The Annals of Probability
1974, Vol. 2, No. 3, 551-552

ON ALMOST SURE CONVERGENCE OF QUADRATIC BROWNIAN VARIATION

BY W. FERNANDEZ DE LA VEGA

Analyse Documentaire et Calcul en Archéologie, C.N.R.S.

We prove that Dudley's condition for a.s. convergence of quadratic Brownian variation on a sequence of partitions of [0, 1] is best possible for the case in which these partitions are restricted to consist of intervals.

Let $(X, \mathscr{S}, \mu)$ be any finite measure space: $\mu > 0$, $\mu(X) < \infty$. Let H be the Hilbert space $L^2(X, \mathscr{S}, \mu)$. We call μ-noise on H the isonormal process L on H: L is a linear map from H into Gaussian random variables with $EL(x) = 0$ and $EL(x)L(y) = (x, y)$ for all $x, y \in H$.

A partition of X will be a finite collection π of disjoint measurable sets whose union is X. The mesh of π is defined by

$$m(\pi) = \max\{\mu(A): A \in \pi\}.$$

We define $L(\pi)^2 = \sum_{A\in\pi} L(\chi_A)^2$. Let $\{\pi_n\}$ be a sequence of partitions of X such that $m(\pi_n) \to 0$. Then $L(\pi_n)^2 \to \mu(X)$ in law and hence in probability. P. Lévy ((1940) Section 4, Théorème 5) proved that $L(\pi_n)^2 \to \mu(X)$ almost surely if the π_n are nested, i.e., for all $A \in \pi_{n+1}$ there is a $B \in \pi_n$ with $A \subset B$. R. M. Dudley (1973) proved that $m(\pi_n) = o(1/\log n)$ suffices for a.s. convergence and that this is best possible: Dudley proved that there exist partitions $\{\pi_n\}$, not consisting of intervals, such that $m(\pi_n) < 1/\log n$ and $L(\pi_n)^2$ does not converge a.s. Dudley asks if $m(\pi_n) = o(1/\log n)$ is also best when the problem is restricted to partitions consisting of intervals, with $X = [0, 1]$. We prove that this is indeed the case, when μ is Lebesgue measure. L is then called white noise, and is the derivative, in the distribution sense, of the standard Brownian motion.

THEOREM. *Let L be white noise on* [0, 1]. *There exist* interval *partitions π_n such that $m(\pi_n) = \mathscr{O}(1/\log n)$ and $L(\pi_n)^2$ does not converge* a.s. *to* 1.

PROOF. We construct the required sequence $\{\pi_n\}$ as follows: π_0 is the partition consisting of the single interval [0, 1]; we take each integer $p \geqq 1$ in turn and add to the sequence in an arbitrary order, all the partitions of [0, 1] each of which contains for each integer k, $0 \leqq k \leqq 2^{p-1} - 1$, either the interval $J_p^k = [2k/2^p, (2k+2)/2^p]$, or both intervals $I_p^{2k} = [2k/2^p, (2k+1)/2^p]$ and $I_p^{2k+1} = [(2k+1)/2^p, (2k+2)/2^p]$. Call Π_p the set consisting of these partitions.

There are, in Π_p, $2^{2^{p-1}} - 1$ partitions of mesh 2^{1-p} and one of mesh 2^{-p}. Their ranks in the sequence $\{\pi_n\}$ are bounded above by $1 + \sum_{0\leqq q\leqq p-1} 2^{2^q} < 2^{1+2^{p-1}}$. We can then verify that the inequality $m(\pi_n) \leqq 3/\log n$ holds for $n \geqq 1$.

Received September 14, 1973.
AMS 1970 *subject classifications.* Primary 60J65; Secondary 60G15.
Key words and phrases. Brownian motion, white noise.

We shall show that the upper limit of the sequence $L(\pi_n)^2$ is a.s. equal to a number greater than one. Define the rv's $M_p = \max\{L(\pi)^2 : \pi \in \Pi_p\}$. The upper limit of the sequence $L(\pi_n)^2$ is equal to the upper limit of the sequence $\{M_p\}$. From the definition of Π_p, we have $M_p = \sum_{0 \leqq k \leqq 2^{p-1}-1} M_p^k$; where: $M_p^k = \max\{L(I_p^{2k})^2 + L(I_p^{2k+1})^2; L(J_p^k)^2\}$, $L(I_p^{2k})$ and $L(I_p^{2k+1})$ are independent rv's with common df $\mathscr{N}(0, 2^{-p})$ while $L(J_p^k)$ is the sum of $L(I_p^{2k})$ and $L(I_p^{2k+1})$. Observe now that if X and Y are independent rv's with common df $\mathscr{N}(0, \rho^2)$, and we define $Z = \max\{X^2 + Y^2; (X+Y)^2\}$, there exist then constants $a > 0$, $b > 0$, such that $EZ = 2(1+a)\rho^2$, $\sigma^2 Z = b\rho^4$. Hence we have $EM_p^k = (1+a)2^{1-p}$, $\sigma^2 M_p^k = b2^{-2p}$. M_p is the sum of the independent i.d. rv's M_p^k and so $EM_p = 1 + a$, $\sigma^2 M_p = b2^{-p-1}$. The series $\{b2^{-p-1}\}$ being convergent, the rv's M_p, $p \geqq 1$, converge a.s. to the number $1 + a$. □

Finally we may observe that if a sequence $\{\pi_n\}$ is such that the condition $m(\pi_n) = \mathscr{O}(1/\log n)$ is satisfied, then the sequence $L(\pi_n)^2$ is almost surely bounded. This assertion is an easy consequence of a theorem of Hanson and Wright (1971).

REFERENCES

DUDLEY, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probability* **1** 66–103.

HANSON, D. L. and WRIGHT F. T. (1971). A bound on tail probabilities for quadratic forms in independent random variables. *Ann. Math. Statist.* **42** 1079–1083.

LÉVY, P. (1940). Le mouvement brownien plan. *Amer. J. Math.* **62** 487–550.

ANALYSE DOCUMENTAIRE ET
CALCUL EN ARCHEOLOGIE
31, CHEMIN JOSEPH AIGUIER
13274 MARSEILLE CEDEX 2, FRANCE