leqq 2\varepsilon + \varepsilon \cdot \frac{n_0 \cdot E((X')^+)^r}{EN^*} \leqq 2\varepsilon + \varepsilon \cdot \frac{n_0 2^{r-1} \cdot (E(X^+)^r + \theta^r)}{\theta^r \cdot (1 - \alpha(\lambda))^r EN^*},$$

where the last inequality follows from the c_r-inequalities. Since $EN^*/\lambda \to 1$ as $c \to \infty$, it follows that $0 \leqq \limsup_{c\to\infty} E(X'_{N^*})^r/\lambda \leqq 2\varepsilon$. Thus $\lim_{c\to\infty} E(X'_{N^*})^r/\lambda = 0$. Since $\lambda < S'_{N^*} = \sum_{\nu=1}^{N^*} X_\nu' \leqq \lambda + X'_{N^*}$, it follows that

$$0 \leqq \frac{E(S'_{N^*} - \lambda)^r}{\lambda} \leqq \frac{E(X'_{N^*})^r}{\lambda} \to 0 \qquad \text{as } c \to \infty.$$

Thus also $\lim_{c\to\infty} (ES'_{N^*} - \lambda)/\lambda^{1/r} = 0$, and since $EX_\nu' = 1$ Wald's lemma implies that $\lim_{c\to\infty} (EN^* - \lambda)/\lambda^{1/r} = 0$. As before $N \leqq N^*$ and hence $\limsup_{c\to\infty} (EN - \lambda)/\lambda^{1/r} \leqq 0$, which proves (d).

REMARK. Although n_0, as chosen in (d), depends on λ it is easily seen that the same n_0 may be chosen for all $\lambda >$ some λ_1.

The asymptotic normality of $N = N(c)$ will now be investigated.

THEOREM 3.5. *Let* $0 \leqq \alpha < 1$ *and assume that* $0 < EX = \theta < \infty$ *and* $\operatorname{Var} X = \sigma^2 < \infty$. *Then*

$$\mathscr{L}\left(\frac{N - \lambda}{\sigma\lambda^{\frac{1}{2}}/\theta(1 - \alpha)}\right) \Rightarrow N(0, 1), \qquad \textit{as } c \to \infty.$$

The case when $a(y) = y^\alpha$, $0 \leqq \alpha < 1$, has been studied by Siegmund [31], where the corresponding result was proved differently.

PROOF. $(c \cdot a(N) - N\theta)/\sigma N^{\frac{1}{2}} < (S_N - N\theta)/\sigma N^{\frac{1}{2}} \leqq (c \cdot a(N) - N\theta + X_N)/\sigma N^{\frac{1}{2}}$. With Lemmas 2.5 and 2.6 applied as in the proof of Theorem 2.5, it is found that $\mathscr{L}(Z_N) \Rightarrow N(0, 1)$ as $c \to \infty$, where $Z_N = (N\theta - c \cdot a(N))/\sigma\lambda^{\frac{1}{2}}$. An expansion of $a(y)$ at the point λ leads to

$$Z_N = \frac{N\theta - c(a(\lambda) + (N-\lambda) \cdot a'(\lambda + \rho(N-\lambda)))}{\sigma\lambda^{\frac{1}{2}}}$$

$$= \frac{N\theta - \lambda\theta - \lambda\theta/a(\lambda) \cdot (N-\lambda) \cdot a'(\lambda + \rho(N-\lambda))}{\sigma\lambda^{\frac{1}{2}}} = \frac{N-\lambda}{\sigma\lambda^{\frac{1}{2}}/\theta(1-\alpha)} \cdot Y_N ,$$

where

$$Y_N = \frac{1 - [\lambda \cdot a'(\lambda + \rho(N-\lambda))]/a(\lambda)}{1-\alpha}, \qquad \text{and } 0 \leqq \rho = \rho(N, c) \leqq 1 .$$

If it were known that $Y_N \to_P 1$ as $c \to \infty$, then, by Cramér's theorem (see [9] page 254), Z_N and $[(N-\lambda)/\sigma\lambda^{\frac{1}{2}}]/\theta(1-\alpha)$ would have the same limit distribution. Thus, with the following lemma the proof of Theorem 3.5 is complete.

LEMMA 3.3. *Let* $0 \leqq \alpha < 1$. *If* Y_N *is as above, then*

$$Y_N \to_{\text{a.s.}} 1 , \qquad \textit{as } c \to \infty .$$

PROOF. It suffices to prove that $[\lambda a'(\lambda + \rho(N-\lambda))]/a(\lambda) \to_{\text{a.s.}} \alpha$, as $c \to \infty$.

Set $A = \{\omega; N/\lambda \to 1 \text{ as } c \to \infty\}$. According to Theorem 3.3a $P(A) = 1$. Let $\omega \in A$ and choose $\varepsilon > 0$. Then there is a c_0, such that $c > c_0$ implies that $|N - \lambda| < \lambda\varepsilon$, and then $\lambda(1-\varepsilon) \leqq \lambda + \rho(N-\lambda) \leqq \lambda(1+\varepsilon)$. Since $a'(y)$ is non-increasing it follows that

$$\alpha(\lambda(1+\varepsilon)) = \frac{\lambda \cdot a'(\lambda(1+\varepsilon))}{a(\lambda)} \leqq \frac{\lambda a'(\lambda + \rho(N-\lambda))}{a(\lambda)}$$

$$\leqq \frac{\lambda \cdot a'(\lambda(1-\varepsilon))}{a(\lambda)} = \alpha(\lambda(1-\varepsilon))$$

if $c > c_0$. From Lemma 3.1a it then follows that

$$\alpha(1+\varepsilon)^{\alpha-1} \leqq \liminf_{c\to\infty} \frac{\lambda \cdot a'(\lambda + \rho(N-\lambda))}{a(\lambda)}$$

$$\leqq \limsup_{c\to\infty} \frac{\lambda \cdot a'(\lambda + \rho(N-\lambda))}{a(\lambda)} \leqq \alpha(1-\varepsilon)^{\alpha-1} .$$

Since ε was arbitrary $\lim_{c\to\infty} [\lambda \cdot a'(\lambda + \rho(N-\lambda))]/a(\lambda) = \alpha$ for that particular ω, but since $\omega \in A$ was chosen arbitrarily and $P(A) = 1$ it follows that

$$\frac{\lambda \cdot a'(\lambda + \rho(N-\lambda))}{a(\lambda)} \to_{\text{a.s.}} \alpha , \qquad \text{as } c \to \infty .$$

Suppose that $\alpha = 0$ and $a(y) \to 1$ as $y \to \infty$. Then $\lambda/(c/\theta) \to 1$ as $c \to \infty$ and $N - \lambda = N - c/\theta + c/\theta - \lambda = N - c/\theta + [\lambda(1 - a(\lambda))]/a(\lambda)$. Thus, if

$$\frac{\lambda(1 - a(\lambda))}{\sigma\lambda^{\frac{1}{2}}/\theta \cdot a(\lambda)} \to 0 \qquad \text{as } c \to \infty ,$$

then

$$\mathscr{L}\left(\frac{N - c/\theta}{\sigma/\theta(c/\theta)^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty$$

by Cramér's theorem and Theorem 3.5, and thus the following corollary is proved.

COROLLARY 3.5.1. *If $\alpha = 0$ and $a(y) \to 1$ as $y \to \infty$, if $0 < EX = \theta < \infty$, $\operatorname{Var} X = \sigma^2 < \infty$, and if $y^{\frac{1}{2}}(1 - a(y)) \to 0$ as $y \to \infty$, then*

$$\mathscr{L}\left(\frac{N - c/\theta}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\right) \Rightarrow N(0, 1), \qquad \textit{as } c \to \infty .$$

In particular, if $a(y) \equiv 1$ the corollary reduces to Theorem 2.5.

As a special case, let $a(y) = 2/\pi \arctan y^{1/s}$, $s \geqq 0$. Then $\alpha = 0$ and $a(y) \to 1$ as $y \to \infty$ and

$$\begin{aligned} y^{\frac{1}{2}}(1 - a(y)) &= y^{\frac{1}{2}}\left(1 - \frac{2}{\pi}\arctan y^{1/s}\right) \\ &= y^{\frac{1}{2}} \cdot \frac{2}{\pi}\arctan y^{-1/s} \to 0 \qquad \text{if } s < 2 \\ &\to \frac{2}{\pi} \qquad \text{if } s = 2 \\ &\to +\infty \qquad \text{if } s > 2 . \end{aligned}$$

Thus

$$\begin{aligned} \mathscr{L}\left(\frac{N - c/\theta}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\right) &\Rightarrow N(0, 1) \qquad \text{if } s < 2 \\ &\Rightarrow N\left(-\frac{2\theta}{\pi\sigma}, 1\right) \qquad \text{if } s = 2 \\ &\Rightarrow \text{does not exist} \qquad \text{if } s > 2 . \end{aligned}$$

As in Section 2 the convergence in distribution does not imply that $EN \sim \lambda$ or $\operatorname{Var} N \sim \sigma^2\lambda/\theta^2(1 - \alpha)^2$ as $c \to \infty$. By Theorem 3.3c it follows, however, that $EN/\lambda \to 1$ as $c \to \infty$, and the next result gives upper bounds for $EN - \lambda$ as $c \to \infty$, but it remains to find the complete solutions of these problems.

THEOREM 3.6. *Let $0 \leqq \alpha < 1$ and suppose that $L(y) \equiv 1$. If $0 < EX = \theta < \infty$ and $E(X^+)^2 < \infty$, then*

$$\limsup_{c\to\infty} EN - \lambda \leqq C(\alpha) ,$$

where $C(\alpha)$ is defined as $E((X - \theta\alpha)^+)^2/2\theta^2(1 - \alpha)^2$ in the non-lattice case and as $E((X - \theta\alpha)^+)^2/2\theta^2(1 - \alpha)^2 + 1/2\theta(1 - \alpha)$ in the lattice case.

If also $\operatorname{Var} X < \infty$, the above inequality is given on page 1645 of Siegmund [30] in the non-lattice case. If $a(y) \equiv 1$ the theorem reduces to the upper bounds of Corollaries 2.6.1 and 2.7.1. Furthermore, the theorem gives a sharper upper bound than Theorem 3.4d does, when $r \geqq 2$.

PROOF. Since $L(y) \equiv 1$ it follows that $\alpha(\lambda) = \lambda \cdot a'(\lambda)/a(\lambda) \equiv \alpha$. Hence

$$N^* = \min\{n;\, S_n > \theta\alpha n + \lambda\theta(1-\alpha)\} = \min\left\{n;\, \sum_{\nu=1}^{n} \frac{X_\nu - \theta\alpha}{\theta(1-\alpha)} > \lambda\right\}.$$

From $N \leqq N^*$ it follows that $EN - \lambda \leqq EN^* - \lambda$, from which the conclusion follows by Corollaries 2.6.1 and 2.7.1.

3.3. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables such that $0 < EX = \theta < \infty$. In the next theorem it is assumed that $E|X|^r < \infty$, where $1 \leqq r < 2$, and in the last theorem it is assumed that the sequence $X_1, X_2, \cdots$ belongs to the domain of attraction of a stable law with exponent β, $1 < \beta \leqq 2$.

THEOREM 3.7. *Let* $0 \leqq \alpha < 1$ *and* $1 \leqq r < 2$. *If* $E|X|^r < \infty$, *then*
(a) $(N - \lambda)/\lambda^{1/r} \to_{\text{a.s.}} 0$, *as* $c \to \infty$;
if $E(X^+)^r < \infty$ *and if also* $L(y) \equiv 1$, *then*
(b) $\limsup_{c\to\infty} (EN - \lambda)/\lambda^{2-r} \leqq 0$.

Note that, because of Theorem 3.5, (a) is not true when $r = 2$. If $r = 1$, (a) reduces to Theorem 3.3 a and (b) is contained in Theorem 3.4 d. Furthermore, (b) is a sharper result than Theorem 3.4 d (if $L(y) \equiv 1$).
For the proof of (a) the following lemma is used.

LEMMA 3.4. *Let* $1 \leqq r < 2$. *If* $E|X^r| < \infty$, *then*

$$\frac{X_N}{\lambda^{1/r}} \to_{\text{a.s.}} 0, \qquad \textit{as } c \to \infty.$$

PROOF. The proof that $X_N/N^{1/r} \to_{\text{a.s.}} 0$, as $c \to \infty$ is the same as in Lemma 2.8. Since $N/\lambda \to_{\text{a.s.}} 1$, as $c \to \infty$, the result follows.

PROOF OF THEOREM 3.7 (a). Obviously

$$\frac{c \cdot a(N) - N\theta}{\lambda^{1/r}} < \frac{S_N - N\theta}{\lambda^{1/r}} \leqq \frac{c \cdot a(N) - N\theta + X_N}{\lambda^{1/r}}.$$

As in the proof of Theorem 2.8 a it follows that

$$\frac{S_N - N\theta}{N^{1/r}} \to_{\text{a.s.}} 0, \qquad \text{as } c \to \infty,$$

and since $N/\lambda \to_{\text{a.s.}} 1$, as $c \to \infty$ also

$$\frac{S_N - N\theta}{\lambda^{1/r}} \to_{\text{a.s.}} 0, \qquad \text{as } c \to \infty.$$

By Lemma 3.4 it follows that $Z_N \to_{\text{a.s.}} 0$, as $c \to \infty$, where $Z_N = (N\theta - c \cdot a(N))/\lambda^{1/r}$. An expansion of $a(y)$ as in the proof of Theorem 3.5 yields

$$Z_N = \frac{N - \lambda}{\lambda^{1/r}/\theta(1-\alpha)} \cdot Y_N,$$

where Y_N is as before. According to Lemma 3.3 $Y_N \to_{\text{a.s.}} 1$, as $c \to \infty$, and thus $(N - \lambda)/\lambda^{1/r} \cdot \theta(1 - \alpha) \to_{\text{a.s.}} 0$, as $c \to \infty$. Since $0 < \theta(1 - \alpha) \leqq \theta$ the result follows.

PROOF OF (b). With the same method as in the proof of Theorem 3.6 it follows that

$$\limsup_{c\to\infty} \frac{EN - \lambda}{\lambda^{2-r}} \leqq \limsup_{c\to\infty} \frac{EN^* - \lambda}{\lambda^{2-r}} = 0$$

by Theorem 2.8 b. Thus (b) is proved.

With arguments similar to those of Corollary 3.5.1 the following corollary is proved.

COROLLARY 3.7.1. *If $\alpha = 0$ and $a(y) \to 1$ as $y \to \infty$, if $E|X|^r < \infty$, $1 \leqq r < 2$, and if $y^{1-r^{-1}}(1 - a(y)) \to 0$ as $y \to \infty$, then*

$$\frac{N - c/\theta}{c^{1/r}} \to_{\text{a.s.}} 0\,, \qquad \text{as } c \to \infty\,.$$

If $a(y) \equiv 1$ the corollary reduces to Theorem 2.8 a and if $a(y) = 2/\pi \cdot \arctan y^{1/s}$, then

$$\begin{aligned} \frac{N - c/\theta}{c^{1/r}} &\to_{\text{a.s.}} 0 && \text{if } \frac{1}{s} + \frac{1}{r} > 1 \\ &\to_{\text{a.s.}} -\frac{2}{\pi \cdot \theta^{1/r}} && \text{if } \frac{1}{s} + \frac{1}{r} = 1 \\ &\to_{\text{a.s.}} \text{does not exist} && \text{if } \frac{1}{s} + \frac{1}{r} < 1\,. \end{aligned}$$

The last theorem of this section is the result corresponding to Theorem 2.9.

THEOREM 3.8. *Let $0 \leqq \alpha < 1$. If all the assumptions of Theorem 2.9 are satisfied, then*

$$P\left(\frac{N - \lambda}{B(\lambda)/\theta(1 - \alpha)} \geqq -x\right) \to G_\beta(x)\,, \qquad \text{as } c \to \infty\,,$$

where the function $B(y)$ is defined as in Theorem 2.9.

The proof is similar to the proof of Theorem 3.5.

LEMMA 3.5. *If the assumptions of Theorem 3.8 are satisfied, then*

(a) $B_N/B(\lambda) \to_p 1$, *as* $c \to \infty$
(b) $X_N/B(\lambda) \to_p 0$, *as* $c \to \infty$.

The proof is carried through in exactly the same way as the proof of Lemma 2.9 and is therefore omitted.

PROOF OF THEOREM 3.8. As in the proof of Theorem 2.9 it is found that $P((S_N - N\theta)/B_N \leqq x) \to G_\beta(x)$ as $c \to \infty$. From $(c \cdot a(N) - N\theta)/B(\lambda) < (S_N - N\theta)/B(\lambda) \leqq (c \cdot a(N) - N\theta + X_N)/B(\lambda)$, Lemma 3.5 a and b and Cramér's theorem, it follows that $P(Z_N \geqq -x) \to G_\beta(x)$ as $c \to \infty$, where $Z_N = (N\theta - c \cdot a(N))/B(\lambda) =$

$(N-\lambda)/[B(\lambda)/\theta(1-\alpha)]\cdot Y_N$, and Y_N is as before. Lemma 3.3 and Cramér's theorem imply that $(N-\lambda)/[B(\lambda)/\theta(1-\alpha)]$ and Z_N have the same limit distribution, i.e.

$$P\left(\frac{N-\lambda}{B(\lambda)/\theta(1-\alpha)} \geqq -x\right) \to G_\beta(x) \qquad \text{as } c \to \infty .$$

In the case of normal attraction, i.e. when $B_n = A \cdot n^{1/\beta}$, the theorem states that $P((N-\lambda)/[A/\theta(1-\alpha)]\cdot\lambda^{1/\beta} \geqq -x) \to G_\beta(x)$ as $c\to\infty$. If $\beta = 2$ this result reduces to Theorem 3.5.

The following corollary is proved with arguments similar to those of Corollaries 3.5.1 and 3.7.1.

COROLLARY 3.8.1. *Suppose that* $\alpha = 0$ *and* $a(y) \to 1$ *as* $y \to \infty$. *If the sequence* $X_1, X_2, \cdots$ *belongs to*

(a) *the domain of attraction of a stable law with exponent* β, $1 < \beta \leqq 2$, *and if* $y(1-a(y))/B(y) \to 0$ *as* $y\to\infty$, *then*

$$P\left(\frac{N-c/\theta}{[B(c/\theta)]/\theta} \geqq -x\right) \to G_\beta(x)\,, \qquad \textit{as } c\to\infty\,,$$

(b) *the domain of normal attraction of a stable law with exponent* β, $1<\beta\leqq 2$, *and if* $y(1-a(y))/y^{1/\beta} \to 0$ *as* $y\to\infty$, *then*

$$P\left(\frac{N-c/\theta}{(A/\theta)(c/\theta)^{1/\beta}} \geqq -x\right) \to G_\beta(x)\,, \qquad \textit{as } c\to\infty\,.$$

4. First passage times for some random processes. In this chapter some of the previous results are generalized to separable, left continuous random processes with independent, stationary increments. Let $\{X(t);\, t \geqq 0\}$ be such a process and suppose that $EX(t) = t\theta$, $\theta > 0$. Suppose also that the process is continuous from above, in the sense that $X(t+0) - X(t) \leqq 0$ for all $t \geqq 0$. This means that $X(t)$ has no positive jumps, or equivalently, if $X(t_1) = x_1$ and $X(t_2) = x_2$, where $x_1 < x_2$ and $t_1 < t_2$, then for any $x \in (x_1, x_2)$ there exists, with probability 1, a $t \in (t_1, t_2)$ such that $X(t) = x$. (See Borovkov [4]). Set $X(0) = 0$, $\mathscr{F}_0 = \{\phi, \Omega\}$ and let $\mathscr{F}_t = \sigma\{X(s);\, s \leqq t\}$, $t \in (0,\infty)$. A finite, nonnegative random variable T, defined on Ω, is called a stopping time if the event $\{T \leqq t\} \in \mathscr{F}_t$ for all $t \in [0,\infty)$.

4.1. In this section the stopping time

$$T = T(c) = \inf\{t;\, X(t) > c\}\,,$$

where $c \geqq 0$, is considered. Since $X(t)$ is continuous from the left and from above it follows that $P(X(T) = c) = 1$. Define $T_N = \inf\{t;\, X(t) > c,\, t \geqq 1$ is an integer$\}$ and set $X_1 = X(1)$ and $X_\nu = X(\nu) - X(\nu-1)$, $\nu = 2, 3, \cdots$. Then $X_1, X_2, \cdots$ are i.i.d. random variables such that $0 < EX = \theta < \infty$ and hence $T_N = \min\{n;\, \sum_{\nu=1}^n X_\nu > c\}$ is a stopping time of the Case I-type considered in Section 2. Obviously $0 \leqq T \leqq T_N$ and thus Theorems 2.1 and 2.2 imply

THEOREM 4.1. (a) *Let* $r \geqq 1$. *Then* $E|X(1)^-|^r < \infty \Rightarrow ET^r < \infty$.

(b) *There exists an* $s_0 > 0$ *such that* $Ee^{sT} < \infty$, $|s| < s_0$ *if there exists an* $s_1 > 0$ *such that* $Ee^{sX(1)^-} < \infty$, $|s| < s_1$.

THEOREM 4.2. (a) $T/c \to_{\text{a.s.}} \theta^{-1}$, *as* $c \to \infty$.

(b) *Let* $r \geqq 1$. *Then* $E|X(1)^-|^r < \infty \Rightarrow E(T/c)^r \to \theta^{-r}$, *as* $c \to \infty$.

PROOF OF (a). Since $P(\bigcap_{c \geqq c_0}\{T(c) > t\}) = P(T(c_0) > t) = P(\sup_{0 \leqq s \leqq t} X(s) \leqq c_0) \to 1$ as $c_0 \to \infty$, it follows that $T(c) \to_{\text{a.s.}} +\infty$ as $c \to \infty$. An investigation of the proof of Lemma 2.7, i.e. Richter [28] Theorem 1, shows that it is applicable and thus, from $X(t)/t \to_{\text{a.s.}} \theta$ as $c \to \infty$ (see Doob [11] page 364), it follows that $X(T)/T \to_{\text{a.s.}} \theta$ as $c \to \infty$. Since $P(X(T) = c) = 1$ the result follows.

PROOF OF (b). (a) and Fatou's lemma imply that $\liminf_{c\to\infty} E(T/c)^r \geqq \theta^{-r}$. By Theorem 2.3 b $\limsup_{c\to\infty} E(T/c)^r \leqq \limsup_{c\to\infty} E(T_N/c)^r = \theta^{-r}$, where T_N is as above. Thus $\lim_{c\to\infty} E(T/c)^r = \theta^{-r}$.

Suppose that $\operatorname{Var} X(t) = \sigma^2 t < \infty$ and set $\varphi_t(s) = Ee^{isX(t)}$. Since $X(t)$ has no positive jumps, the canonical representation of $\varphi_1(s) = Ee^{isX(1)}$ is $\varphi_1(s) = e^{\psi(s)}$, where

$$\psi(s) = \log \varphi_1(s) = i\theta s - Ds^2 + \int_{-\infty}^{0} (e^{isu} - 1 - isu)\frac{dG(u)}{u^2},$$

where $D \geqq 0$, $G(-\infty) = 0$, and $G(u)$ is a distribution function except for a multiplicative constant. Furthermore $\varphi_t(s) = (\varphi_1(s))^t$, and thus $\log \varphi_t(s) = t \cdot \varphi(s)$. (See Doob [11] page 417 ff, and Gnedenko and Kolmogorov [15] pages 84–85.)

Set $z = x + iy$. Since $\int_{-\infty}^{0} (e^{izu} - 1 - izu)\, dG(u)/u^2$ is analytic when $\operatorname{Im}(z) < 0$, it follows from Esseen [12] Theorem 2, page 67, that the canonical representation is valid in the domain $\operatorname{Im}(z) \leqq 0$. In particular, it follows that

$$Ee^{\lambda X(t)} = e^{t\zeta(\lambda)}, \qquad \lambda \geqq 0,$$

where

$$\zeta(\lambda) = \theta\lambda + D\lambda^2 + \int_{-\infty}^{0} (e^{\lambda u} - 1 - \lambda u)\frac{dG(u)}{u^2}.$$

THEOREM 4.3. *Let* $0 < EX(t) = \theta t < \infty$ *and* $\operatorname{Var} X(t) = \sigma^2 t < \infty$. *Then*

(a) $Ee^{-\lambda T(c)} = e^{-c\zeta^{-1}(\lambda)}$, $\lambda \geqq 0$, *where* $\zeta^{-1}(\lambda)$ *is the inverse function of* $\zeta(\lambda)$.

(b) $ET = c/\theta$ *and* $\operatorname{Var} T = \sigma^2 c/\theta^3$.

PROOF OF (a). Define $Z_t = e^{\lambda \cdot X(t)}/Ee^{\lambda \cdot X(t)} = e^{\lambda X(t) - t\cdot\zeta(\lambda)}$, $\lambda \geqq 0$. Then $\{Z_t;\, t \geqq 0\}$ is a martingale and $EZ_t = 1$, i.e. $Ee^{\lambda X(t) - t\zeta(\lambda)} = 1$. Define $T_n = \min\{T, n\}$. According to Doob [11] Theorem 11.8, page 376, it follows that $EZ_{T_n} = 1$, i.e.

$$\begin{aligned} 1 &= Ee^{\lambda X(T_n) - T_n\cdot\zeta(\lambda)} \\ &= E(e^{\lambda X(T) - T\zeta(\lambda)} \cdot I\{T \leqq n\}) + E(e^{\lambda X(n) - n\zeta(\lambda)} \cdot I\{T > n\}). \end{aligned}$$

The derivatives of $\zeta(\lambda)$ show that $\zeta(\lambda)$ is a positive, increasing, convex function. Furthermore, $X(n) \leqq c$ when $T > n$. Thus

$$E(e^{\lambda X(n) - n\zeta(\lambda)} \cdot I\{T > n\}) \to 0 \qquad \text{as } n \to \infty$$

and hence $Ee^{\lambda X(T)-T\zeta(\lambda)} = 1$. From $P(X(T) = c) = 1$ it follows that $Ee^{-T\zeta(\lambda)} = e^{-c\lambda}$, and from the convexity of $\zeta(\lambda)$ it follows that the inverse function $\xi^{-1}(\lambda)$ exists. Hence

$$Ee^{-\lambda T} = e^{-c\zeta^{-1}(\lambda)}, \qquad \lambda \geqq 0 .$$

A proof of (b) is obtained by taking derivatives of the Laplace-transform. (b) is also proved in e.g. Borovkov [4] page 334, and follows also from results of Hall [17] pages 61 and 63. If $X(t)$ is the Wiener process, (b) follows from Shepp [29] Theorem 1, page 1912.

THEOREM 4.4. *Let* $0 < EX(t) = \theta t < \infty$ *and* $\operatorname{Var} X(t) = \sigma^2 t < \infty$. *Then*

$$\mathscr{L}\left(\frac{T(c) - c/\theta}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\right) \Rightarrow N(0, 1), \qquad \textit{as } c \to \infty .$$

PROOF. $T(c) = T((n-1)/n \cdot c) + \inf\{t;\ X(t) - X(T((n-1)/n \cdot c)) > c/n|$

$$\left|X\left(T\left(\frac{n-1}{n}c\right)\right) = \frac{n-1}{n}c\right\} = T\left(\frac{n-1}{n}c\right) + T_1 .$$

Thus $T(c) = T_1 + T_2 + \cdots + T_n$, where $T_1, T_2, \cdots, T_n$ are i.i.d. random variables distributed as $T(c/n)$. Since this partition is possible for $n = 1, 2, \cdots$, it follows that $T(c)$ has an infinitely divisible distribution. (See also Borovkov [4] page 333.) Hence $Ee^{isT(c)} = (Ee^{isT(1)})^c$, $-\infty < s < \infty$.

Also, $E\exp\{is(T(c) - c/\theta)\} = (E\exp\{is(T(1) - \theta^{-1})\})^c$. Since $E(T(1) - \theta^{-1}) = 0$ and $E(T(1) - \theta^{-1})^2 = \sigma^2/\theta^3$, it follows that

$$\begin{aligned} E\exp\left\{is\frac{T(c) - c/\theta}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\right\} &= \left(E\exp\left\{i \cdot \frac{s}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}}\left(T(1) - \frac{1}{\theta}\right)\right\}\right)^c \\ &= \left(1 + \frac{is}{(\sigma^2 c/\theta^3)^{\frac{1}{2}}} \cdot 0 - \frac{1}{2} \cdot \frac{s^2}{\sigma^2 c/\theta^3} \cdot \frac{\sigma^2}{\theta^3} + o\left(\frac{s^2}{c}\right)\right)^c \\ &= \left(1 - \frac{s^2}{2c} + o\left(\frac{1}{c}\right)\right)^c \\ &\to e^{-s^2/2} \qquad \text{as } c \to \infty,\ -\infty < s < \infty . \end{aligned}$$

The result follows from the continuity theorem.

REMARK. If $X(t)$ has a density $f_{X(t)}(y)$, $t \geqq 0$, then, according to a result of e.g. Borovkov [4],

$$f_{T(c)}(t) = \frac{c}{t} \cdot f_{X(t)}(c) ,$$

where $f_{T(c)}(t)$ is the density of $T(c)$. If, for example, $X(t)$ is the separable Wiener process with $EX(t) = \theta t > 0$ and $\operatorname{Var} X(t) = \sigma^2 t$, then

$$f_{T(c)}(t) = \frac{c}{t} \cdot \frac{1}{(2\pi\sigma^2 t)^{\frac{1}{2}}} \exp\left(-\frac{(c - t\theta)^2}{2\sigma^2 t}\right).$$

Set $T_1(c) = (T(c) - c/\theta)/(\sigma^2 c/\theta^3)^{\frac{1}{2}}$. It follows that

$$f_{T_1(c)}(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \cdot \frac{1}{(1 + \sigma t/(c\theta)^{\frac{1}{2}})^{\frac{1}{2}}} \cdot \exp\left(-\frac{t^2}{2} \frac{1}{1 + \sigma t/(c\theta)^{\frac{1}{2}}}\right)$$
$$\rightarrow \frac{1}{(2\pi)^{\frac{1}{2}}} \cdot e^{-t^2/2} \qquad \text{as } c \rightarrow \infty ,$$

and the result follows from Scheffé's theorem. (See e.g. Billingsley [2] page 224).

4.2. In the second part of this chapter the stopping time

$$T = T(c) = \inf\{t;\ X(t) > c \cdot a(t)\}$$

is considered, where $c \geqq 0$, and $a(y)$, $y \in [0, \infty)$, is a positive, continuous function such that $a(y)/y \rightarrow 0$ as $y \rightarrow \infty$. Since $X(t)$ is continuous from the left and from above, it follows that

$$P(X(T) = c \cdot a(T)) = 1 .$$

Define

$$T_N = \inf\{t;\ X(t) > c \cdot a(t),\ t \geqq 1 \text{ is an integer}\}$$
$$= \min\{n;\ \textstyle\sum_{\nu=1}^{n} X_\nu > c \cdot a(n)\} ,$$

where $X_1, X_2, \cdots$ are as above. Then $0 \leqq T \leqq T_N$, and since T_N is a stopping time of the Case II-type, Theorems 3.1 and 3.2 imply

THEOREM 4.5. (a) *Let* $r \geqq 1$. *Then* $E|X(1)^-|^r < \infty \Rightarrow ET^r < \infty$.

(b) *There exists an* $s_0 > 0$ *such that* $Ee^{sT} < \infty$, $|s| < s_0$, *if there exists an* $s_1 > 0$ *such that* $Ee^{sX(1)^-} < \infty$, $|s| < s_1$.

Now assume that $a(y)$ also satisfies the regularity conditions set up when Case III was studied. Again, let $\lambda = \lambda(c)$ denote the solution of the equation $c \cdot a(y) = \theta y$.

THEOREM 4.6. (a) $T/\lambda \rightarrow_{\text{a.s.}} 1$, *as* $c \rightarrow \infty$.

(b) *Let* $r \geqq 1$. *Then* $E|X(1)^-|^r < \infty \Rightarrow E(T/\lambda)^r \rightarrow 1$ *as* $c \rightarrow \infty$.

PROOF OF (a). As in the proof of Theorem 4.2a it follows that

$$\frac{X(T)}{T} \rightarrow_{\text{a.s.}} \theta , \qquad \text{as } c \rightarrow \infty .$$

Since $P(X(T) = c \cdot a(T)) = 1$, the conclusion follows as in Theorem 3.3a when $\alpha = 0$, and as in Siegmund [30] Lemma 4, when $0 < \alpha < 1$.

PROOF OF (b). (a) and Fatou's lemma imply that

$$\liminf_{c\rightarrow\infty} E\left(\frac{T}{\lambda}\right)^r \geqq 1 .$$

By Theorem 3.3b

$$\limsup_{c\rightarrow\infty} E\left(\frac{T}{\lambda}\right)^r \leqq \limsup_{c\rightarrow\infty} E\left(\frac{T_N}{\lambda}\right)^r = 1 .$$

Thus

$$\lim_{c\to\infty} E\left(\frac{T}{\lambda}\right)^r = 1 .$$

THEOREM 4.7. *Let* $0 \leqq \alpha < 1$. *If* $0 < EX(t) = t\theta < \infty$ *and* $\operatorname{Var} X(t) = \sigma^2 t < \infty$, *then*

$$\mathscr{L}\left(\frac{T-\lambda}{\sigma\lambda^{\frac{1}{2}}/\theta(1-\alpha)}\right) \Rightarrow N(0, 1) , \qquad as \ c \to \infty .$$

The proof is similar to the proof of Theorem 3.5.

LEMMA 4.1. *Let* $0 \leqq \alpha < 1$, *define* $\alpha(\lambda t) = \lambda \cdot a'(\lambda t)/a(\lambda)$ *and*

$$Y_T = \frac{1 - [\lambda \cdot a'(\lambda + \rho(T-\lambda))/a(\lambda)]}{1-\alpha} ,$$

where $0 \leqq \rho = \rho(T, c) \leqq 1$. *Then*

(a) $\alpha(\lambda t) \to \alpha t^{\alpha-1}$, *as* $c \to \infty$

(b) $Y_T \to_{\text{a.s.}} 1$, *as* $c \to \infty$

(c) $\mathscr{L}((X(T) - T\theta)/\sigma T^{\frac{1}{2}}) \Rightarrow N(0, 1)$, *as* $c \to \infty$.

PROOF. The proofs of (a) and (b) are the same as the proofs in Section 3. Since $\mathscr{L}((X(t) - t\theta)/\sigma t^{\frac{1}{2}}) \Rightarrow N(0, 1)$ as $t \to \infty$, (c) may be proved exactly as Lemma 2.5, i.e. Rényi [27] Theorem 1, was proved. The details are therefore omitted.

PROOF OF THE THEOREM. According to Lemma 4.1 c

$$\mathscr{L}\left(\frac{X(T) - T\theta}{\sigma T^{\frac{1}{2}}}\right) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty .$$

Since $P(X(T) = c \cdot a(T)) = 1$ it follows that

$$\mathscr{L}(Z_T) \Rightarrow N(0, 1) \qquad \text{as } c \to \infty ,$$

where

$$Z_T = \frac{T\theta - c \cdot a(T)}{\sigma\lambda^{\frac{1}{2}}} = \frac{T-\lambda}{\sigma\lambda^{\frac{1}{2}}/\theta(1-\alpha)} \cdot Y_T .$$

Now the conclusion follows from Lemma 4.1 b and Cramér's theorem ([9] page 254).

REMARK. If $a(y) \equiv 1$, Theorem 4.7 reduces to Theorem 4.4 and thus another proof of this theorem has been obtained.

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