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SOJOURN TIME PROBLEMS

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It is supposed that in the time interval $(0, \infty)$ a stochastic process is alternately in states A and B. Denote by $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ the lengths of the successive intervals spent in states A and B respectively. In this paper the distribution and the asymptotic distribution of the total time spent in state A (B) in the interval $(0, t)$ are determined in the case where (α_1, β_1), $(\alpha_2, \beta_2), \cdots$ are mutually independent and identically distributed vector variables.

1. Introduction. Let $\{\eta(u), 0 \leqq u < \infty\}$ be a stochastic process with state space $A \cup B$ where A and B are disjoint sets. If $\eta(u) \in A$, then we say that the process is in state A at time u, and if $\eta(u) \in B$, then we say that the process is in state B at time u. Let us assume that in any finite interval $(0, t)$ the process changes states only a finite number of times with probability one. Let $\mathbf{P}\{\eta(0) \in A\} = 1$ and denote by $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ the lengths of the successive intervals spent in states A and B respectively in the interval $(0, \infty)$. Denote by $\alpha(t)$ the total time spent in state A in the interval $(0, t)$ and by $\beta(t)$ the total time spent in state B in the interval $(0, t)$. Obviously $\alpha(t)$ and $\beta(t)$ are random variables and $\alpha(t) + \beta(t) = t$ for all $t \geqq 0$.

In this paper we determine the distributions of $\alpha(t)$ and $\beta(t)$ in the general case, and the asymptotic distributions of $\alpha(t)$ and $\beta(t)$ in the case where (α_1, β_1), $(\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n), \cdots$ are mutually independent and identically distributed vector random variables which belong to the domain of normal attraction of a two-dimensional distribution function. The case where $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences has been considered earlier by the author [3].

2. The distributions of $\alpha(t)$ and $\beta(t)$. Let us introduce the notation $\gamma_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ for $n = 1, 2, \cdots$ and $\gamma_0 = 0$, furthermore, $\delta_n = \beta_1 + \beta_2 + \cdots + \beta_n$ for $n = 1, 2, \cdots$ and $\delta_0 = 0$.

Theorem 1. *If* $0 < x \leqq t$, *then*

$$(1)\qquad \mathbf{P}\{\alpha(t) < x\} = \sum_{n=1}^{\infty} [\mathbf{P}\{\gamma_n < x, \delta_{n-1} \leqq t - x\} - \mathbf{P}\{\gamma_n < x, \delta_n \leqq t - x\}]$$

and if $0 \leqq x < t$, *then*

$$(2)\qquad \mathbf{P}\{\beta(t) \leqq x\} = \sum_{n=0}^{\infty} [\mathbf{P}\{\delta_n \leqq x, \gamma_n < t - x\} - \mathbf{P}\{\delta_n \leqq x, \gamma_{n+1} < t - x\}] .$$

Proof. Since $\mathbf{P}\{\alpha(t) < x\} = 1 - \mathbf{P}\{\beta(t) \leqq t - x\}$ for $0 \leqq x \leqq t$, it is sufficient to prove (2). For $0 \leqq x < t$ denote by $\tau = \tau(t - x)$ the smallest $u \in [0, \infty)$ for

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which $\alpha(u) = t - x$, provided that such a u exists. Then $\eta(\tau) \in A$ and we have

(3) $$\{\beta(t) \leqq x\} \equiv \{\beta(\tau) \leqq x\} .$$

This follows from the following identities

(4) $$\{\beta(t) \leqq x\} \equiv \{\alpha(\tau) \leqq \alpha(t)\} \equiv \{\tau \leqq t\} \equiv \{\alpha(\tau) + \beta(\tau) \leqq t\} \equiv \{\beta(\tau) \leqq x\} .$$

Here we used that $\alpha(t) + \beta(t) = t$ for all $t \geqq 0$, and that $\alpha(t)$ and $\beta(t)$ are non-decreasing functions of t for $0 \leqq t < \infty$.

Since $\beta(\tau) = \delta_n$ $(n = 0, 1, \cdots)$ if $\gamma_n < t - x \leqq \gamma_{n+1}$, it follows from (3) that

(5) $$\mathbf{P}\{\beta(t) \leqq x\} = \sum_{n=0}^{\infty} \mathbf{P}\{\delta_n \leqq x \text{ and } \gamma_n < t - x \leqq \gamma_{n+1}\}$$

for $0 \leqq x < t$ which proves (2).

If for each $t \geqq 0$ we define $\omega(t)$ as a discrete random variable taking on positive integers only and satisfying the relation

(6) $$\{\omega(t) \leqq n\} \equiv \{\delta_n > t\}$$

for all $t \geqq 0$ and $n = 0, 1, 2, \cdots$, then we can write that

(7) $$\mathbf{P}\{\alpha(t) < x\} = \mathbf{P}\{\gamma_{\omega(t-x)} < x\}$$

for $0 \leqq x \leqq t$. We note that $\mathbf{P}\{\omega(0) = 1\} = 1$.

If for each $t \geqq 0$ we define $\rho(t)$ as a discrete random variable taking on non-negative integers only and satisfying the relation

(8) $$\{\rho(t) < n\} \equiv \{\gamma_n \geqq t\}$$

for all $t \geqq 0$ and $n = 1, 2, \cdots$, then we can write that

(9) $$\mathbf{P}\{\beta(t) \leqq x\} = \mathbf{P}\{\delta_{\rho(t-x)} \leqq x\}$$

for $0 \leqq x \leqq t$. We note that $\mathbf{P}\{\rho(0) = 0\} = 1$.

3. The asymptotic distributions of $\alpha(t)$ and $\beta(t)$. Formulas (7) and (9) make it possible to determine the asymptotic distributions of $\alpha(t)$ and $\beta(t)$ as $t \to \infty$ if we know the asymptotic distribution of $\gamma_{\omega(t)}$ as $t \to \infty$ or the asymptotic distribution of $\delta_{\rho(t)}$ as $t \to \infty$. In our case the asymptotic distributions of $\alpha(t)$ and $\beta(t)$ can be determined by Theorem 2. In Theorem 2 and in the rest of the paper if we say that a family of distribution functions converges to a limiting distribution function then by this we mean that the distribution functions converge in every continuity point of the limiting distribution function.

THEOREM 2. *Let us suppose that either* $0 < d < 1$, $D_1 > 0$, $D_2 > 0$ *or* $d \geqq 1$, $D_1 = 0$, $D_2 > 0$. *If*

(10) $$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\gamma_{\omega(t)} - D_1 t}{D_2 t^d} \leqq x\right\} = \mathbf{P}\{\theta \leqq x\} ,$$

then

(11) $$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\alpha(t) - M_1 t}{M_2 t^m} \leqq x\right\} = R(x) ,$$

and if

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\delta_{\rho(t)} - D_1 t}{D_2 t^d} \leqq x\right\} = \mathbf{P}\{\theta \leqq x\}, \tag{12}$$

then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t) - M_1 t}{M_2 t^m} \leqq x\right\} = R(x), \tag{13}$$

where the constants M_1, M_2, m and the distribution function $R(x)$ are given in the following table.

TABLE 1

d	M_1	M_2	m	$R(x)$
$d > 1$	1	$D_2^{-1/d}$	$\frac{1}{d}$	$\mathbf{P}\{-\theta^{-1/d} \leqq x\}$
$d = 1$	0	1	1	$\mathbf{P}\left\{\frac{D_2\theta}{1+D_2\theta} \leqq x\right\}$
$d < 1$	$\frac{D_1}{1+D_1}$	$\frac{D_2}{(1+D_1)^{1+d}}$	d	$\mathbf{P}\{\theta \leqq x\}$

PROOF. Both (11) and (13) can be proved in a similar way. Let us prove (13). Let us define

$$u = t + D_1 t + xD_2 t^d \tag{14}$$

for $t \geqq 0$. If x is such that $u \geqq t$, then by (9) we have

$$\mathbf{P}\{\beta(u) \leqq u - t\} = \mathbf{P}\{\delta_{\delta(t)} \leqq D_1 t + xD_2 t^d\}. \tag{15}$$

If $d \geqq 1$ and $x \geqq 0$ or $d < 1$, $-\infty < x < \infty$, and t is sufficiently large, then $u \geqq t$ is satisfied and $t \to \infty$ as $u \to \infty$. Thus we have

$$\lim_{u\to\infty} \mathbf{P}\{\beta(u) \leqq u - t\} = \mathbf{P}\{\theta \leqq x\} \tag{16}$$

where $t = t(u)$, which satisfies $0 \leqq t \leqq u$ for sufficiently large u, can be obtained by (14).

If $d > 1$, then $D_1 = 0$ and for $x > 0$ we obtain that

$$t = \left(\frac{u}{xD_2}\right)^{1/d} + o(u^{1/d}) \tag{17}$$

as $u \to \infty$. Thus by (16)

$$\lim_{u\to\infty} \mathbf{P}\left\{\beta(u) \leqq u - \left(\frac{u}{xD_2}\right)^{1/d}\right\} = \mathbf{P}\{\theta \leqq x\} \tag{18}$$

for $x > 0$.

If $d = 1$, then $D_1 = 0$ and for $x \geqq 0$ we obtain that

$$t = \frac{u}{1 + xD_2} \tag{19}$$

for $u \geqq 0$. Thus by (16)

$$\lim_{u\to\infty} \mathbf{P}\left\{\beta(u) \leqq \frac{uxD_2}{1+xD_2}\right\} = \mathbf{P}\{\theta \leqq x\} \tag{20}$$

for $x \geqq 0$.

If $d < 1$, then $D_1 > 0$ and we obtain that

$$t = \frac{u}{1+D_1} - \frac{xD_2}{1+D_1}\left(\frac{u}{1+D_1}\right)^d + o(u^d) \tag{21}$$

as $u \to \infty$. Thus by (16)

$$\lim_{u\to\infty} \mathbf{P}\left\{\beta(u) \leqq \frac{D_1 u}{1+D_1} + \frac{xD_2}{1+D_1}\left(\frac{u}{1+D_1}\right)^d\right\} = \mathbf{P}\{\theta \leqq x\}\,. \tag{22}$$

The limit relations (18), (20), (22) prove (13).

Our next aim is to find the asymptotic distributions of $\gamma_{\omega(t)}$ and $\delta_{\rho(t)}$ as $t \to \infty$ in the case where $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n), \cdots$ are mutually independent and identically distributed vector random variables which belong to the domain of normal attraction of a two-dimensional distribution function $F(x, y)$. In this case we have

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\gamma_n - A_1 n}{A_2 n^a} \leqq x\,, \frac{\delta_n - B_1 n}{B_2 n^b} \leqq y\right\} = F(x, y) \tag{23}$$

where the normalizing constants satisfy the conditions $\frac{1}{2} \leqq a < 1$, $A_1 > 0$, $A_2 > 0$ or $a \geqq 1$, $A_1 = 0$, $A_2 > 0$ and $\frac{1}{2} \leqq b < 1$, $B_1 > 0$, $B_2 > 0$ or $b \geqq 1$, $B_1 = 0$, $B_2 > 0$.

We shall use the following auxiliary theorem due to F. J. Anscombe [1].

LEMMA 1. *Let us suppose that* $\nu(t)$ $(0 \leqq t < \infty)$ *are discrete random variables taking on nonnegative integers only and that*

$$\lim_{t\to\infty} \frac{\nu(t)}{t} = c \tag{24}$$

in probability where c is a positive constant. Let $\zeta(n)$ $(n = 0, 1, 2, \cdots)$ *be a sequence of real random variables for which*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\zeta(n)}{b(n)} \leqq x\right\} = Q(x) \tag{25}$$

and

$$\lim_{\varepsilon\to 0} \liminf_{m\to\infty} \mathbf{P}\{\max_{|n-m|<m\delta(\varepsilon)} |\zeta(n) - \zeta(m)| < \varepsilon b(m)\} = 1 \tag{26}$$

for some $\delta(\varepsilon) > 0$ *such that* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Then*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\zeta(\nu(t))}{b([ct])} \leqq x\right\} = Q(x) \tag{27}$$

regardless of whether $\{\nu(t)\}$ *depends on* $\{\zeta(n)\}$ *or not.*

F. J. Anscombe [1] demonstrated that if $\zeta(n)$ is the nth partial sum of a sequence of mutually independent and identically distributed random variables and if $0 < Q(0) < 1$, then (26) is satisfied and thus Lemma 1 is applicable. We can easily show that Lemma 1 is still applicable if $Q(0) = 0$ or $Q(0) = 1$.

THEOREM 3. *Let us suppose that* (α_n, β_n) $(n = 1, 2, \cdots)$ *are mutually independent, and identically distributed vector variables for which* (23) *holds with* $a \geqq 1$ *and* $b \geqq 1$. *Let*

$$\Phi(s, q) = \int_0^\infty \int_0^\infty e^{-sx-qy}\, d_x d_y F(x, y) \tag{28}$$

for $\operatorname{Re}(s) \geqq 0$ *and* $\operatorname{Re}(q) \geqq 0$. *Then*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\gamma_{\omega(t)}}{A_2 B_2^{-a/b} t^{a/b}} \leqq x\right\} = Q(x) \tag{29}$$

exists and

$$\int_0^\infty x^s\, dQ(x) = \frac{1}{\Gamma(1-s)\Gamma(1+as/b)} \int_0^\infty x^s\, dV(x) \tag{30}$$

for sufficiently small $|\operatorname{Re}(s)|$ *where*

$$V(s) = 1 - \frac{\log \Phi\left(\frac{1}{s}, 0\right)}{\log \Phi\left(\frac{1}{s}, 1\right)} \tag{31}$$

for $\operatorname{Re}(s) > 0$.

PROOF. In proving this theorem we may assume without loss of generality that $A_2 = B_2 = 1$. Let

$$\Psi(s, q) = \mathbf{E}\{e^{-s\alpha_n - q\beta_n}\} \tag{32}$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(q) \geqq 0$. By (23) it follows that

$$\lim_{n\to\infty} \left[\Psi\left(\frac{s}{n^a}, \frac{q}{n^b}\right)\right]^n = \Phi(s, q) \tag{33}$$

and

$$\lim_{n\to\infty} n\left[\Psi\left(\frac{s}{n^a}, \frac{q}{n^b}\right) - 1\right] = \log \Phi(s, q) \tag{34}$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(q) \geqq 0$. We note that necessarily $\log \Phi(s, 0) = -As^{1/a}$ and $\log \Phi(0, q) = -Bq^{1/b}$ where A and B are positive constants.

For simplicity let us write $\xi(t) = \gamma_{\omega(t)}$ for $t \geqq 0$ and denote by $I(A)$ the indicator variable of the event A, that is, $I(A) = 1$ if A occurs and $I(A) = 0$ if A does not occur. By (6) we have

$$\mathbf{E}\{e^{-s\xi(t)}\} = 1 - [1 - \Psi(s, 0)]M(t, s) \tag{35}$$

for $\operatorname{Re}(s) \geqq 0$ where

$$M(t, s) = \sum_{n=0}^\infty \mathbf{E}\{e^{-s\gamma_n} I(\delta_n \leqq t)\}\,. \tag{36}$$

If we express the sum in (36) in the form of an integral, then we can write that

$$M(t^b, st^{-a}) = t \int_0^\infty \mathbf{E}\{\exp[-st^{-a}\gamma_{[ut]}]I(\delta_{[ut]} \leqq t^b)\}\, du \tag{37}$$

for Re $(s) \geqq 0$ and $t > 0$.

We shall prove that if Re $(s) \geqq 0$, then

$$\lim_{t\to\infty} \frac{M(t, st^{-a/b})}{t^{1/b}} = \lim_{t\to\infty} \frac{M(t^b, st^{-a})}{t} = \mu(s) \tag{38}$$

exists and

$$\mu(s) = \int_0^\infty \mathbf{E}\{e^{-su^a\gamma} I(\delta \leqq u^{-b})\}\, du \tag{39}$$

where $\mathbf{P}\{\gamma \leqq x, \delta \leqq y\} = F(x, y)$.

For Re $(s) \geqq 0$ we have

$$|\mu(s)| \leqq \mu(0) = \int_0^\infty \mathbf{P}\{\delta \leqq u^{-b}\}\, du = \mathbf{E}\{\delta^{-1/b}\} = \frac{1}{B\Gamma(1 + 1/b)} \tag{40}$$

where the last equality follows from $\mathbf{E}\{e^{-s\delta}\} = e^{-Bs^{1/b}}$ for Re $(s) \geqq 0$.

Since by (34) $\lim_{s\to+0}[1 - \Psi(s, 0)]s^{-1/a} = A$, it follows from (35) and (38) that

$$\lim_{t\to\infty} \mathbf{E}\{\exp[-s\xi(t)t^{-a/b}]\} = 1 - As^{1/a}\mu(s) \tag{41}$$

for Re $(s) \geqq 0$.

It remains to prove (38). First, let $s = 0$. Since

$$\int_0^\infty e^{-qt}\, dM(t, 0) = \frac{1}{1 - \Psi(0, q)} \tag{42}$$

for Re $(q) > 0$ and since $\lim_{q\to+0}[1 - \Psi(0, q)]q^{-1/b} = B$, it follows from a Tauberian theorem of O. Szász [2] that

$$\lim_{t\to\infty} \frac{M(t, 0)}{t^{1/b}} = \frac{1}{B\Gamma(1 + 1/b)}\,. \tag{43}$$

This proves (38) for $s = 0$.

By (23) it follows that if $t \to \infty$, then the integrand in (37) tends to the integrand in (39) for $u > 0$ and Re $(s) \geqq 0$. Since (38) holds for $s = 0$, we can conclude that for any $K > 0$ and Re $(s) \geqq 0$ we have

$$\left|\int_K^\infty \mathbf{E}\{\exp[-st^{-a}\gamma_{[ut]}]I(\delta_{[ut]} \leqq t^b)\}\, du\right| \leqq \int_K^\infty \mathbf{P}\{\delta_{[ut]} \leqq t^b\}\, du \to \int_K^\infty \mathbf{P}\{\delta \leqq u^{-b}\}\, du \quad \text{as } t \to \infty\,, \tag{44}$$

and the extreme right member is arbitrarily close to 0 if K is sufficiently large. Thus by the dominated convergence theorem it follows that in (37) the integral tends to $\mu(s)$ for Re $(s) \geqq 0$ as $t \to \infty$. This proves (38).

By (40) it follows that the right-hand side of (41) tends to 1 as $s \to +0$. Thus by the continuity theorem of Laplace–Stieltjes transforms we can conclude that the limiting distribution

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\xi(t)}{t^{a/b}} \leqq x\right\} = Q(x) \tag{45}$$

exists, and

$$\int_0^\infty e^{-sx}\, dQ(x) = 1 - As^{1/a}\mu(s) \tag{46}$$

for $\text{Re}(s) \geqq 0$. However, we can also obtain $Q(x)$ in another way.

By (35) we have

$$q \int_0^\infty e^{-qt}\mathbf{E}\{e^{-s\xi(t)}\}\, dt = 1 - \frac{1 - \Psi(s, 0)}{1 - \Psi(s, q)} \tag{47}$$

for $\text{Re}(s) \geqq 0$ and $\text{Re}(q) > 0$. Now let ν be a positive real random variable which is independent of the process $\{\xi(t), 0 \leqq t < \infty\}$ and for which $\mathbf{P}\{\nu \leqq x\} = 1 - e^{-x}$ if $x \geqq 0$. Then by (47) we have

$$\mathbf{E}\{e^{-s\xi(\nu/q)}\} = 1 - \frac{1 - \Psi(s, 0)}{1 - \Psi(s, q)} \tag{48}$$

for $\text{Re}(s) \geqq 0$ and $q > 0$. By (34) and (48) we get

$$\begin{aligned} \lim_{q\to 0} \mathbf{E}\{\exp[-sq^{a/b}\xi(\nu/q)]\} &= 1 - \lim_{q\to 0} \frac{[1 - \Psi(sq^{a/b}, 0)]q^{-1/b}}{[1 - \Psi(sq^{a/b}, q)]q^{-1/b}} \\ &= 1 - \frac{\log \Phi(s, 0)}{\log \Phi(s, 1)} = V\left(\frac{1}{s}\right) \end{aligned} \tag{49}$$

for $\text{Re}(s) \geqq 0$ where we used the definition (31).

If ξ, ν_1, ν_2 are mutually independent random variables for which $\mathbf{P}\{\xi \leqq x\} = Q(x)$ and $\mathbf{P}\{\nu_1 \leqq x\} = \mathbf{P}\{\nu_2 \leqq x\} = 1 - e^{-x}$ for $x \geqq 0$, then by (49) we have

$$\mathbf{P}\{\xi\nu_1^{-1}\nu_2^{a/b} \leqq x\} = V(x) \tag{50}$$

for $x > 0$. Hence it follows that

$$\mathbf{E}\{\xi^s\}\mathbf{E}\{\nu_1^{-s}\}\mathbf{E}\{\nu_2^{as/b}\} = \int_0^\infty x^s\, dV(x) \tag{51}$$

for sufficiently small $|\text{Re}(s)|$. This proves (30). From (30) we can obtain $Q(x)$ by Mellin's inversion formula.

In the particular case when $a = b$ we have

$$\int_0^\infty x^s\, dQ(x) = \frac{\sin \pi s}{\pi s} \int_0^\infty x^s\, dV(x) \tag{52}$$

for sufficiently small $|\text{Re}(s)|$, and hence it follows by inversion that

$$\frac{dQ(x)}{dx} = \frac{V(xe^{\pi i}) - V(xe^{-\pi i})}{2\pi i x} \tag{53}$$

for $x > 0$ where the definition of $V(x)$ is extended by analytical continuation to the complex plane cut along the negative real axis from 0 to ∞.

In the particular case when γ and δ are independent random variables, that is, $F(x, y) = \mathbf{P}\{\gamma \leqq x\}\mathbf{P}\{\delta \leqq y\}$ we have

$$Q(x) = \mathbf{P}\{\gamma\delta^{-a/b} \leqq x\}\,. \tag{54}$$

This follows easily from (46). Conversely, we can prove that if (54) is true, then γ and δ are necessarily independent random variables.

To prove this last statement let us suppose that the vector variable (γ, δ) and ν_1 and ν_2 are mutually independent. Let $\mathbf{P}\{\gamma \leqq x, \delta \leqq y\} = F(x, y)$ with Laplace–Stieltjes transform $\Phi(s, q)$ defined by (28), and $\mathbf{P}\{\nu_1 \leqq x\} = \mathbf{P}\{\nu_2 \leqq x\} = 1 - e^{-x}$ for $x \geqq 0$. Then we have

$$\mathbf{P}\{\gamma\nu_1^{-1} \leqq x, \delta\nu_2^{-1} \leqq y\} = \Phi\left(\frac{1}{x}, \frac{1}{y}\right) \tag{55}$$

for $x > 0$ and $y > 0$. Hence we can deduce that

$$\mathbf{P}\{\gamma\delta^{-a/b}\nu_1^{-1}\nu_2^{a/b} \leqq x\} = \frac{axV'(x)}{[1 - V(x)]} \tag{56}$$

for $x > 0$ where $V(x)$ is given by (31). If we compare (50) and (56), then we can conclude that

$$\frac{axV'(x)}{1 - V(x)} = V(x) \tag{57}$$

is a necessary and sufficient condition for the validity of (54). The general solution of (57) is

$$V(x) = \frac{Cx^{1/a}}{1 + Cx^{1/a}} \tag{58}$$

for $x > 0$ where C is a positive constant. This implies that

$$\Phi(s, q) = \exp[-A(s^{1/a} + Cq^{1/b})] \tag{59}$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(q) \geqq 0$. Hence it follows that $C = B/A$ and that γ and δ are independent.

Finally, the asymptotic distribution of $\alpha(t)$ is given by (11) where now $d = a/b$, $D_1 = 0$, $D_2 = A_2B_2^{-a/b}$, and $\mathbf{P}\{\theta \leqq x\} = Q(x)$.

We note that in a similar way we can prove that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\delta_{\rho(t)}}{B_2A_2^{-b/a}t^{b/a}} \leqq x\right\} = Q^*(x) \tag{60}$$

exists and

$$\int_0^\infty x^s\, dQ^*(x) = \frac{1}{\Gamma(1 - s)\Gamma(1 + bs/a)} \int_0^\infty x^s\, dV^*(x) \tag{61}$$

for sufficiently small $|\operatorname{Re}(s)|$ where

$$V^*(s) = \frac{\log \Phi(1, 0)}{\log \Phi(1, 1/s)} \tag{62}$$

for $\operatorname{Re}(s) > 0$. The asymptotic distribution of $\beta(t)$ is given by (13) where now $d = b/a$, $D_1 = 0$, $D_2 = B_2A_2^{-b/a}$, and $\mathbf{P}\{\theta \leqq x\} = Q^*(x)$.

We observe that

$$V^*(x) = 1 - V(x^{-a/b}) \tag{63}$$

for $x > 0$.

The following theorem contains the case $a \geqq 1$, $\frac{1}{2} \leqq b < 1$ as a particular case.

THEOREM 4. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ *are mutually independent and identically distributed random variables for which*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\gamma_n}{A_2 n^a} \leqq x\right\} = \mathbf{P}\{\gamma \leqq x\} \tag{64}$$

where $a \geqq 1$ *and* $A_2 > 0$, *and if*

$$\lim_{n\to\infty} \frac{\delta_n}{n} = B_1 \tag{65}$$

in probability where $B_1 > 0$, *then we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\gamma_{\omega(t)} B_1^a}{A_2 t^a} \leqq x\right\} = \mathbf{P}\{\gamma \leqq x\}\,. \tag{66}$$

PROOF. By (6) and (65) it follows that

$$\lim_{t\to\infty} \frac{\omega(t)}{t} = \frac{1}{B_1} \tag{67}$$

in probability. Thus (66) immediately follows from Lemma 1.

In this case the asymptotic distribution of $\alpha(t)$ is given by (11) where now $d = a$, $D_1 = 0$, $D_2 = A_2 B_1^{-a}$ and $\mathbf{P}\{\theta \leqq x\} = \mathbf{P}\{\gamma \leqq x\}$.

The following theorem contains the case $b \geqq 1$, $\frac{1}{2} \leqq a < 1$ as a particular case.

THEOREM 5. *If* $\beta_1, \beta_2, \cdots, \beta_n, \cdots$ *are mutually independent and identically distributed random variables for which*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\delta_n}{B_2 n^b} \leqq x\right\} = \mathbf{P}\{\delta \leqq x\} \tag{68}$$

where $b \geqq 1$ *and* $B_2 > 0$, *and if*

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = A_1 \tag{69}$$

in probability where $A_1 > 0$, *then we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\delta_{\rho(t)} A_1^b}{B_2 t^b} \leqq x\right\} = \mathbf{P}\{\delta \leqq x\}\,. \tag{70}$$

PROOF. By (8) and (69) it follows that

$$\lim_{t\to\infty} \frac{\rho(t)}{t} = \frac{1}{A_1} \tag{71}$$

in probability. Thus (70) immediately follows from Lemma 1.

In this case the asymptotic distribution of $\beta(t)$ is given by (13) where now $d = b$, $D_1 = 0$, $D_2 = B_2 A_1^{-b}$ and $\mathbf{P}\{\theta \leqq x\} = \mathbf{P}\{\delta \leqq x\}$.

THEOREM 6. *If* (α_n, β_n) $(n = 1, 2, \cdots)$ *are mutually independent and identically distributed vector variables for which* (23) *holds with* $\frac{1}{2} \leqq a < 1$ *and* $\frac{1}{2} \leqq b < 1$, *then*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{A_1\delta_{\rho(t)} - B_1 t}{A_1^{-d}t^d} \leqq x\right\} = Q(x) \tag{72}$$

exists where $d = \max(a, b)$,

$$\begin{aligned} Q(x) &= \mathbf{P}\{A_1 B_2 \delta \leqq x\} && \textit{for}\quad b > a\,, \\ &= \mathbf{P}\{A_1 B_2 \delta - B_1 A_2 \gamma \leqq x\} && \textit{for}\quad b = a\,, \\ &= \mathbf{P}\{-B_1 A_2 \gamma \leqq x\} && \textit{for}\quad b < a\,, \end{aligned} \tag{73}$$

and $\mathbf{P}\{\gamma \leqq x, \delta \leqq y\} = F(x, y)$.

PROOF. By (23) it follows that

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{A_1\delta_n - B_1\gamma_n}{n^d} \leqq x\right\} = Q(x) \tag{74}$$

where $d = \max(a, b)$ and $Q(x)$ is given by (73).

By (8) and (23) it follows that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\rho(t) - t/A_1}{A_2 A_1^{-(1+a)} t^a} \leqq x\right\} = \mathbf{P}\{-\gamma \leqq x\}\,, \tag{75}$$

and

$$\lim_{t\to\infty} \frac{\rho(t)}{t} = \frac{1}{A_1} \tag{76}$$

in probability. If we spply Lemma 1 to the random variables $\zeta(n) = A_1\delta_n - B_1\gamma_n$ $(n = 0, 1, 2, \cdots)$, and $\{\rho(t), 0 \leqq t < \infty\}$, then we obtain that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)}}{(t/A_1)^d} \leqq x\right\} = Q(x)\,. \tag{77}$$

It remains to show that (77) implies (72). This follows from the inequalities

$$A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)+1} \leqq A_1\delta_{\rho(t)} - B_1 t \leqq A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)} \tag{78}$$

for $t \geqq 0$ and from the fact that

$$\lim_{t\to\infty} \frac{\alpha_{\rho(t)+1}}{t^a} = 0 \tag{79}$$

in probability. The relation (79) follows from the inequality

$$\mathbf{P}\left\{\frac{\alpha_{\rho(t)+1}}{t^a} > \varepsilon\right\} \leqq \mathbf{P}\left\{\left|\rho(t) - \frac{t}{A_1}\right| > Kt^a\right\} + 2Kt^a\mathbf{P}\{\alpha_1 > t^a\varepsilon\} \tag{80}$$

which holds for $\varepsilon > 0$ and $K > 0$. Since $\mathbf{P}\{\alpha_1 \leqq x\}$ belongs to the domain of normal attraction of a stable distribution function with characteristic exponent $1/a$, it follows that

$$\lim_{t\to\infty} \mathbf{P}\{\alpha_1 > t^a\varepsilon\}(t^a\varepsilon)^{1/a} = c \tag{81}$$

where c is a nonnegative constant. ($c = 0$ if $a = \frac{1}{2}$.) This implies that the second term on the right-hand side of (80) tends to 0 as $t \to \infty$. If $t \to \infty$ and $K \to \infty$, then by (75) the first term on the right-hand side of (80) tends to 0. Since $\varepsilon > 0$ is arbitrary, this implies (79). This completes the proof of the theorem.

Now the asymptotic distribution of $\beta(t)$ is given by (13) where $d = \max(a, b)$, $D_1 = B_1$, $D_2 = 1/A_1^d$ and $\mathbf{P}\{\theta \leqq x\} = Q(x)$ is given by (73).

4. Examples. First, let us suppose that (23) holds with $a = b = 1/\alpha$ where $0 < \alpha < 1$, $A_1 = B_1 = 0$, $A_2 > 0$, $B_2 > 0$ and that

$$\Phi(s, q) = e^{-s^\alpha - q^\alpha} \tag{82}$$

for $\mathrm{Re}(s) \geqq 0$ and $\mathrm{Re}(q) \geqq 0$ where $\Phi(s, q)$ is defined by (28). Then by (31) we have

$$V(x) = \frac{x^\alpha}{1 + x^\alpha} \tag{83}$$

for $x \geqq 0$ and (29) holds with

$$\frac{dQ(x)}{dx} = \frac{x^\alpha \sin \alpha\pi}{\pi x(1 + 2x^\alpha \cos \alpha\pi + x^{2\alpha})} \tag{84}$$

for $x > 0$. This follows from (53). Thus by Theorem 2 we obtain that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\alpha(t)}{t} \leqq x\right\} = Q\left(\frac{B_2 x}{A_2(1 - x)}\right) \tag{85}$$

for $0 \leqq x \leqq 1$.

Second, let us suppose that (23) holds with $a = b = 1/\alpha$ where $0 < \alpha < 1$, $A_1 = B_1 = 0$, $A_2 > 0$, $B_2 > 0$ and that

$$\Phi(s, q) = e^{-(s+q)^\alpha} \tag{86}$$

for $\mathrm{Re}(s + q) \geqq 0$ where $\Phi(s, q)$ is defined by (28). Then by (31) we have

$$V(x) = 1 - \frac{1}{(1 + x)^\alpha} \tag{87}$$

for $x \geqq 0$ and (29) holds with

$$\begin{aligned} \frac{dQ(x)}{dx} &= \frac{\sin \alpha\pi}{\pi x(x - 1)^\alpha} \qquad \text{for} \quad x > 1\,, \\ &= 0 \qquad\qquad\qquad \text{for} \quad x \leqq 1\,. \end{aligned} \tag{88}$$

This follows from (53). Thus by Theorem 2 we obtain that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\alpha(t)}{t} \leqq x\right\} = Q\left(\frac{B_2 x}{A_2(1 - x)}\right) \tag{89}$$

for $0 \leqq x \leqq 1$.

We note that in the second example by (62) we obtain that

$$V^*(x) = \frac{x^\alpha}{(1 + x)^\alpha} \tag{90}$$

for $x \geqq 0$ and thus (60) holds with

$$\frac{dQ^*(x)}{dx} = \frac{\sin \alpha\pi}{\pi x^{1-\alpha}(1-x)^{\alpha}} \quad \text{for} \quad 0 < x < 1\,, \tag{91}$$
$$= 0 \quad \text{for} \quad x \geqq 1\,.$$

Now by Theorem 2 we obtain that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t)}{t} \leqq x\right\} = Q^*\left(\frac{A_2 x}{B_2(1-x)}\right) \tag{92}$$

for $0 \leqq x \leqq 1$. Of course (89) and (92) are merely different versions of the same limit theorem.

Third, let us suppose that (23) holds with $a = b = \frac{1}{2}$, $A_1 > 0$, $B_1 > 0$, $A_2 > 0$, $B_2 > 0$ and that $F(x, y)$ is a two-dimensional normal distribution function of type

$$N\left(\left\|\begin{matrix}0\\0\end{matrix}\right\|, \left\|\begin{matrix}1 & r\\ r & 1\end{matrix}\right\|\right)\,. \tag{93}$$

Then by (72) and (13) we obtain that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t) - \dfrac{B_1 t}{A_1 + B_1}}{\left(\dfrac{A_1}{A_1 + B_1}\right)^{\frac{3}{2}} t^{\frac{1}{2}}} \leqq x\right\} = \mathbf{P}\left\{\frac{A_1 B_2 \delta - B_1 A_2 \gamma}{A_1^{\frac{3}{2}}} \leqq x\right\} \tag{94}$$

where the random variables γ and δ have a two-dimensional normal distribution of type (93). By (94) we can write also that

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t) - M_1 t}{M_2 t^{\frac{1}{2}}} \leqq x\right\} = \Phi(x) \tag{95}$$

where $M_1 = B_1/(A_1 + B_1)$,

$$M_2 = \frac{(A_1^2 B_2^2 + B_1^2 A_2^2 - 2r A_1 B_1 A_2 B_2)^{\frac{1}{2}}}{(A_1 + B_1)^{\frac{3}{2}}}\,, \tag{96}$$

and $\Phi(x)$ is the normal distribution function of type $N(0, 1)$.

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