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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE[1]

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Let $\{\xi_k : k \geqq 1\}$ be a sequence of independent, identically distributed random variables with $E\{\xi_1\} = 0$ and $E\{\xi_1^2\} = \sigma^2$, $0 < \sigma^2 < \infty$. Form the random walk $\{S_n : n \geqq 0\}$ by setting $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, $n \geqq 1$. Let T denote the hitting time of the set $(-\infty, 0]$ by the random walk. The main result in this paper is a functional central limit theorem for the random functions $S_{[nt]}/\sigma n^{\frac{1}{2}}$, $0 \leqq t \leqq 1$, conditional on $T > n$. The limit process, W^+, is identified in terms of standard Brownian motion. Similar results are obtained for random partial sums and renewal processes. Finally, in the case where $E\{\xi_1\} = \mu > 0$, it is shown that the conditional (on $T > n$) and unconditional weak limit for $(S_{[nt]} - \mu nt)/\sigma n^{\frac{1}{2}}$ is the same, namely, Brownian motion.

1. Introduction. Let $\{\xi_k : k \geqq 1\}$ be a sequence of independent, identically distributed random variables with $E\{\xi_1\} = 0$ and $E\{\xi_1^2\} = \sigma^2$, $0 < \sigma^2 < \infty$. Form the random walk $\{S_n : n \geqq 0\}$ by setting $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geqq 1$. Define the random function X_n by

$$X_n(t) = S_{[nt]}/\sigma n^{\frac{1}{2}}\,, \qquad 0 \leqq t \leqq 1\,,$$

where $[x]$ is the greatest integer in x. Next let T be the hitting time of the set $(-\infty, 0]$ by the random walk:

$$T = \inf\{n > 0 : S_n \geqq 0\}\,,$$

where the infimum of the empty set is taken to be $+\infty$.

Our goal in this paper is to obtain a functional central limit theorem (f.c.l.t.), or so-called invariance principle, for the random function X_n, conditioned on $T > n$.

To be more specific we assume that $\{\xi_k : k \geqq 1\}$ are the coordinate functions defined on the product space $(\Omega, \mathscr{F}, P)$. If $\Lambda_n = \{T > n\}$, then we let $(\Lambda_n, \Lambda_n \cap \mathscr{F}, P_n)$ be the trace of $(\Omega, \mathscr{F}, P)$ on Λ_n, where $\Lambda_n \cap \mathscr{F} = \{\Lambda_n \cap F : F \in \mathscr{F}\}$ and $P_n(A) = P(A)/P(\Lambda_n)$ for $A \in \Lambda_n \cap \mathscr{F}$. The expectation with respect to P_n is denoted by $E_n\{\cdot\}$. Next let $D \equiv D[0, 1]$ be the space of real-valued, right-continuous functions on $[0, 1]$ having left limits and $\mathscr{D}$ be the σ-field of Borel sets generated by the open sets of the Skorohod J_1-topology. Let

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$D_+ = \{x \in D : x \geqq 0\}$, and $\mathscr{D}_+ = D_+ \cap \mathscr{D}$. The measurable mapping X_n^+: $(\Lambda_n, \Lambda_n \cap \mathscr{F}) \to (D_+, \mathscr{D}_+)$ is defined by

$$X_n^+(\cdot, \omega) = S_{[n\cdot]}(\omega)/\sigma n^{\frac{1}{2}}, \qquad \omega \in \Lambda_n .$$

The random function X_n^+ induces a p.m. μ_n^+ on $\mathscr{D}_+$: for $A \in \mathscr{D}_+$,

$$\mu_n^+(A) = P_n\{X_n^+ \in A\} = P\{X_n^+ \in A\}/P\{\Lambda_n\} .$$

The main result of this paper is that $\mu_n^+ \Rightarrow \mu^+$, the p.m. of a random function W^+, if $\mu_3 = E\{|\xi_1|^3\}$ is finite and ξ_1 is nonlattice or integer-valued with span 1; the symbol $\Rightarrow$ means weak convergence. Alternatively, we write $X_n^+ \Rightarrow W^+$ for this result. The finite-dimensional distribution (f.d.d.'s) of W^+ will be given explicitly and W^+ identified in terms of Brownian motion W. Roughly speaking, W^+ is the first-half of the absolute value of a Brownian excursion. This result for conditioned partial sums can easily be converted into a result for conditioned renewal processes using the methods of Vervaat (1972), or alternatively Iglehart and Whitt (1971). We also give a conditioned f.c.l.t. for random partial sums.

These results have an application to queueing theory, as well as the obvious interpretation as the fortune of a gambler or insurance company prior to ruin. For the general single server queue the waiting time of the nth customer, W_n, can be expressed in terms of the random walk as

$$W_n = \max\{S_n - S_r : 0 \leqq r \leqq n\}, \qquad n \geqq 0 ;$$

see [9] for this relation as well as other aspects of the sample-path behavior of $\{W_n : n \geqq 0\}$. In this context T is the number of customers served in the first busy-period. Observe that $W_n = S_n$ on the set $\{T > n\}$. Thus the result $X_n^+ \Rightarrow W^+$ can be thought of as a limit law for the waiting time process in the first busy period of a general single server queue with traffic intensity equal to 1.

The ordinary central limit theorem corresponding to $X_n^+ \Rightarrow W^+$ is

$$(1.1) \qquad \lim_{n\to\infty} P\{S_n/\sigma n^{\frac{1}{2}} \leqq x \mid T > n\} = 1 - \exp(-x^2/2), \qquad x \geqq 0 .$$

This result (without the finite third absolute moment condition) was announced by Spitzer (1960), page 162, in a footnote "added in proof." Apparently the proof of (1.1) was never published. The form of the limit distribution of (1.1) was given in Dwass and Karlin (1963), equation (52) page 1159 and page 1160, under the assumption that the limit exists. A complete proof of (1.1) is given here in Proposition 2.1. Note that the limit is the Rayleigh distribution.

Related results have previously been obtained by Belkin (1970), (1972) and Stone (1971). Belkin considers an integer-valued random walk in the domain of attraction of a stable law and conditions on the walk not having entered a finite set of integers. The proof of tightness given here follows that of Belkin; however, the W_n variables must be used rather than the S_n's. Port and Stone consider infinitely divisible processes on a locally compact Abelian group and condition on not having entered a compact set.

This paper is organized in the following manner. Section 2 contains a proof of the convergence of the f.d.d.'s of X_n^+ to those of W^+. In Section 3 the sequence $\{X_{n+} : n \geqq 1\}$ is shown to be tight and hence converge weakly to W^+. Also the identification of W^+ in terms of Brownian motion is given in Section 3. Conditioned random partial sum and conditioned renewal processes are treated in Section 4. Finally, in Section 5 a simple proof is given which shows that when $\mu = E\{\xi_1\} > 0$ the conditioned (on $\{T > n\}$) and unconditioned limit of $(S_{[nt]} - \mu nt)/\sigma n^{\frac{1}{2}}$ is the same, namely Brownian motion W.

2. Convergence of finite-dimensional distributions. Our goal in this section is to show that the finite-dimensional distributions (f.d.d.'s) of the random functions X_n^+ converge to those of the random function W^+. The first step in this direction is the ordinary central limit theorem.

(2.1) PROPOSITION. *If $\mu = 0$, $0 < \sigma^2 < \infty$, $E\{|\xi_1|^3\} < \infty$, and ξ_1 is nonlattice or integer-valued with span 1, then for all $x \geqq 0$*

$$\lim_{n\to\infty} P_n\{X_n^+(1) \leqq x\} = 1 - \exp(-x^2/2)\,. \tag{2.2}$$

PROOF. First recall that $S_n = W_n$ on $\{T > n\}$. So if we can show (2.2) with $X_n^+(1)$ replaced by $W_n/\sigma n^{\frac{1}{2}}$, the result will be established. Let $f_n = P\{T = n\}$ and $r_n = P\{T > n\}$. A simple path decomposition yields

$$P_n\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} = r_n^{-1}P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} - r_n^{-1} \sum_{k=1}^{n} P\{W_n/\sigma n^{\frac{1}{2}} \leqq x, T = k\}\,. \tag{2.3}$$

Using the fact that T is an almost everywhere finite, optional random variable and that $W_T = 0$, we can write after a simple manipulation

$$\begin{aligned} P_n\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} &= P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} - \textstyle\sum_{k=1}^{n} (f_k/r_n)[P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} \\ &\quad - P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\}]\,. \end{aligned} \tag{2.4}$$

It is well-known that when $\mu = 0$ and $0 < \sigma^2 < \infty$

$$r_n \sim cn^{-\frac{1}{2}} \qquad \text{as } n \to \infty\,, \tag{2.5}$$

where c is a positive constant whose precise value need not concern us; see Spitzer (1960), Theorem 3.5, for this result. Furthermore, under the hypotheses of (2.1)

$$f_n \sim (c/2)n^{-\frac{3}{2}} \qquad \text{as } n \to \infty\,; \tag{2.6}$$

This result is contained in Borovkov (1970), Corollary 9. Hence, combining (2.5) and (2.6) we see that for an arbitrary $\varepsilon > 0$, $f_k/r_n \sim (\frac{1}{2})(k/n)^{-\frac{3}{2}}n^{-1}$ as $n \to \infty$, uniformly for $[\varepsilon n] \leqq k \leqq n$. If we let $M_n = \max\{S_r : 0 \leqq r \leqq n\}$, then W_n and M_n are known to have the same distribution; cf. Feller (1971), page 198. Also when $\mu = 0$ and $0 < \sigma^2 < \infty$, $M_n/\sigma n^{\frac{1}{2}} \Rightarrow |N|$, the positive normal with density $(2/\pi)^{\frac{1}{2}} \exp\{-x^2/2\}$, $x \geqq 0$. If in addition $E\{|\xi_1|^3\} < \infty$, Nagaev (1969) page 443, has obtained the Berry–Esséen type bound

(2.7) $$\sup_{x\geq 0}|P\{M_n/\sigma n^{\frac{1}{2}} \leqq x\} - |N|(x)| \leqq Kn^{-\frac{1}{2}}$$

for all $n \geqq 1$, where $|N|(x) = (2/\pi)^{\frac{1}{2}} \int_0^x \exp(-u^2/2)\,du$ and K is a finite, positive constant. We shall in general use K for such a constant without further mention.

Returning to (2.4), the first term on the right-hand side of the equation converges to $|N|(x)$. Hence, to obtain (2.2) we must show that the sum in (2.4) converges to $\exp(-x^2/2) + |N|(x) - 1$. Select $\varepsilon > 0$ and consider the sum

(2.8) $$\sum_{k=[\varepsilon n]}^{[(1-\varepsilon)n]} (f_k/r_n)[P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} - P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\}] .$$

Using (2.7), we see that this sum is majorized for n sufficiently large by the expression

(2.9) $$[(1+\varepsilon)/2] \sum_{k=[\varepsilon n]}^{[(1-\varepsilon)n]} (k/n)^{-\frac{3}{2}} \times [|N|(x[1-k/n]^{-\frac{1}{2}}) - |N|(x) + K([n-k]^{-\frac{1}{2}} + n^{-\frac{1}{2}})] \cdot \frac{1}{n} .$$

The principal term of this sum is the Riemann approximating sum for the integral

(2.10) $$[(1+\varepsilon)/2] \int_\varepsilon^{1-\varepsilon} v^{-\frac{3}{2}}[|N|(x[1-v]^{-\frac{1}{2}}) - |N|(x)]\,dv .$$

The remainder term of (2.9) is dominated by

(2.11) $$[(1+\varepsilon)/2] \int_\varepsilon^{1-\varepsilon} v^{-\frac{3}{2}}\,dv \cdot O(n^{-\frac{1}{2}}) .$$

Combining (2.10) and (2.11) we see that for n large (2.8) is majorized by a term which is arbitrarily close to the finite integral

(2.12) $$(2\pi)^{-\frac{1}{2}} \int_\varepsilon^{1-\varepsilon} v^{-\frac{3}{2}}\,dv \int_x^{x(1-v)^{-\frac{1}{2}}} \exp(-u^2/2)\,du ,$$

for n sufficiently large. The fact that (2.12) provides an arbitrarily close lower bound for (2.8) is shown in the same way. A simple interchange of integrals shows that

(2.13) $$(2\pi)^{-\frac{1}{2}} \int_0^1 v^{-\frac{3}{2}}\,dv \int_x^{x(1-v)^{-\frac{1}{2}}} \exp(-u^2/2)\,du = \exp(-x^2/2) + |N|(x) - 1 .$$

Thus, we will have completed the proof of (2.2) if we can show that

(2.14) $$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} |\sum_{k=1}^{[\varepsilon n]} + \sum_{k=[(1-\varepsilon)n]+1}^{n} (f_k/r_n) \times [P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} - P\{W_n/\sigma n^{\frac{1}{2}}\} \leqq x]| = 0 .$$

First take the sum $\sum_{k=[(1-\varepsilon)n]+1}^{n}$. It is easily bounded by $2[r_{[(1-\varepsilon)n]} - r_n]/r_n$ which converges to $(1-\varepsilon)^{-\frac{1}{2}} - 1$ as $n \to \infty$. Consequently,

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} |\sum_{k=[(1-\varepsilon)n]+1}^{n}| = 0 .$$

To handle the sum $\sum_{k=0}^{[\varepsilon n]}$ we first select M so large that for $M \leqq k \leqq [\varepsilon n]$ $(f_k/r_n) \leqq (k/n)^{-\frac{3}{2}} \cdot n^{-1}$ and $\sum_{k=M}^{\infty} k^{-\frac{3}{2}} < \varepsilon$. So we obtain for n large

$$\sum_{k=M}^{[\varepsilon n]} (f_k/r_n)|P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} - P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\}| \leqq \sum_{k=M}^{[\varepsilon n]} (k/n)^{-\frac{3}{2}}[|N|(x[1-k/n]^{-\frac{1}{2}}) - |N|(x) + K([n-k]^{\frac{1}{2}} + n^{-\frac{1}{2}})] \cdot \frac{1}{n} .$$

The term

$$\sum_{k=M}^{[\varepsilon n]} (k/n)^{-\frac{3}{2}}\left[|N|\left(x\left[1-\frac{k}{n}\right]^{-\frac{1}{2}}\right)-|N|(x)\right]\cdot\frac{1}{n}$$

is the Riemann approximating sum for

$$(2/\pi)^{\frac{1}{2}} \int_0^\varepsilon v^{-\frac{3}{2}} \int_x^{x(1-v)^{-\frac{1}{2}}} \exp(-u^2/2)\, du$$

which can be made arbitrarily small by selecting ε small by virtue of the finiteness of the integral in (2.13). The term

$$\sum_{k=M}^{[\varepsilon n]} (k/n)^{-\frac{3}{2}}([n-k]^{-\frac{1}{2}}+n^{-\frac{1}{2}})\cdot\frac{1}{n}$$

is easily bounded by a finite positive constant times

$$\sum_{k=M}^{\infty} k^{-\frac{3}{2}} < \varepsilon$$

by the selection of M. This disposes of the term $\sum_{k=M}^{[\varepsilon n]}$ in (2.14).

To treat the term $\sum_{k=0}^{M}$ in (2.14) we need to establish for $x > 0$, $1 \leqq k \leqq M$, and $n \geqq 1$ the bound

$$[P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} - P\{W_{n-k+1}/\sigma n^{\frac{1}{2}} \leqq x\}] = a_{n,k} \leqq K/n^{1-\delta/2} \tag{2.15}$$

where $0 < \delta < 1$ may be arbitrarily small. Using the fact that W_n and M_n have the same distribution, it is easy to see that

$$a_{n,k} = P\{M_{n-k} \leqq \sigma n^{\frac{1}{2}}x,\ S_{n-k+1} > \sigma n^{\frac{1}{2}}x\}\,.$$

Next note that for $m \geqq 1$

$$\begin{aligned}
P&\left\{M_{n-k} \leqq \sigma n^{\frac{1}{2}}x,\ S_{n-k+1} > \sigma n^{\frac{1}{2}}\left(x+\frac{1}{m}\right)\right\}\\
&= a_{n,k}(m)\\
&= \int_{-\infty}^{x} P\left\{\xi_1 > \sigma n^{\frac{1}{2}}\left(x+\frac{1}{m}-y\right)\right\}\cdot P\{M_{n-k} \leqq \sigma n^{\frac{1}{2}}x,\ S_{n-k}/\sigma n^{\frac{1}{2}} \in dy\}\,.
\end{aligned}$$

Use Chebyshev's inequality to obtain

$$\begin{aligned}
a_{n,k}(m) \leqq \frac{E|\xi_1|^{2-\delta}}{(\sigma n^{\frac{1}{2}})^{2-\delta}} \int_{-\infty}^{x}\left(x+\frac{1}{m}-y\right)^{-(2-\delta)}&\\
\times P\{M_{n-k} \leqq \sigma n^{\frac{1}{2}}x,\ S_{n-k}/\sigma n^{\frac{1}{2}} \varepsilon\, dy\}\,.&
\end{aligned} \tag{2.16}$$

From Donsker's theorem we know that $(M_{n-k}/\sigma n^{\frac{1}{2}}, S_{n-k}/\sigma n^{\frac{1}{2}}) \Rightarrow (M, W(1))$, where $\{W(t): t \geqq 0\}$ is Brownian motion and $M = \sup\{W(t): 0 \leqq t \leqq 1\}$; see Billingsley (1968) page 77. On $(-\infty, x]$, $(x+m^{-1}-y)^{-(2-\delta)}$ is a bounded continuous function of y and thus by weak convergence the integral in (2.16) converges to

$$\begin{aligned}
\int_{-\infty}^{x}&\left(x+\frac{1}{m}-y\right)^{-(2-\delta)} P\{M \leqq x,\ W(1) \in dy\}\\
&\leqq \int_{-\infty}^{x} (x-y)^{-(2-\delta)} P\{M \leqq x,\ W(1) \varepsilon\, dy\}\\
&= (2/\pi)^{\frac{1}{2}} \int_{-\infty}^{x} (x-y)^{-(2-\delta)} \int_{y\vee 0}^{x} (2z-y)\exp[-(2z-y)^2/2]\, dz\, dy < \infty\,;
\end{aligned}$$

for the joint density of $(M, W(1))$ see Itô–McKean (1965) page 27. Hence, we conclude that for all $m, n \geqq 1, \varepsilon > 0, 1 \leqq k \leqq M$

$$a_{n,k}(m) \leqq K/n^{1-\delta/2},$$

where K does not depend on m. Since $a_{n,k}(m) \nearrow a_{n,k}$ as $m \to \infty$, we have the bound (2.15).

The term

$$\sum_{k=1}^{M} (f_k/r_n)[P\{W_{n-k}/\sigma n^{\frac{1}{2}} \leqq x\} - P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\}] \leqq Kn^{\frac{1}{2}} \sum_{k=1}^{M} a_{n,k} \leqq Kn^{-\frac{1}{2}+\delta/2} \to 0$$

as $n \to \infty$ provided $\delta < 1$. The second inequality above uses (2.15). This completes the demonstration of (2.14) and establishes (2.2).

(2.17) REMARK. Without conditioning $P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} \to |N|(x)$, the limit distribution having mean $(2/\pi)^{\frac{1}{2}}$; with conditioning $P_n\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} \to 1 - \exp(-x^2/2)$, the limit distribution having mean $(\pi/2)^{\frac{1}{2}}$. Hence, conditioning to stay positive increases the mean of the limit distribution by a factor of $\pi/2$.

Next we state a standard result similar to one used by Belkin (1972) page 54. Alternatively, see Billingsley (1968) Theorem 5.5.

(2.18) LEMMA. *Let $\{\mu_n : n \geqq 1\}$ be a sequence of finite measures on $\mathscr{R}$, the Borel sets of $R = (-\infty, +\infty)$. Suppose $\mu_n \Rightarrow \mu$, finite. If $\{f_n : n \geqq 1\}$ is a sequence of uniformly bounded, Borel measurable functions converging uniformly on compact sets to an everywhere bounded continuous limit f, then*

$$\lim_{n\to\infty} \int_B f_n(x)\mu_n(dx) = \int_B f(x)\mu(dx)$$

for $B \in \mathscr{R}$ provided $\mu(\partial B) = 0$.

For convenience we let

$$g(t, x_1, x_2) = (2\pi t)^{-\frac{1}{2}}[\exp(-(x_2 - x_1)^2/2t) - \exp(-(x_1 + x_2)^2/2t)]$$

for $x_1, x_2 > 0$ and $0 < t \leqq 1$. Also set

$$p(0, 0; t, x) = t^{-\frac{3}{2}} x \exp(-x^2/2t)|N|(x/(1 - t)^{\frac{1}{2}})$$

for $x > 0$, $0 < t \leqq 1$, and

$$p(t_1, x_1; t_2, x_2) = g(t_2 - t_1, x_1, x_2)|N|(x_2/(1 - t_2)^{\frac{1}{2}})/|N|(x_1/(1 - t_1)^{\frac{1}{2}})$$

for $x_1, x_2 > 0$, $0 < t_1 < t_2 \leqq 1$.

The next step in showing that the f.d.d.'s converge is

(2.19) PROPOSITION. *For all $x \geqq 0$ and $0 < t < 1$, under the conditions of* (2.1)

$$\lim_{n\to\infty} P_n\{X_n^+(t) \leqq x\} = \int_0^x p(0, 0; t, y)\, dy. \tag{2.20}$$

PROOF. An elementary calculation shows that

$$P_n\{X_n^+(t) \leqq x\} = r_n^{-1} \int_{(0, xn^{\frac{1}{2}}/[nt]^{\frac{1}{2}}]} P\{X_{[nt]}(1) \in dy; S_1 > 0, \cdots, S_n > 0\}. \tag{2.21}$$

Now use the path structure of $\{S_n : n \geqq 0\}$ to write

$$\begin{aligned} &P\{X_{[nt]}(1) \in dy; S_1 > 0, \cdots, S_n > 0\} \\ (2.22) \qquad &= r_{[nt]} P\{X_{[nt]}(1) \in dy \mid T > [nt]\} \\ &\times P\left\{\max_{0 \leqq k \leqq n-[nt]}(-S_k/\sigma(n-[nt])^{\frac{1}{2}}) \leqq y\left(\frac{nt}{n-[nt]}\right)^{\frac{1}{2}}\right\}. \end{aligned}$$

Under the conditions of (2.1) we know from (2.7) that the last factor on the right-hand side of (2.22) converges uniformly in y to $|N|\{y(t/(1-t))^{\frac{1}{2}}\}$. On the other hand, from (2.5) $r_{[nt]}/r_n \to t^{-\frac{1}{2}}$ and the probability measure represented by the second factor on the right-hand side of (2.22) converges weakly to the Rayleigh distribution by (2.3). Now appealing to (2.18) yields

$$\lim_{n\to\infty} P_n\{X_n^+(t) \leqq x\} = t^{-\frac{1}{2}} \int_0^{xt^{-\frac{1}{2}}} ye^{-y^2/2}|N|\{y(t/(1-t))^{\frac{1}{2}}\}\, dy .$$

Finally, make the change of variables $t^{\frac{1}{2}}y = u$ to obtain (2.20).

The final step in showing convergence of the f.d.d.'s is

(2.23) THEOREM. *For all* $k \geqq 1$, $x_1, \cdots, x_k > 0$, *and* $0 < t_1 < t_2 < \cdots < t_k \leqq 1$, *under the conditions of* (2.1)

$$\begin{aligned} &\lim_{n\to\infty} P_n\{X_n^+(t_1) \leqq x_1, \cdots, X_n^+(t_k) \leqq x_k\} \\ (2.24) \qquad &= \int_0^{x_1} \cdots \int_0^{x_k} p(0, 0; t_1, y_1)p(t_1, y_1; t_2, y_2) \cdots \\ &\qquad p(t_{k-1}, y_{k-1}; t_k, y_k)\, dy_k \cdots dy_1 . \end{aligned}$$

PROOF. The proof is by induction on k. This result holds for $k = 1$ by virtue of (2.20). Suppose (2.24) is true for $k = m-1$, we show next that it can be extended to $k = m$. We begin by writing

$$\begin{aligned} &P_n\{X_n^+(t_1) \leqq x_1, \cdots, X_n^+(t_m) \leqq x_m\} \\ &= r_n^{-1} \int_{0+}^{x_{m-1}} \int_{0+}^{x_m} P\{X_n(t_1) \leqq x_1, \cdots, X_n(t_{m-2}) \leqq x_{m-2}, \\ &\qquad X_n(t_{m-1}) \in dy_{m-1}, X_n(t_m) \in dy_m; \\ &\qquad S_1 > 0, S_2 > 0, \cdots, S_n > 0\} \\ (2.25) \qquad &= \frac{r_{[nt_{m-1}]}}{r_n} \int_{0+}^{x_{m-1}} \int_{0+}^{x_m} P\{X_n(t_1) \leqq x_1, \cdots, \\ &\qquad X_n(t_{m-1}) \in dy_{m-1} \mid T > [nt_{m-1}]\} \\ &\qquad \times P^{\sigma n^{\frac{1}{2}} y_{m-1}}\left\{\frac{S_{[nt_m]-[nt_{m-1}]}}{\sigma n^{\frac{1}{2}}} \in dy_m, \right. \\ &\qquad \left. \min_{0 \leqq k \leqq [nt_m]-[nt_{m-1}]} (S_k/\sigma n^{\frac{1}{2}}) > 0\right\} \\ &\qquad \times P^{\sigma n^{\frac{1}{2}} y_m}\{\min_{0 \leqq k \leqq n-[nt_m]} (S_k/\sigma n^{\frac{1}{2}}) > 0\} , \end{aligned}$$

where $P^x\{\cdot\}$ is the p.m. for $\{S_n : n \geqq 0\}$ when $S_0 = x$.

We have used in the proof of (2.19) the fact that

$$\begin{aligned} &\lim_{n\to\infty} P^{\sigma n^{\frac{1}{2}} y_m}\{\min_{0 \leqq k \leqq n-[nt_m]} (S_k/\sigma n^{\frac{1}{2}}) \geqq 0\} \\ (2.26) \qquad &= \lim_{n\to\infty} P\{\max_{0 \leqq k \leqq n-[nt_m]} (-S_k/\sigma n^{\frac{1}{2}}) < y_m\} \\ &= |N|(y_m/(1-t_m)^{\frac{1}{2}}) \end{aligned}$$

uniformly in y_m. Furthermore, for $x > 0$

$$\lim_{n\to\infty} P^{\sigma n^{\frac{1}{2}} y_{m-1}} \left\{ \frac{S_{[nt_m]-[nt_{m-1}]}}{\sigma n^{\frac{1}{2}}} \leqq x, \right. \tag{2.27}$$

$$\left. \min_{0 \leqq k \leqq [nt_m]-[nt_{m-1}]} (S_k/\sigma n^{\frac{1}{2}}) > 0 \right\}$$

$$= \int_0^x g(t_m - t_{m-1}, y_{m-1}, y)\, dy ,$$

uniformly in y_{m-1}; cf. Itô–McKean (1965) page 30, for the value of the limit and Billingsley and Topsøe (1967) Theorem 2, for the uniformity.

Finally, by our induction assumption

$$\begin{aligned} &\lim_{n\to\infty} P\{X_n(t_1) \leqq x_1, \cdots, X_n(t_{m-1}) \leqq x_{m-1} \mid T > [nt_{m-1}]\} \\ &\quad = \int_0^{x_1/t_{m-1}^{\frac{1}{2}}} \cdots \int_0^{x_{m-1}/t_{m-1}^{\frac{1}{2}}} p(0, 0; t_1/t_{m-1}, y_1) \\ &\qquad \times p(t_1/t_{m-1}, y_1; t_2/t_{m-1}, y_2) \cdots \\ &\qquad\quad p(t_{m-2}/t_{m-1}, y_{m-2}; 1, y_{m-1})\, dy_{m-1} \cdots dy_1 . \end{aligned} \tag{2.28}$$

Combining (2.25), (2.26), (2.27), (2.28), and using (2.18) twice plus the Lebesgue bounded convergence theorem, we obtain

$$\begin{aligned} &\lim_{n\to\infty} P_n\{X_n^+(t_1) \leqq x_1, \cdots, X_n^+(t_m) \leqq x_m\} \\ &\quad = t_{m-1}^{-\frac{1}{2}} \int_0^{x_{m-1}} \int_0^{x_m} \int_0^{x_1/t_{m-1}^{\frac{1}{2}}} \cdots \int_0^{x_{m-2}/t_{m-1}^{\frac{1}{2}}} \\ &\qquad \times p(0, 0; t_1/t_{m-1}, y_1) p(t_1/t_{m-1}, y_1; t_2/t_{m-1}, y_2) \cdots \\ &\qquad\quad p(t_{m-2}/t_{m-1}, y_{m-2}; 1, y_{m-1}/t_{m-1}^{\frac{1}{2}})\, dy_{m-1} \cdots dy_1 \\ &\qquad \times g(t_m - t_{m-1}, y_{m-1}, y_m) |N| (y_m/(1 - t_m)^{\frac{1}{2}})\, dy_m . \end{aligned} \tag{2.29}$$

Now make the change of variables $t_{m-1}^{\frac{1}{2}} y_1 = u_1, \cdots, t_{m-1}^{\frac{1}{2}} x_{m-2} = u_{m-2}$ in (2.29) and one obtains (2.24) for $k = m$.

The f.d.d.'s on the right-hand side of (2.24) are those of a random function W^+, whose existence will be shown in the next section.

3. Weak convergence in D. In this section we prove that the sequence of processes $\{X_n^+ : n \geqq 1\}$ converge weakly as random functions in D_+ to W^+, the random function in D_+ whose f.d.d.'s are given in Section 2. We use Theorems 15.1 and 15.5 of Billingsley (1968) as the basic tool in the proof.

To this end we define the following modulus of continuity for functions $x \in D$:

$$w_x(\delta, a, b) = \sup \{|x(s) - x(t)|\} ;$$

where $0 \leqq a < b \leqq 1$, $0 < \delta < 1$, and the supremum extends over s and t satisfying $a \leqq s \leqq t \leqq b$ and $t - s \leqq \delta$. In Section 2 we showed that the f.d.d.'s of X_n^+ converge to those of W^+. Thus to complete the proof that $X_n^+ \Rightarrow W^+$ is suffices from Theorems 15.1 and 15.5 of [3] to show for every $\varepsilon > 0$ that

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} P_n\{w_{X_n^+}(\delta, 0, 1) \geqq \varepsilon\} = 0 , \tag{3.1}$$

since $X_n(0) = 0$. This will show at the same time that W^+ exists as a random function on $(D_+, \mathscr{D}_+)$ and that $P\{W^+ \in C\} = 1$.

The first step in demonstrating (3.1) is the next lemma, whose proof follows the argument of Belkin (1972) page 49, and will be omitted.

(3.2) LEMMA. *For every* $\varepsilon > 0$,

$$\lim_{\tau \downarrow 0} \limsup_{n\to\infty} r_n^{-1} P\{\sup_{0\leq s\leq \tau} X_n(s) \geqq \varepsilon, T > [n\tau]\} = 0 . \tag{3.3}$$

With this lemma in hand it is an easy matter to show

(3.4) THEOREM. *A random function* W^+, *on* $(D_+, \mathscr{D}^+)$ *exists with finite-dimensional distributions given by* (2.24). $X_n^+ \Rightarrow W^+$ *as* $n \to \infty$ *and* $P\{W^+ \in C\} = 1$.

PROOF. As remarked above, it suffices to show (3.1). For every $\tau \in (0, 1]$, $\varepsilon > 0$, and $0 < \delta < \tau$

$$\begin{aligned} P_n\{w_{X_n^+}(\delta, 0, 1) \geqq \varepsilon\} & \\ \leqq r_n^{-1} P\{w_{X_n}(\delta, 0, 1) \geqq \varepsilon, T > [n\tau]\} & \\ \leqq r_n^{-1} P\{w_{X_n}(\delta, 0, 1) \geqq \varepsilon, \sup_{0\leq s\leq\tau} X_n(s) < \varepsilon, T > [n\tau]\} & \\ + r_n^{-1} P\{\sup_{0\leq s\leq\tau} X_n(s) \geqq \varepsilon, T > [n\tau]\} . & \end{aligned} \tag{3.5}$$

Using the path structure of the random walk we have

$$\begin{aligned} r_n^{-1} P\{w_{X_n}(\delta, 0, 1) \geqq \varepsilon, \sup_{0\leq s\leq\tau} X_n(s) < \varepsilon, T > [n\tau]\} & \\ \leqq r_n^{-1} P\{w_{X_n}(\delta, \tau - \delta, 1) \geqq \varepsilon, T > [n\tau]\} & \\ \leqq r_n^{-1} P\{w_{X_n}(\delta, \tau - \delta, 1) \geqq \varepsilon, T > [n(\tau - \delta)]\} & \\ = \frac{r_{[n(\tau-\delta)]}}{r_n} P\{w_{X_n}(\delta, \tau - \delta, 1) \geqq \varepsilon\} & \\ \leqq \frac{r_{[n(\tau-\delta)]}}{r_n} P\{w_{X_n}(\delta, 0, 1) \geqq \varepsilon\} . & \end{aligned} \tag{3.6}$$

Since $X_n \Rightarrow W$ and $P\{W \in C\} = 1$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} P\{w_{X_n}(\delta, 0, 1) \geqq \varepsilon\} = 0 ;$$

see, for example, [10] Lemma 3. Thus combining (3.5), (3.3), and (3.6) we obtain (3.1) which completes the proof of (3.4).

Next we relate W^+ to W, Brownian motion. This job has essentially been done by Belkin (1972). Let

$$\tau = \sup\{t \in [0, 1] : W(t) = 0\}$$

and set

$$\Delta = 1 - \tau .$$

Then the f.d.d.'s of W^+ coincide with those of

$$\{|W(\tau + \Delta t)|/\Delta^{\frac{1}{2}} : 0 \leqq t \leqq 1\} ;$$

see [2] page 61. Since the class of finite-dimensional sets is a determining class, [3] page 123,

$$W^+(\cdot) = |W(\tau + \Delta\cdot)/\Delta^{\frac{1}{2}}| .$$

Note that this identification also shows that $W^+(\cdot)$ is continuous a.e.

4. Conditioned renewal and random partial sum processes. Assume now that the random variables $\{\xi_k : k \geqq 1\}$ are nonnegative with finite mean $\mu > 0$ and otherwise satisfy the conditions of (2.1). Let

$$Y_n(t, \omega) = (S_{[nt]}(\omega) - \mu nt)/\sigma n^{\frac{1}{2}} , \qquad 0 \leqq t \leqq 1 ,$$

and $\tilde{\Lambda}_n = \{S_k - \mu k > 0 : 1 \leqq k \leqq n\}$. Denote the restriction of Y_n to $[0, 1] \times \tilde{\Lambda}_n$ by Y_n^+. Then our previous results, (3.12), can be reinterpreted as $Y_n^+ \Rightarrow W^+$. Next let $\{N(t) : t \geqq 0\}$ be the renewal process, with rate $\mu^{-1} \equiv \lambda$, associated with the sequence $\{\xi_k : k \geqq 1\}$:

$$N(t, \omega) = \#\{n \geqq 1 : S_n(\omega) \leqq t\} .$$

Then form the random functions

$$N_n(t, \omega) = (\lambda nt - N(nt, \omega))/\sigma\lambda^{\frac{3}{2}} n^{\frac{1}{2}} , \qquad 0 \leqq t \leqq 1 ,$$

and let $\Theta_n = \{\lambda\tau - N(\tau) > 0 : 0 < \tau \leqq n\}$. We let N_n^+ denote the restriction of N_n^+ to $[0, 1] \times \Theta_n$. The f.c.l.t. for N_n^+ is

(4.1) PROPOSITION. *$N_n^+ \Rightarrow W^+$ as $n \to \infty$.*

PROOF. We use the method of Vervaat (1972). From (3.12) the random functions $(S_{[n\lambda\cdot]} - n\cdot)/\sigma(n\lambda)^{\frac{1}{2}}$ restricted to $\tilde{\Lambda}_{[n\lambda]}$ converge weakly to W^+, hence so does $(n^{-1}S_{[n\lambda\cdot]} - \cdot)/\sigma\lambda^{\frac{1}{2}}n^{-\frac{1}{2}}$. Now use the method of Vervaat (1972) page 251, plus the fact that $P\{W^+ \in C\} = 1$ and $\tilde{\Lambda}_{[n\lambda]} \subset \Theta_n \subset \tilde{\Lambda}_{[n\lambda]+1}$ to conclude that $N_n^+ \Rightarrow W^+$.

Now we return to the original case $\mu = 0$ together with the conditions of (2.1) and assume our random walk $\{S_n : n \geqq 0\}$ and a renewal process $\{N(t) : t \geqq 0\}$ are both defined on a common probability space. Let the times between renewal epochs be $\{u_i : i \geqq 1\}$ with $E\{u_i\} = \lambda^{-1}$, $0 < \lambda < \infty$, and $E\{u_i^2\} < \infty$. Define the random function

$$Z_n(t, \omega) = S_{N(nt,\omega)}(\omega)/\sigma(\lambda n)^{\frac{1}{2}} , \qquad 0 \leqq t \leqq 1 ,$$

and let $\Gamma_n = \{T > N(n)\}$. Denote the restriction of Z_n to $[0, 1] \times \Gamma_n$ by Z_n^+. Our goal now is to show that $Z_n^+ \Rightarrow W^+$. We shall simply sketch the proof. The proof of the next result is straightforward and therefore omitted.

(4.2) LEMMA. *$s_n = P\{T > N(n)\} \sim c(\lambda n)^{-\frac{1}{2}}$ as $n \to \infty$, where c is the same positive constant appearing in* (2.5).

To show that the f.d.d.'s of Z_n^+ converge to those of W^+ we condition on the values of $N(nt_1), \cdots, N(nt_k)$ and $N(n)$. Using essentially the same proof employed in (4.2) together with the continuity of the joint density of $(W^+(t_1), \cdots, W^+(t_k))$ as a function of $(t_1, \cdots, t_k)$ we obtain

(4.3) LEMMA. *If $0 < t_1 < \cdots < t_k \leqq 1$, then*

$$(Z_n^+(t_1), \cdots, Z_n^+(t_k)) \Rightarrow (W^+(t_1), \cdots, W^+(t_k)) .$$

To complete the proof that $Z_n^+ \Rightarrow W^+$ we must, of course, show that the sequence $\{Z_n^+ : n \geqq 1\}$ is tight. For that proof we need the following result which is proved using Kolmogorov's inequality.

(4.4) LEMMA. *For every* $\varepsilon > 0$, $\alpha < 1$, *and* $\tau > 0$

$$\lim_{n\to\infty} n^\alpha P\left\{\sup_{0\leqq s<t\leqq \tau}\left|\frac{N(nt)-N(ns)}{n} - \lambda(t-s)\right| > \varepsilon\right\} = 0 . \tag{4.5}$$

The tightness of $\{Z_n^+ : n \geqq 1\}$ follows from [3], Theorem 15.5 and the next

(4.6) LEMMA. *Let* $w_x(\delta, 0, 1) \equiv w_x(\delta)$ *for* $x \in D$. *For every* $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\{w_{Z_n}(\delta) \geqq \varepsilon \mid T > N(n)\} = 0 . \tag{4.7}$$

PROOF. It will suffice to take $0 < \varepsilon < \lambda$. Decomposing on the value of $N(n)$ we obtain

$$P\{w_{Z_n}(\delta) \geqq \varepsilon \mid T > N(n)\} = s_n^{-1} \sum_{k=0}^{\infty} P\{w_{Z_n}(\delta) \geqq \varepsilon, T > k, N(n) = k\} .$$

Now use the easily proved fact that $P\{|N(n)/n - \lambda| > \varepsilon\} = o(n^{-\frac{1}{2}})$, (4.2), (4.4), (3.12), and standard arguments to conclude (4.7).

Now since $Z_n^+(0, \omega) \equiv 0$, (4.6) plus Theorem 15.5 of [3] yields

(4.8) THEOREM. $Z_n^+ \Rightarrow W^+$ *as* $n \to \infty$.

5. The case of positive mean. Suppose now that $\mu = E\{\xi_1\} > 0$ and let $Y_n(t, \omega) = (S_{[nt]}(\omega) - \mu nt)/\sigma n^{\frac{1}{2}}$, $0 \leqq t \leqq 1$. Recall that $\Lambda_n = \{T > n\}$ and let Y_n^+ denote the restriction of Y_n to $[0, 1] \times \Lambda_n$. In this case $\lim_{n\to\infty} r_n = c_1$, $0 < c_1 < 1$; see, for example, Chung (1968), proof of Theorem 8.4.4. Since the random walk is drifting to $+\infty$, one would not expect the conditioning on $\{T > n\}$ to affect the weak limit of Y_n. We proceed to sketch the proof of this fact; namely, that $X_n^+ \Rightarrow W$ as $n \to \infty$.

From (2.3) we have for all real x

$$\begin{aligned} r_n P_n\{(W_n - n\mu)/\sigma n^{\frac{1}{2}} \leqq x\} &= r_n P\{(W_0 - n\mu)/\sigma n^{\frac{1}{2}} \leqq x\} - \sum_{k=0}^{n-1} r_k[P\{(W_{n-k-1} - n\mu)/\sigma n^{\frac{1}{2}} \leqq x\} \\ &\quad - P\{(W_{n-k} - n\mu)/\sigma n^{\frac{1}{2}} \leqq x\}] . \end{aligned} \tag{5.1}$$

It is well known that $(W_n - n\mu)/\sigma n^{\frac{1}{2}} \Rightarrow N(0, 1)$, a standard normal random variable. The first term on the right-hand side of (5.1) converges to c_1 and the sum converges to $c_1(1 - \Phi(x))$, where Φ is the standard normal distribution function. Hence

$$\lim_{n\to\infty} P_n\{Y_n(1) \leqq x\} = \Phi(x) , \qquad -\infty < x < \infty .$$

The convergence of the f.d.d.'s of Y_n to those of W is obtained using the methods of (2.19) and (2.23) and the fact that $\min_{0\leqq k\leqq n} S_k$ converges a.e. to a finite random variable. Finally, for tightness observe that

$$P_n\{w_{Y_n^+}(\delta, 0, 1) \geqq \varepsilon\} \leqq r_n^{-1} P\{w_{Y_n}(\delta, 0, 1) \geqq \varepsilon\}$$

and thus $\lim_{\delta \downarrow 0} \limsup_{n\to\infty} P_n\{w_{Y_n^+}(\delta, 0, 1) \geqq \varepsilon\} = 0$. Hence we obtain

(5.2) PROPOSITION. *If* $\mu > 0$ *and* $0 < \sigma^2 < \infty$, *then* $Y_n^+ \Rightarrow W$ *as* $n \to \infty$.

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