The Annals of Probability
1974, Vol. 2, No. 4, 729-737

SERIES OF RANDOM PROCESSES WITHOUT DISCONTINUITIES OF THE SECOND KIND

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For series of independent random processes in the space $D[0, 1]$ endowed with the Skorohod topology, convergence in distribution is shown to imply almost sure convergence. Under mild conditions, such as e.g. when the limiting process has no jumps of fixed size and location, the latter convergence is uniform. As an application, we discuss a representation by Ferguson and Klass of processes with independent increments.

0. Introduction. It is a well-known fact (see e.g. [7] page 251) that a series of independent random variables converges in distribution if and only if it is almost surely (a.s.) convergent. For series of random elements in Banach spaces and topological groups, results of the same type have been established by Tortrat (1965), Itô, Nisio (1968) and others. The purpose of the present paper is to prove extensions to series of random processes without discontinuities of the second kind. Applications are given in Section 2 below to a representation by Ferguson and Klass (1972) of processes with independent increments, and in [5], Theorem 2.1, to the canonical representation of processes on [0, 1] with interchangeable increments.

1. Main results. We shall restrict our attention to processes in $D[0, 1]$, since the extensions to $D[0, \infty)$ and $D(-\infty, \infty)$ are immediate from results in [6]. Let us therefore assume throughout this section that $X_1, X_2, \cdots$ are independent random elements in $D[0, 1]$ in the sense of [1], and define $S_n = X_1 + \cdots + X_n$, $n \in N$. For functions x in $D[0, 1]$ we put $||x|| = \sup_t |x(t)|$, and we define the moduli of continuity $w_x = w(x, \cdot)$ and $w_x' = w'(x, \cdot)$ as in [1], pages 109–110. It will further be convenient to write $x\{t\} = x(t) - x(t-)$ for the jump size of x at $t \in [0, 1]$. Convergence in probability will be denoted by $\rightarrow_P$, and we shall write $=_d$ and $\rightarrow_d$ for equality and convergence in distribution respectively, in $D[0, 1]$ always with respect to the Skorohod (J_1) topology [1]. Finally, we write 1_A for the indicator of the set A.

THEOREM 1. *In* $D[0, 1]$ *endowed with the Skorohod topology, convergence* a.s. *and in distribution of* S_n *are equivalent.*

This theorem can not be strengthened to the effect that Skorohod sense convergence in distribution imply a.s. uniform convergence, since this is clearly

Received January 2, 1973; revised January 31, 1974.
AMS 1970 *subject classifications.* Primary 60G99; Secondary 60J30.
Key words and phrases. Series of random processes, processes without discontinuities of the second kind, convergence almost surely and in distribution, Skorohod and uniform topologies, processes with independent increments.

not even true for nonrandom S_n. To see this, we may take X_n to be the indicator of the set $[\frac{1}{2} + 1/(n+1), \frac{1}{2} + 1/n)$. However, we shall be able to prove the (perhaps surprising) fact that the convergence in Theorem 1 is a.s. uniform, provided we exclude situations which are close to this simple example:

THEOREM 2. *If S_n converges in distribution with respect to the Skorohod topology in $D[0, 1]$, then S_n is* a.s. *uniformly convergent if and only if there does not exist any nonrandom sub-sequence $N' \subset N$ and real numbers $h > 0$ and $t, t_n, t_n' \in (0, 1)$, $n \in N'$, such that $t_n \to t$, $t_n' \to t$, $X_n\{t_n\} \to_P h$, $X_n\{t_n'\} \to_P -h$ as $n \to \infty$ through N'.*

It is sometimes possible to decide from the properties of the limiting process whether the convergence is uniform:

THEOREM 3. *Let S be a fixed process in $D[0, 1]$. The following statements are equivalent*:

(i) *Every sequence $\{S_n\}$ such that $S_n \to_d S$ in the Skorohod topology is* a.s. *uniformly convergent.*

(ii) *S has no jump in $(0, 1)$ of nonrandom size and location.*

Our proof of Theorem 1 is based on that of Theorem 2, which is therefore given first.

PROOF OF THEOREM 2. The necessity of our condition follows from the fact that $\|S_n - S\| \to 0$ implies $\|X_n\| \to 0$. Conversely, suppose that $S_n \to_d S$ in the Skorohod topology, and that our condition is fulfilled. Call a jump of modulus $> \varepsilon$ an ε-jump, and introduce for $\varepsilon, \delta > 0$ and $T \subset [0, 1]$ the $D[0, 1]$-sets

$$J_1(\varepsilon, T) = \{x : x \text{ has an } \varepsilon\text{-jump in } T\},$$
$$J_2(\varepsilon, \delta, T) = \{x : x \text{ has two } \varepsilon\text{-jumps in } T, < \delta \text{ apart}\},$$
$$J_2'(\varepsilon, \delta, T) = \{x : x \text{ has two } \varepsilon\text{-jumps } < \delta \text{ apart, one of which lies in } T\},$$
$$J_2(\varepsilon, \delta) = J_2(\varepsilon, \delta, [0, 1]).$$

The main difficulty consists in proving that

$$\lim_{\delta \to 0} \limsup_{m,n \to \infty} P\{S_m - S_n \in J_2(\varepsilon, \delta)\} = 0, \qquad \varepsilon > 0. \tag{1}$$

Anticipating the proof of (1), let $\varepsilon, \eta > 0$ be arbitrary, and choose $\delta > 0$ and $n_0 \in N$ such that

$$P\{S_m - S_n \in J_2(\varepsilon, \delta)\} \leq \eta, \qquad m, n > n_0, \tag{2}$$

$$P\{w'(S_n, \delta) \geq \varepsilon\} \leq \eta, \qquad n \in N, \tag{3}$$

which is possible by (1) and by [1], page 125. Let us further choose a partitioning $0 = t_0 < t_1 < \cdots < t_p = 1$ such that $S\{t_i\} = 0$ a.s., $i = 1, \cdots, p-1$, and

$$t_i - t_{i-1} \leq \delta, \qquad i = 1, \cdots, p. \tag{4}$$

For $i = 0, \cdots, p$ we then have $S_n(t_i) \to_d S(t_i)$ by [1], page 124, and hence, by the classical result for series of random variables ([7] page 251), $S_m(t_i) - S_n(t_i) \to_P 0$,

$m, n \to \infty$, so for some $n_0' \geqq n_0$,

(5) $$P\{|S_m(t_i) - S_n(t_i)| \leqq \varepsilon, i = 0, \cdots, p\} \geqq 1 - \eta , \qquad m, n > n_0' .$$

By (2), (3) and (5), we have with probability $\geqq 1 - 4\eta$ for fixed $m, n > n_0'$

(6) $$S_m - S_n \notin J_2(\varepsilon, \delta)$$

(7) $$w'(S_m, \delta) < \varepsilon , \qquad w'(S_n, \delta) < \varepsilon ,$$

(8) $$|S_m(t_i) - S_n(t_i)| \leqq \varepsilon , \qquad i = 0, 1, \cdots, p .$$

Consider an elementary event such that (6)—(8) are satisfied. For any $i \in \{1, \cdots, p\}$, there exist by (4) and (7) some $u, v \in (t_{i-1}, t_i]$ such that

(9) $$w_{S_m}[t_{i-1}, u) \leqq \varepsilon , \quad w_{S_m}[u, t_i] \leqq \varepsilon , \quad w_{S_n}[t_{i-1}, v) \leqq \varepsilon , \quad w_{S_n}[v, t_i] \leqq \varepsilon .$$

If $u \neq v$, it is seen from (6) that either $|(S_m - S_n)\{u\}| \leqq \varepsilon$ or $|(S_m - S_n)\{v\}| \leqq \varepsilon$, and in the first of these cases (the second is symmetric), we obtain from (9)

$$w_{S_m - S_n}[t_{i-1}, v) \leqq 5\varepsilon , \qquad w_{S_m - S_n}[v, t_i] \leqq 5\varepsilon ,$$

which is also true if $u = v$. Hence by (8) $||S_m - S_n|| \leqq 6\varepsilon$. This proves that $P\{||S_m - S_n|| > 6\varepsilon\} \leqq 4\eta$ for $m, n > n_0'$, and so $||S_m - S_n|| \to_P 0$ as $m, n \to \infty$. By considering some a.s. convergent sub-sequence, it follows easily that $||S' - S_n|| \to_P 0$ for some random element S' in $D[0, 1]$. To see that this is also true in the sense of a.s. convergence, let $n \to \infty$ and then $m \to \infty$ in the elementary inequality

$$P\{\max_{m<k\leqq n} ||S_k - S_m|| > 2\varepsilon\} \leqq \frac{P\{||S_n - S_m|| > \varepsilon\}}{1 - \max_{m<k\leqq n} P\{||S_n - S_k|| > \varepsilon\}} , \qquad m, n \in N, m < n ,$$

from [4], page 38, to conclude that $\sup_{k>m} ||S_m - S_k|| \to_P 0$ and hence $\sup_{k>m} ||S' - S_k|| \to_P 0$ as $m \to \infty$. It follows that, for any $\varepsilon > 0$,

$$\begin{aligned} P\{\limsup_{n\to\infty} ||S' - S_n|| > \varepsilon\} &= P \textstyle\bigcap_m \{\sup_{k>m} ||S' - S_k|| > \varepsilon\} \\ &= \lim_{m\to\infty} P\{\sup_{k>m} ||S' - S_k|| > \varepsilon\} = 0 , \end{aligned}$$

and since ε is arbitrary, we get $\limsup_{n\to\infty} ||S' - S_n|| = 0$ a.s. as desired.

Turning to the proof of (1), let us assume that (1) is false. Then there exist some $\varepsilon, \eta > 0$ and some $\delta_k > 0$ and $m_k, n_k \in N$, $k \in N$, such that $\delta_k \to 0$, $n_k > m_k \to \infty$ and

$$P\{S_{m_k} - S_{n_k} \in J_2(\varepsilon, \delta_k)\} \geqq \eta , \qquad k \in N .$$

Let $\varepsilon' < \varepsilon/100$, say, and choose $\delta > 0$ such that

$$P\{w'(S_n, \delta) \geqq \varepsilon'\} \leqq \eta/4 , \qquad n \in N .$$

Then

$$P\{S_{m_k} - S_{n_k} \in J_2(\varepsilon, \delta_k), w'(S_{m_k}, \delta) < \varepsilon', w'(S_{n_k}, \delta) < \varepsilon'\} \geqq \eta/2 , \qquad k \in N .$$

Hence there exists for each $q, k \in N$ some interval $T_{q,k}$ of the form $[(i - 1)2^{-q}, i2^{-q}]$, $i = 1, \cdots, 2^q$, such that

$$P\{S_{m_k} - S_{n_k} \in J_2'(\varepsilon, \delta_k, T_{q,k}), w'(S_{m_k}, \delta) < \varepsilon', w'(S_{n_k}, \delta) < \varepsilon'\} \geqq \eta 2^{-q-1} .$$

For $q = 1$, one of the two possible intervals, say T_1', must occur infinitely often, say for $k \in N_1'$. Proceeding recursively, we can construct a *decreasing* sequence $\{T_q'\}$ of intervals and a corresponding sequence $\{N_q'\}$ of N-subsequences, such that $T_{q,k} = T_q'$, $k \in N_q'$, $q \in N$. Expanding each T_q' symmetrically to an interval T_q of length 2^{-q+1}, and defining $N_q = \{k \in N_q' : \delta_k < 2^{-q-1}\}$, we get for $k \in N_q$, $q \in N$,

$$P\{S_{m_k} - S_{n_k} \in J_2(\varepsilon, \delta_k, T_q), w'(S_{m_k}, \delta) < \varepsilon', w'(S_{n_k}, \delta) < \varepsilon'\} \geqq \eta 2^{-q-1} . \tag{10}$$

In particular, we have by independence and tightness

$$\begin{aligned} P\{S_{m_k} \notin J_1(\varepsilon/2, T_q)\}\eta 2^{-q-1} &\leqq P\{S_{m_k} \notin J_1(\varepsilon/2, T_q), S_{m_k} - S_{n_k} \in J_2(\varepsilon, \delta_k, T_q)\} \\ &\leqq P\{S_{n_k} \in J_2(\varepsilon/2, \delta_k, T_q)\} \leqq P\{w'(S_{n_k}, \delta_k) > \varepsilon/2\} \to 0 \end{aligned}$$

as $k \to \infty$ through N_q, so

$$1 = \limsup_{n\to\infty} P\{S_n \in J_1(\varepsilon/2, T_q)\} \leqq P\{S \in J_1(\varepsilon/3, T_q)\} .$$

Since this holds for every $q \in N$, it follows that $|S\{u\}| > \varepsilon/3$ a.s., where $\{u\} = \bigcap_q T_q$, and hence for any open interval G containing u,

$$P\{S_n \in J_1(\varepsilon/3, G)\} \to 1 , \qquad n \to \infty . \tag{11}$$

From (11) we shall draw the (apparently much stronger) conclusion that, for some *nonrandom* $u_n \in [0, 1]$, $n \in N$, with $u_n \to u$,

$$P\{|S_n\{u_n\}| > \varepsilon/3\} \to 1 , \qquad n \to \infty . \tag{12}$$

To this end, let $\eta' > 0$ be arbitrary, and choose the G in (11) so small that

$$P\{w'(S_n, |G|) \geqq \varepsilon'\} \leqq \eta' , \qquad n \in N . \tag{13}$$

By (11), we may then choose (another) $n_0 \in N$ such that

$$P\{S_n \in J_1(\varepsilon/3, G)\} \geqq 1 - \eta' , \qquad n > n_0 . \tag{14}$$

Let us now consider any fixed $n > n_0$ and any open interval G' satisfying $u \in G' \subset G$. By (11), we may then choose $m > n$ such that

$$P\{S_m \in J_1(\varepsilon/3, G')\} \geqq 1 - \eta' . \tag{15}$$

Using (13) and (15), we get

$$\begin{aligned} P\{S_n \in J_1(\varepsilon/3, G\backslash G')\} &\leqq P\{S_n \in J_1(\varepsilon/3, G\backslash G'), S_m \in J_1(\varepsilon/3, G'), w'(S_m, |G|) < \varepsilon'\} + 2\eta' \\ &\leqq P\{S_m - S_n \in J_1(\varepsilon/4, G\backslash G')\} + 2\eta' , \end{aligned}$$

and also, by independence,

$$\begin{aligned} &P\{S_m - S_n \in J_1(\varepsilon/4, G\backslash G')\}P\{S_n \in J_1(\varepsilon/3, G')\} \\ &\quad \leqq P\{S_m - S_n \in J_1(\varepsilon/4, G\backslash G'), S_n \in J_1(\varepsilon/3, G'), S_m \in J_1(\varepsilon/3, G'), \\ &\qquad w'(S_m, |G|) < \varepsilon', w'(S_n, |G|) < \varepsilon'\} + 3\eta' = 3\eta' , \end{aligned}$$

since the last event is contradictory. From these two inequalities it follows by elimination of $P\{S_m - S_n \in J_1(\varepsilon/4, G\backslash G')\}$ that

$$P\{S_n \in J_1(\varepsilon/3, G')\}[P\{S_n \in J_1(\varepsilon/3, G\backslash G')\} - 2\eta'] \leqq 3\eta' ,$$

and hence

(16) $$P\{S_n \in J_1(\varepsilon/3, G')\} \wedge P\{S_n \in J_1(\varepsilon/3, G\backslash G')\} \leqq 2\eta' + (3\eta')^{\frac{1}{2}} < \eta'' ,$$

where by definition $\eta'' = 3\eta' + (3\eta')^{\frac{1}{2}}$. On the other hand, we have by (14)

$$P\{S_n \in J_1(\varepsilon/3, G')\} + P\{S_n \in J_1(\varepsilon/3, G\backslash G')\} \geqq P\{S_n \in J_1(\varepsilon/3, G)\} \geqq 1 - \eta' ,$$

and so, by combination,

(17) $$P\{S_n \in J_1(\varepsilon/3, G')\} \vee P\{S_n \in J_1(\varepsilon/3, G\backslash G')\} \geqq 1 - 3\eta' - (3\eta')^{\frac{1}{2}} = 1 - \eta'' .$$

Varying the endpoints of G' continuously over G, it is easily seen from (16) and (17), that for some $u_n \in G$, $n > n_0$,

(18) $$P\{|S_n\{u_n\}| > \varepsilon/3\} \geqq 1 - 2\eta'' , \qquad n > n_0 .$$

Since η' and G can be chosen arbitrarily small (subject to the restriction (13)), we may assume that $u_n \to u$ and that the probability in (18) tends to 1, which completes the proof of (12). If $u_n = u$ for all sufficiently large n, it would follow from (10) and (12) that, for fixed $q \in N$ with $2^{-q+1} < \delta$ and for $k \in N_q$,

$$\begin{aligned} \eta 2^{-q-1} &\leqq P\{S_{m_k} - S_{n_k} \in J_1(\varepsilon, T_q\backslash\{u\}), w'(S_{m_k}, \delta) < \varepsilon', w'(S_{n_k}, \delta) < \varepsilon'\} \\ &\leqq P\{S_{m_k} \in J_1(\varepsilon/2, T_q\backslash\{u\}), w'(S_{m_k}, \delta) < \varepsilon'\} \\ &\qquad + P\{S_{n_k} \in J_1(\varepsilon/2, T_q\backslash\{u\}), w'(S_{n_k}, \delta) < \varepsilon'\} \\ &\leqq P\{|S_{m_k}\{u\}| < \varepsilon'\} + P\{|S_{n_k}\{u\}| < \varepsilon'\} \to 0 , \qquad k \to \infty , \end{aligned}$$

which is impossible. Hence $u_{n-1} \neq u_n$ for infinitely many n, say for $n \in N'$, and moreover, $u \in (0, 1)$, as is easily seen from the tightness of $\{S_n\}$.

Next consider any closed set $F \subset R$, let $\varepsilon', \eta' > 0$ be arbitrary, and choose a neighborhood G of u satisfying (13). For any S-continuity points $s, t \in G$ with $0 \leqq s < u < t \leqq 1$, we get by (12) and (13), writing $F_{2\varepsilon'} = \{x \in R : \inf_{y \in F} |x - y| \leqq 2\varepsilon'\}$,

$$\begin{aligned} \limsup_{n\to\infty} P\{S_n\{u_n\} \in F\} &\leqq \limsup_{n\to\infty} P\{S_n(t) - S_n(s) \in F_{2\varepsilon'}\} + \eta' \\ &\leqq P\{S(t) - S(s) \in F_{2\varepsilon'}\} + \eta' , \end{aligned}$$

and letting $s, t \to u$ and then $\varepsilon', \eta' \to 0$, the last member tends to $P\{S\{u\} \in F\}$. By [1], page 11, this proves that $S_n\{u_n\} \to_d S\{u\}$, and hence for the corresponding characteristic functions, $\varphi_n \to \varphi$. By (12) and the tightness of $\{S_n\}$, it follows easily that

(19) $$S_{n-1}\{u_{n-1}\} + X_n\{u_{n-1}\} = S_n\{u_{n-1}\} \to_P 0 , \qquad n \in N' ,$$

(20) $$S_n\{u_n\} - X_n\{u_n\} = S_{n-1}\{u_n\} \to_P 0 , \qquad n \in N' .$$

By independence, (19) implies $|\varphi_{n-1}| \to 1$, $n \in N'$, and so $|\varphi| \equiv 1$, which means that $S\{u\} = h \neq 0$ is nonrandom. But then $S_n\{u_n\} \to_P h$, so by (19) and (20)

$$X_n\{u_{n-1}\} \to_P -h , \qquad X_n\{u_n\} \to_P h , \qquad n \in N' .$$

This contradicts the hypothesis of the theorem, so (1) must be true, and the proof is completed.

PROOF OF THEOREM 3. Suppose that S satisfies (ii), and let $S_n \to_d S$. Then proceed as above up to the point where we conclude that $S\{u\} = h$ is nonrandom. This contradiction of (ii) proves (1), and it follows that S_n is a.s. uniformly convergent. The converse is easily proved by modifying the counterexample following Theorem 1.

PROOF OF THEOREM 1. Suppose that $S_n \to_d S$. Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary and let $u_1, \cdots, u_p$ be the (finitely many) values of $t \in (0, 1)$ for which the hypothesis of Theorem 2 is violated with $h > \varepsilon$. For fixed $j \in \{1, \cdots, p\}$ and any open set G containing u_j, we get

$$\lim_{\delta\to0} \limsup_{m,n\to\infty} P\{S_m - S_n \in J_2(\varepsilon, \delta, G)\} = 1 .$$

Proceeding as in the proof of Theorem 2, we obtain $|S\{u\}| > \varepsilon/3$ a.s. for some $u \in G$, and since G was arbitrary, we may assume that $u = u_j$. Furthermore, $S\{u_j\} = h_j$ is nonrandom, and for some $u_{nj} \in (0, 1)$, $n \in N$, with $u_{nj} \to u_j$, we have $S_n\{u_{nj}\} \to_P h_j$. Let us define

$$S_n' = S_n + \textstyle\sum_{j=1}^p h_j 1_{[u_j, u_{nj})} , \qquad n \in N ,$$

$$X_1' = S_1' , \qquad X_n' = S_n' - S_{n-1}' = X_n + \textstyle\sum_{j=1}^p h_j 1_{[u_{n-1,j}, u_{nj})} , \qquad n = 2, 3, \cdots ,$$

where $1_{[t,s)} = -1_{[s,t)}$ for $t \geqq s$. Then $X_1', X_2', \cdots$ are independent processes in $D[0, 1]$ with partial sums $S_1', S_2', \cdots$. For any $\varepsilon', \delta > 0$, we easily obtain

$$P\{w'(S_n', \delta) \geqq \varepsilon'\} \leqq \textstyle\sum_{j=1}^p P\{|S_n\{u_{nj}\} - h_j| \geqq \varepsilon'/5\} + P\{w'(S_n, \delta + 2 \max_j |u_{nj} - u_j|) \geqq \varepsilon'/5\} ,$$

and since $\{S_n\}$ is tight, it follows by Theorem 15.2 in [1] that

$$\lim_{\delta\to0} \limsup_{n\to\infty} P\{w'(S_n', \delta) \geqq \varepsilon'\} \leqq \lim_{\delta\to0} \limsup_{n\to\infty} P\{w'(S_n, \delta) \geqq \varepsilon'/5\} = 0 , \qquad \varepsilon' > 0 ,$$

proving tightness of $\{S_n'\}$. Moreover, for any $t \notin \{u_1, \cdots, u_q\}$, we have $S_n'(t) = S_n(t)$ for all but finitely many $n \in N$, so $S_{n'} \to_d S$ by Theorem 15.1 in [1]. Next verify that $S_n'\{u_j\} \to_P h_j$ as $n \to \infty$, by considering separately the n-values with $u_{nj} = u_j$ and those with $u_{nj} \neq u_j$. If the hypothesis of Theorem 2, with $\{X_n'\}$ in place of $\{X_n\}$, were violated for some $t \in (0, 1)$ and $h > \varepsilon$, then clearly $t = u_j$ for some j, and we would get $S_n'\{u'_{nj}\} \to_P h_j$ as before for some $u'_{nj} \to u_j$ with $u'_{nj} \neq u_j$ infinitely often, contradicting the tightness of $\{S_n'\}$. Therefore the hypothesis must hold for $\{X_n'\}$, except possibly for h-values $\leqq \varepsilon$. Arguing as in the proof of Theorem 2, we may conclude successively that

$$\lim_{\delta\to0} \limsup_{m,n\to\infty} P\{S_m' - S_n' \in J_2(3\varepsilon, \delta)\} = 0 ,$$

$$\lim_{m,n\to\infty} P\{||S_m' - S_n'|| > 18\varepsilon\} = 0 ,$$

$$\lim_{m\to\infty} P\{\sup_{n>m} ||S_n' - S_m'|| > 36\varepsilon\} = 0 . \tag{21}$$

For arbitrary $\eta > 0$, let $\delta > 0$ be so small that the intervals $I_j \doteq [u_j - \delta, u_j + \delta]$, $j = 1, \cdots, p$, are disjoint and contained in $[0, 1]$, and such that also

$$P\{w'(S_n', \delta) \geqq \varepsilon/6\} \leqq \eta , \qquad n \in N . \tag{22}$$

Then choose $m \in N$ so large that

$$|u_{nj} - u_j| \leqq \varepsilon\delta\,, \qquad n \geqq m, j = 1, \cdots, p\,, \tag{23}$$

and such that with probability $\geqq 1 - \eta$,

$$|S_m'\{u_j\} - h_j| \leqq \varepsilon/6\,, \qquad j = 1, \cdots, p\,, \tag{24}$$

$$||S_n' - S_m'|| \leqq 37\varepsilon\,, \qquad n > m\,, \tag{25}$$

the latter being possible by (21). By (22), the relations (24), (25) and

$$w'(S_m', \delta) < \varepsilon/6 \tag{26}$$

hold simultaneously with probability $\geqq 1 - 2\eta$. Consider an elementary event satisfying (24)—(26), and define

$$S_n'' = S_n' - \sum_{j=1}^p h_j 1_{[u_j,1]} = S_n - \sum_{j=1}^p h_j 1_{[u_{nj},1]}\,, \qquad n \in N\,.$$

For $j = 1, \cdots, p$, it follows from (24), (26) and the relation $|h_j| > \varepsilon/3$ that $w(S_m'', I_j) < \varepsilon/2$, and hence by (25), for any $s, t \in I_j$ and $n > m$,

$$|S_m''(s) - S_n''(t)| \leqq |S_m''(s) - S_m''(t)| + |S_m''(t) - S_n''(t)| \leqq \varepsilon/2 + 37\varepsilon < 38\varepsilon\,.$$

For $n \geqq m$ we next define the continuous mapping λ_n from [0, 1] onto itself by

$$\lambda_n(0) = 0\,, \quad \lambda_n(1) = 1\,, \quad \lambda_n(u_j) = u_{nj}\,, \quad \lambda_n(u_j \pm \delta) = u_j \pm \delta\,, \quad j = 1, \cdots, p\,, \tag{28}$$

and the requirement that λ_n be linear on the intervals between these $3p + 2$ points. By (25) and (27) we get

$$||S_m \circ \lambda_m - S_n \circ \lambda_n|| = ||S_m'' \circ \lambda_m - S_n'' \circ \lambda_n|| \leqq 38\varepsilon\,, \tag{29}$$

since for t outside all the I_j,

$$S_m'' \circ \lambda_m(t) - S_n'' \circ \lambda_n(t) = S_m''(t) - S_n''(t) = S_m'(t) - S_n'(t)\,.$$

Furthermore, we have by (23) and (28), for any $n \geqq m$,

$$\sup_{s \neq t}\left|\log \frac{\lambda_n(t) - \lambda_n(s)}{t - s}\right| \leqq \left|\log \frac{\delta + \varepsilon\delta}{\delta}\right| \vee \left|\log \frac{\delta - \varepsilon\delta}{\delta}\right| = -\log(1 - \varepsilon) < 2\varepsilon\,,$$

so by [1], page 113,

$$\sup_{s \neq t}\left|\log \frac{\lambda_m \lambda_n^{-1}(t) - \lambda_m \lambda_n^{-1}(s)}{t - s}\right| < 4\varepsilon\,, \qquad n > m\,. \tag{30}$$

Defining the metric d_0 in $D[0, 1]$ as in [1], page 113, it follows from (29) and (30) that $d_0(S_m, S_n) \leqq 38\varepsilon$ for $n > m$, so we have in fact proved that

$$P\{\sup_{n>m} d_0(S_m, S_n) > 38\varepsilon\} \leqq 2\eta\,.$$

Since η was arbitrary, we get

$$\begin{aligned} P\{\limsup_{m,n\to\infty} d_0(S_m, S_n) > 76\varepsilon\} &= \lim_{k\to\infty} P\{\sup_{m,n>k} d_0(S_m, S_n) > 76\varepsilon\} \\ &\leqq 2 \lim_{k\to\infty} P\{\sup_{m>k} d_0(S_m, S_k) > 38\varepsilon\} = 0\,, \end{aligned}$$

and since even ε was arbitrary, it follows that $d_0(S_m, S_n) \to 0$ a.s. as $m, n \to \infty$. The space $D[0, 1]$ being complete in d_0, this completes the proof.

2. On a representation of independent increment processes. Let X be a separable infinitely divisible process on $[0, 1]$ with independent increments, and suppose that X is right continuous except possibly at the fixed jumps (cf. [7] page 540). It has been shown by Ferguson and Klass (1972) that the jump part of X is distributed as the sum of certain suitably centered one-jump processes, the sum being interpreted in the sense of pointwise a.s. convergence. The authors conjecture that the convergence holds with probability one, simultaneously on the whole interval. Indeed, we shall show that the convergence is a.s. uniform. Furthermore, it will be shown that, if X has no fixed jumps, the series may be chosen so as to represent X with probability one.

As in [2] it suffices to consider the component of X corresponding to the positive jumps, so let us assume that

$$\log Ee^{iuX(t)} = \int_0^\infty \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right)\lambda_t(dz), \qquad u \in R,\ t \in [0, 1], \tag{31}$$

where $\{\lambda_t\}$ has the properties stated (for $\{N_t\}$) in [2]. By the relations

$$\lambda([0, t) \times dz) \equiv \lambda_{t-}(dz), \qquad \lambda([0, t] \times dz) \equiv \lambda_{t+}(dz),$$

we may define a measure λ on the space $T \times (0, \infty)$, where T is obtained from $[0, 1]$ by counting the fixed discontinuity points of X twice. (Hence, each fixed discontinuity point $s \in [0, 1]$ splits into two points in T which we denote by s and $s+$.) Now consider a Poisson process on $T \times (0, \infty)$ with intensity λ, and denote its unit atom positions by (τ_j, β_j), where we assume that $\beta_1 \geqq \beta_2 \geqq \cdots > 0$ and that the order among atoms with the same β_j is determined at random. Furthermore, define the centering functions c_j, $j \in N$, as in [2], page 1641, or roughly by

$$c_j(t) = \int_{z_j}^{z_{j-1}} \frac{z}{1+z^2}\,\lambda_t(dz), \qquad t \in [0, 1], \tag{32}$$

where the z_j are determined, if possible, by $\lambda_1[z_j, \infty) = j$, and otherwise by a suitable linear interpolation. Put $1_+ = 1_{R_+}$ and interpret $1_+(s - (s+))$ as 0.

PROPOSITION. *The series*

$$Y(t) = \sum_j \{\beta_j 1_+(t - \tau_j) - c_j(t)\}, \qquad t \in [0, 1],$$

is a.s. *uniformly convergent with the same finite-dimensional distributions as* X. *If* X *is continuous in probability, we may choose the* β_j *and* τ_j *as the jump sizes and positions, and then* $X = Y$ a.s.

PROOF. To prove the first assertion, it suffices by the arguments in [2] to prove the a.s. uniform convergence of the sequence $\{V^{(n)}\}$ defined there. Since X has at most countably many fixed jumps, we may replace X in the proof by a process in $D[0, 2]$ obtained by separating the left- and right-hand jumps by intervals of length 2^{-j}, $j \in N$, so we can assume from the beginning that X lies in $D[0, 1]$.

Clearly, X has no jumps of fixed size and location, so it suffices by Theorem 3 to prove that $V^{(n)} - V^{(1)} \to_d X - V^{(1)}$. But this is equivalent to proving that, after a suitable truncation of λ, $V^{(n)} \to_d X$. We may therefore assume that λ has bounded support, in which case all moments exist and the denominator $1 + z^2$ in (31) and (32) may be replaced by 1. Now convergence of the finite-dimensional distributions is a standard fact ([7] page 300), and since

$$F(t) \equiv \text{Var}\, X(t) = \textstyle\int_0^\infty z^2 \lambda_t(dz)$$

is bounded and non-decreasing, the tightness of $\{V^{(n)}\}$ follows by [1], page 133, from the fact that, for any $r < s < t$,

$$\begin{aligned} E(V_s^{(n)} - V_r^{(n)})^2 (V_t^{(n)} - V_s^{(n)})^2 &= E(V_s^{(n)} - V_r^{(n)})^2 E(V_t^{(n)} - V_s^{(n)})^2 \\ &= \text{Var}\,(V_s^{(n)} - V_r^{(n)})\,\text{Var}\,(V_t^{(n)} - V_s^{(n)}) \\ &\leqq \{F(s) - F(r)\}\{F(t) - F(s)\}\,. \end{aligned}$$

To prove the second assertion, note that the point process on $[0, 1] \times (0, \infty)$, whose atom positions are determined by the jump positions and sizes of X, is actually a Poisson process with intensity λ ([7] page 550). To see that $X = Y$ a.s., it suffices by right continuity to show that $X_t = Y_t$ a.s. for each fixed $t \in [0, 1]$. But this follows from the fact that $V_t^{(n)} \to X(t)$ in L_2.

REMARK. The fact that the processes $V^{(n)}$ of [2] are a.s. uniformly convergent is essentially a classical result, due to Itô (1942). The above proof is given merely to illustrate the usefulness of our general theorems.

Acknowledgment. I would like to thank the referee for several useful suggestions.

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