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SUMS OF INDEPENDENT RANDOM VARIABLES ON PARTIALLY ORDERED SETS

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Let $(\mathscr{A}, \leqq)$ be a partially ordered set, $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ a collection of i.i.d. random variables with mean zero, indexed by $\mathscr{A}$. Let $S_\beta = \sum_{\alpha \leqq \beta} X_\alpha$, $|\beta| = \text{card}\{\alpha \in \mathscr{A}: \alpha \leqq \beta\}$. We study the a.s. convergence to zero of $Z_\beta = S_\beta/|\beta|$, when $|\beta| \mapsto \infty$.

We first derive a Hájek–Rényi inequality for $K^r = \{(k_1, k_2, \cdots, k_r): k_i$ a positive integer$\}$. This is used to derive a sufficient condition for the convergence of Z_β for a class of partially ordered sets including K^r. For many of these sets (and certain other sets as well) this condition is shown to be necessary. Finally a weaker sufficient condition is derived for a much larger class of sets, giving a theorem analogous to one of Hsu and Robbins for the linearly ordered case.

0. Introduction. This paper contains complements to and generalizations of the results of [9]. In Section 1 we prove a Hájek–Rényi type inequality for n-dimensional integer lattices, verifying a conjecture of Pyke [8]; in Section 2 this inequality is used to prove a strong law of large numbers for a class of partially ordered sets, including the integer lattices considered in [9]. The present proof is completely elementary and offers some insight into the need for the moment conditions found to be necessary and sufficient in [9]. Section 3 contains a partial converse to the law of large numbers of Section 2. Finally in Section 4 we prove an analogue of a theorem of Hsu and Robbins which gives sufficient conditions for a strong law of large numbers on a large class of partially ordered sets.

1. A Hájek–Rényi inequality for integer lattices. In 1955 Hájek and Rényi proved the following inequality [6]:

(1.1) Let $X_1, X_2, \cdots, X_n, \cdots$ be independent random variables with mean zero and $\text{Var}(X_k) = \sigma_k^2 < \infty$ for each k. Let $\{b_k\}_{k=1,2,\dots}$ be a non-decreasing sequence of positive numbers, c any positive number. Then

$$P\{\max_{1 \leqq k \leqq n} |S_k/b_k| \geqq c\} \leqq (1/c^2) \sum_{j=1}^n \sigma_j^2/b_j^2 ,$$

where $S_k = \sum_{i=1}^n X_i$.

In this section we will extend this inequality to sums of random variables indexed by an integer lattice. Let K_r be the set of r-tuples $\mathbf{k} = (k_1, k_2, \cdots, k_r)$ with positive integers for coordinates; let $\leqq$ denote the coordinate-wise partial

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ordering on K_r. We consider a set $\{X_{\mathbf{k}}: \mathbf{k} \in K_r\}$ of random variables indexed by K_r, with mean zero and Var $(X_{\mathbf{k}}) = \sigma_{\mathbf{k}}^2$ for each $\mathbf{k}$; let $S_{\mathbf{k}} = \sum_{\mathbf{j} \leq \mathbf{k}} X_{\mathbf{j}}$.

We will impose two conditions on the $X_{\mathbf{k}}$, which stop short of requiring their mutual independence.

(1.2) $$E\{X_{\mathbf{k}} \mid \sigma[(X_{\mathbf{j}}); \mathbf{j} < \mathbf{k}]\} = 0 \qquad \text{for each } \mathbf{k} \in K_r .$$

(1.3) If $\mathscr{F}_{\mathbf{k}} = \sigma\{(X_{\mathbf{j}}); \mathbf{j} \leqq \mathbf{k}\}$, and $\mathbf{m} \wedge \mathbf{n}$ denotes the coordinatewise infimum, then $E\{\cdot \mid \mathscr{F}_{\mathbf{m}} \mid \mathscr{F}_{\mathbf{n}}\} = E\{\cdot \mid \mathscr{F}_{\mathbf{m} \wedge \mathbf{n}}\}$.

Conditions (1.2) and (1.3) guarantee in particular that the partial sums form a martingale on K_r. Clearly mutual independence of the $X_{\mathbf{k}}$ will imply both conditions.

For an array of constants $\{b_{\mathbf{k}}; \mathbf{k} \in K_r\}$ we define $\Delta b_{\mathbf{k}}$ to be the rth-order finite difference of the b's at the point $\mathbf{k}$. Thus for example if $r = 2$, $\Delta b_{k,m} = b_{k,m} - b_{k-1,m} - b_{k,m-1} + b_{k-1,m-1}$ (where $b_{ij} \equiv 0$ if i or $j = 0$).

THEOREM 1.1 *Let* $\{X_{\mathbf{k}}; \mathbf{k} \in K_r\}$ *satisfy* (1.2) *and* (1.3), *with* $E(X_{\mathbf{k}}) = 0$ *and* $\sigma_{\mathbf{k}}^2 < \infty$ *for each* $\mathbf{k} \in K_r$. *Let* $\{b_{\mathbf{k}}; \mathbf{k} \in K_r\}$ *be an array of positive constants such that* $\Delta b_{\mathbf{k}} \geqq 0$ *for each* $\mathbf{k} \in K_r$. *Then*

$$P\{\max_{\mathbf{k} \leqq \mathbf{n}} |S_{\mathbf{k}}/b_{\mathbf{k}}| \geqq c\} \leqq (4^{2r-1}/c^2) \sum_{\mathbf{k} \leqq \mathbf{n}} \sigma_{\mathbf{k}}^2/b_{\mathbf{k}}^2 .$$

REMARK. We note that for $r = 1$ the theorem gives the usual Hájek–Rényi inequality with a factor of 4. This loss is due to the fact that our proof does not reduce to the direct proof of Hájek and Rényi for the case $r = 1$, but rather proceeds through a submartingale inequality of Doob, a detour necessitated by the lack of linear ordering when $r \geqq 2$.

We begin with a lemma which improves slightly some known martingale inequalities of the Kolmogorov type (cf. [1], [11], [12]).

LEMMA 1.1. *Let* $\{M_{\mathbf{k}}, \mathscr{F}_{\mathbf{k}}\}$ *be a martingale or a positive submartingale such that the* $\mathscr{F}_{\mathbf{k}}$ *satisfy condition* (1.3). *Then*

$$P\{\max_{\mathbf{k} \leqq \mathbf{n}} |M_{\mathbf{k}}| \geqq c\} \leqq (4^{r-1}/c^2) \max_{\mathbf{k} \leqq \mathbf{n}} E\{M_{\mathbf{k}}^2\} .$$

PROOF. ($r = 2$). Let $\mathbf{n} = (n_1, n_2)$. Following Zimmerman ([12]), set $Z_i(\omega) = \max_{1 \leqq j \leqq n_2} |M_{ij}(\omega)|$, $I(\omega) = \inf\{i: Z_i(\omega) \geqq c\}$ (or ∞ if no such i exists), $J(\omega) = \inf\{j: |M_{Ij}| \geqq c\}$. Let $\Lambda = \{\max_{i,j} |M_{ij}| \geqq c\}$; then

$$\Lambda = \bigcup_{i=1}^{n_1} \bigcup_{j=1}^{n_2} B_{ij} \qquad \text{where} \quad B_{ij} = \{I(\omega) = i, J(\omega) = j\} .$$

Then

$$\int_{B_{ij}} M_{n_1 j}^2 = \int_{B_{ij}} \{M_{ij} + [M_{n_1 j} - M_{ij}]\}^2 \geqq \int_{B_{ij}} M_{ij}^2$$

since $E\{M_{ij}(M_{n_1 j} - M_{ij})1_{B_{ij}}\} = E\{M_{ij} 1_{B_{ij}} E[M_{n_1 j} - M_{ij} \mid \mathscr{F}_{i n_2}]\} \geqq 0$ by (1.3). Thus

$$E[M_{n_1 J}^2] \geqq \int_\Lambda M_{n_1 J}^2 = \sum_{i,j} \int_{B_{ij}} M_{n_1 J}^2 \geqq \sum_{i,j} \int_{B_{ij}} M_{ij}^2 \geqq c^2 P(\Lambda)$$

and since $\{M_{n_1 j}; \mathscr{F}_{n_1 j}\}$ is a martingale (or positive submartingale):

$$E[M_{n_1 J}^2] \leqq E\{\max_{1 \leqq j \leqq n_2} M_{n_1 J}^2\} \leqq 4E\{M_{n_1 n_2}^2\}$$

by an inequality of Doob ([2] page 319).

The extension to $r > 2$ is as follows: we define $I_1, I_2, \cdots, I_r$ as I, J are defined above; we define

$$\Lambda = \bigcup_{i_1=1}^{n_1} \cdots \bigcup_{i_r=1}^{n_r} B_{i_1\cdots i_r}$$

where $B_{i_1\cdots i_r} = \{I_1(\omega) = i_1, \cdots, I_r(\omega) = i_r\}$.
We therefore have

$$\int_{B_{i_1\cdots i_r}} M^2_{n_1 i_2\cdots i_r} \geqq \int_{B_{i_1\cdots i_r}} M^2_{i_1\cdots i_r}$$

as above. Thus

$$E\{M^2_{n_1 I_2\cdots I_r}\} \geqq c^2 P(\Lambda)$$

and we have

$$E\{M^2_{n_1 I_2\cdots I_r}\} \leqq E\{\max_{1\leqq i_2\leqq n_2} \cdots \max_{1\leqq i_r\leqq n_r} M^2_{n_1 i_2\cdots i_r}\}\,.$$

Using (1.3) above, $\max_{1\leqq i_3\leqq n_3} \cdots \max_{1\leqq i_r\leqq n_r} M^2_{n_1 i_2\cdots i_r}$ is a positive submartingale indexed by i_2 (with respect to the fields $\mathscr{F}_{n_1 i_2 n_3\cdots n_r}$). Using Doob's inequality again, we have by induction that

$$P(\Lambda) \leqq (4^{r-1}/c^2)E\{M^2_{n_1 n_2\cdots n_r}\} = (4^{r-1}/c^2)\max_{\mathbf{k}\leqq\mathbf{n}} E\{M_{\mathbf{k}}^2\}\,. \qquad \square$$

PROOF OF THEOREM 1.1. Let $T_{\mathbf{k}} = \sum_{\mathbf{j}\leqq\mathbf{k}} X_{\mathbf{j}}/b_{\mathbf{j}}$. Then

$$\begin{aligned} S_{\mathbf{k}} &= \textstyle\sum_{\mathbf{j}\leqq\mathbf{k}} X_{\mathbf{j}} = \sum_{\mathbf{j}\leqq\mathbf{k}} b_{\mathbf{j}}(\Delta T_{\mathbf{j}}) = \sum_{\mathbf{j}\leqq\mathbf{k}}(\Delta T_{\mathbf{j}}) \sum_{\mathbf{i}\leqq\mathbf{j}} (\Delta b_{\mathbf{i}}) \\ &= \textstyle\sum_{\mathbf{i}\leqq\mathbf{k}} (\Delta b_{\mathbf{i}}) \sum_{\mathbf{i}\leqq\mathbf{j}\leqq\mathbf{k}} (\Delta T_{\mathbf{j}}) = \sum_{\mathbf{i}\leqq\mathbf{k}} (\Delta b_{\mathbf{i}}) \sum_{\mathbf{i}\leqq\mathbf{j}\leqq\mathbf{k}} X_{\mathbf{j}}/b_{\mathbf{j}}\,. \end{aligned}$$

Since $\sum_{\mathbf{i}\leqq\mathbf{k}} (\Delta b_{\mathbf{i}})/b_{\mathbf{k}} = 1$ we have $\{|S_{\mathbf{k}}/b_{\mathbf{k}}| \geqq c\} \subset \{\max_{\mathbf{i}} |\sum_{\mathbf{i}\leqq\mathbf{j}\leqq\mathbf{k}} X_{\mathbf{j}}/b_{\mathbf{j}}| \geqq c\}$. Thus $\{\max_{\mathbf{k}} |S_{\mathbf{k}}/b_{\mathbf{k}}| \geqq c\} \subset \{\max_{\mathbf{i}} \max_{\mathbf{k}} |\sum_{\mathbf{i}\leqq\mathbf{j}\leqq\mathbf{k}} (X_{\mathbf{j}}/b_{\mathbf{j}})| \geqq c\}$. Now the sum in the right-hand side above is equal to the rth dimensional difference of the partial sums $\sum_{\mathbf{i}\leqq\mathbf{j}} (X_{\mathbf{i}}/b_{\mathbf{i}})$ taken over the 2^r vertices of the "rectangle" $\mathbf{i} \leqq \mathbf{j} \leqq \mathbf{k}$; thus we have that

$$\{\max_{\mathbf{i},\mathbf{k}} |\textstyle\sum_{\mathbf{i}\leqq\mathbf{j}\leqq\mathbf{k}} (X_{\mathbf{j}}/b_{\mathbf{j}})/| \geqq c\} \subset \{\max_{\mathbf{j}} |\sum_{\mathbf{i}\leqq\mathbf{j}} (X_{\mathbf{i}}/b_{\mathbf{i}})| \geqq c/2^r\}\,.$$

It therefore follows by Lemma 1.1, since $(\sum_{\mathbf{i}\leqq\mathbf{j}} (X_{\mathbf{i}}/b_{\mathbf{i}}), \mathscr{F}_{\mathbf{i}})$ is a martingale, that

$$P\{\max_{\mathbf{k}\leqq\mathbf{n}} |S_{\mathbf{k}}/b_{\mathbf{k}}| \geqq c\} \leqq (4^{2r-1}/c^2)E\{\textstyle\sum_{\mathbf{i}\leqq\mathbf{n}} (X_{\mathbf{i}}/b_{\mathbf{i}})\}^2\,.$$

But (1.2) and (1.3) imply that the random variables $\{X_{\mathbf{i}}/b_{\mathbf{i}}\}$ are orthogonal; thus

$$E\{\textstyle\sum_{\mathbf{i}\leqq\mathbf{n}} (X_{\mathbf{i}}/b_{\mathbf{i}})\}^2 = \sum_{\mathbf{i}\leqq\mathbf{n}} \sigma^2(X_{\mathbf{i}})/b_{\mathbf{i}}^2$$

and the theorem is established.

2. A law of large numbers. The inequality of Theorem 1.1 can be used to prove a strong law of large numbers for integer lattices—in fact for partially ordered sets somewhat more general than the lattices K_r.

Let $(\mathscr{A}, \leqq)$ be a denumerably infinite, partially ordered set. We define, for $\alpha \in \mathscr{A}$, $|\alpha| = \operatorname{card}\{\mu \in \mathscr{A} \mid \mu \leqq \alpha\}$. We want to define a class of sets which look, locally, like the lattices K_r with missing points.

DEFINITION 2.1. $\mathscr{A}$ will be called a *local lattice* if

(i) $\{\alpha : |\alpha| = j\}$ is finite for each j;

(ii) for each $\alpha \in \mathscr{A}$, there exists a one-to-one function $\varphi_\alpha: \{\beta \in \mathscr{A} \mid \beta \leqq \alpha\} \to K_r$ for some $r \geqq 1$ which preserves the order relation (we require that r be the same for each α, but not that the φ_α be consistent);

(iii) $\mathscr{A}$ is filtering to the right.

EXAMPLE 2.1. Trivially, K_r itself and any infinite subset of K_r which filters to the right are local lattices.

EXAMPLE 2.2. The partially ordered subset of R^2 defined by $\bigcup_{k=0}^\infty \bigcup_{n=0}^\infty (n/2^k, k)$ is a local lattice (for $r = 2$). On the other hand, the set $\bigcup_{n=0}^\infty \bigcup_{n=1}^\infty (n/2^k, k)$ satisfies (ii) and (iii) of the definition but violates (i).

Let $\mathscr{A}$ be a local lattice. Let $\{X_\alpha: \alpha \in \mathscr{A}\}$ be a collection of independent random variables with mean zero, and define $S_\beta = \sum_{\alpha \leqq \beta} X_\alpha$. We wish to study the a.s. convergence of $Z_\beta = S_\beta/|\beta|$ when $|\beta| \to \infty$.

DEFINITION 2.2.

(a) $d(j) = \text{card}\,\{\alpha \in \mathscr{A} \mid |\alpha| = j\}$ $j = 1, 2, \cdots$

(b) $M(x) = \sum_{j \leqq x} d(j)$ $x \geqq 1$; $M(x) = 1$ $0 \leqq x < 1$.

LEMMA 2.1. *For any random variable X,*

$$E\{M(|X|)\} < \infty \leftrightarrow \sum_{j=1}^\infty d(j)\, P\{|X| \geqq j\} < \infty\,.$$

PROOF.

$$\begin{aligned} M(x) &= \textstyle\sum_{j=1}^\infty d(j)\, 1_{[j,\infty)}(x) + 1_{[0,1)}(x) \qquad \text{so that} \quad E\{M(|X|)\} \\ &= \textstyle\lim_{N\to\infty} \sum_{j=1}^N d(j)\, P\{|X| \geqq j\} + P\{0 \leqq |X| < 1\} \\ &= \textstyle\sum_{j=1}^\infty d(j)\, P\{|X| \geqq j\} + P\{0 \leqq |X| < 1\} \end{aligned}$$

by monotone convergence.

From this point on we will assume that *the X_α are identically distributed.* Let

$$\begin{aligned} Y_\alpha' &= X_\alpha \qquad \text{if} \quad |X_\alpha| \leqq |\alpha| \\ &= 0 \qquad \text{if} \quad |X_\alpha| > |\alpha| \end{aligned}$$

and define $Y_\alpha = Y_\alpha' - E(Y_\alpha')$. Then

$$\textstyle\sum_\alpha P\{X_\alpha \neq Y_\alpha'\} = \sum_\alpha P\{|X_\alpha| > |\alpha|\} = \sum_{j=1}^\infty d(j)\, P\{|X| > j\}\,.$$

Set $T_\beta' = \sum_{\alpha \leqq \beta} Y_\alpha'$. Then if $EM(|X|) < \infty$, it follows from Lemma 2.1 and the remark above that $\sum_\alpha P\{X_\alpha \neq Y_\alpha'\} < \infty$; thus by the Borel-Cantelli lemma, the convergence of $T_\beta'/|\beta|$ will imply the corresponding convergence for Z_β.

In order to prove our main result some constraints on the growth of $M(x)$ will be needed. We recall a concept due to Feller, that of *dominated variation* [5]. A non-decreasing function U varies dominatedly (at infinity) if there exist constants C, γ, and t_0 such that

$$U(tx)/U(t) < Cx^\gamma \qquad \text{for } x > 1,\ t > t_0\,. \tag{2.1}$$

Feller shows in [5] Theorem 1 that, whenever $\sum_{j=1}^\infty d(j)/j^2 < \infty$, our function

$M(x)$ varies dominatedly with index $\gamma < 2$ if and only if

$$\limsup_{t\to\infty} [t^2/M(t)] \sum_{x=t}^{\infty} d(x)/x^2 < \infty . \tag{2.2}$$

This turns out to be precisely the class of M-functions for which we can prove a strong law of large numbers. (Clearly dominated variation is a weaker condition than its more familiar parent, regular variation ([5] page 107).)

LEMMA 2.2. *Let the X_α be identically distributed as X. Suppose that*

(i) $\sum_{j=1} d(j)/j^2 < \infty$
(ii) *M varies dominatedly with index < 2.*
(iii) $EM\{|X|\} < \infty$.

Then $\sum_\alpha \sigma^2(Y_\alpha')/|\alpha|^2 < \infty$.

PROOF. Let F be the common distribution function of the X_α. We have

$$\begin{aligned}\sum_\alpha \sigma^2(Y_\alpha')/|\alpha|^2 &\leqq \sum_{n=1}^{\infty} d(n)/n^2 \int_{|x|\leqq n} x^2\, dF(x) \\ &= \sum_{n=1}^{\infty} d(n)/n^2 \sum_{m=1}^{n} \int_{m-1<|x|\leqq m} x^2\, dF(x) \\ &= \sum_{n=1}^{\infty} d(n)/n^2 \sum_{m=1}^{n} \int_{m-1<|x|\leqq m} M(|x|)[x^2/M(|x|)]\, dF(x) \\ &\leqq \sum_{n=1}^{\infty} d(n)/n^2 \sum_{m=1}^{n} [m^2/M(m-1)] \int_{m-1<|x|\leqq m} M(|x|)\, dF(x) \\ &= \sum_{m=1}^{\infty} \int_{m-1<|x|\leqq m} M(|x|)\, dF(x) [m^2/M(m-1)] \sum_{n=m}^{\infty} d(n)/n^2 .\end{aligned}$$

But by hypotheses (i) and (ii), $[m^2/M(m-1)] \sum_{n=m}^{\infty} d(n)/n^2$ is bounded uniformly in m. Therefore $\sum_\alpha \sigma^2(Y_\alpha')/|\alpha|^2 \leqq K \sum_{m=1}^{\infty} \int_{m-1<|x|\leqq m} M(|x|)\, dF(x) < \infty$. $\square$

THEOREM 2.1. *Let $\mathscr{A}$ be a local lattice satisfying conditions* (i), (ii), *and* (iii) *of Lemma* 2.2. *Then given any* $\varepsilon > 0$,

$$P\{|Z_\beta| > \varepsilon \text{ finitely often}\} = 1 .$$

PROOF. We know from Lemma 2.2 that under the hypotheses of Theorem 2.1, $\sum_\alpha E(Y_\alpha^2)/|\alpha|^2 < \infty$; thus in particular $\sum_{|\alpha|>N} E(Y_\alpha^2)/|\alpha|^2 \to_{N\uparrow\infty} 0$, and by Kronecker's lemma, $1/N^2 \sum_{|\alpha|\leqq N} E(Y_\alpha^2) \to_{N\uparrow\infty} 0$. Let $c > 0$ be fixed and let r be the "dimension" of the local lattice $\mathscr{A}$. Given $\varepsilon > 0$, let N be chosen so large that

$$(4^{2r-1}/c^2)\{1/N^2 \sum_{|\alpha|\leqq N} E(Y_\alpha^2) + \sum_{|\alpha|>N} E(Y_\alpha^2)/|\alpha|^2\} < \varepsilon . \tag{2.3}$$

Let β be such that $|\beta| > N$; let $E_\beta = \{\alpha \in \mathscr{A} \mid \alpha \leqq \beta\}$. By means of φ_β we map E_β into a portion $\mathbf{k} \leqq \mathbf{n}$ of K^r, where $\varphi_\beta(\beta) = \mathbf{n}$. Define, for $\mathbf{k} \leqq \mathbf{n}$,

$$\begin{aligned} W_{\mathbf{j}} &= Y_\gamma \quad \text{if } \varphi_\beta(\gamma) = \mathbf{j} \quad \text{for some } \gamma \leqq \beta \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Let

$$T_{\mathbf{j}} = \sum_{\mathbf{i}\leqq\mathbf{j}} W_{\mathbf{i}} , \qquad T_\beta = \sum_{\alpha\leqq\beta} Y_\alpha .$$

Define now for $\mathbf{j} \leqq \mathbf{n}$,

$$\bar{b}_{\mathbf{j}} = \operatorname{card}\{\mathbf{i} \leqq \mathbf{j} \mid \mathbf{i} \in \varphi_\beta(E_\beta)\} ,$$

i.e., $\bar{b}_{\mathbf{j}}$ is the number of $\mathbf{i} \leqq \mathbf{j}$ which are image points under φ_β of some $\gamma \leqq \beta$.

One verifies without difficulty that $\Delta \bar{b}_{\mathbf{j}} \geqq 0$ for $\mathbf{j} \leqq \mathbf{n}$, and that if $\mathbf{j} = \varphi_\beta(\gamma)$, then $\bar{b}_{\mathbf{j}} = |\gamma|$.

We are almost ready to apply Theorem 1.1; but first we need to modify the $\bar{b}_{\mathbf{j}}$. We define

$$b_{\mathbf{j}} = \bar{b}_{\mathbf{j}} + N \qquad \text{for } \mathbf{j} \leqq \mathbf{n}\,. \tag{2.4}$$

It is clear that $\Delta b_{\mathbf{j}} \geqq 0$, and that for $\alpha \leqq \beta$, $|\alpha| > N$ implies $b_\alpha < 2|\alpha|$ (here $b_\alpha \equiv b_{\varphi_\beta(\alpha)}$). We thus have

$$\begin{aligned} &P\{\max_{\alpha \leqq \beta, |\alpha| > N} |T_\alpha|/|\alpha| \geqq 2c\} \\ &\quad \leqq P\{\max_{\alpha \leqq \beta, |\alpha| > N} |T_\alpha|/b_\alpha \geqq c\} \leqq P\{\max_{\alpha \leqq \beta} |T_\alpha|/b_\alpha \geqq c\} \\ &\quad \leqq P\{\max_{\mathbf{j} \leqq \mathbf{n}} |T_{\mathbf{j}}|/b_{\mathbf{j}} \geqq c\} \leqq (4^{2r-1}/c^2) \textstyle\sum_{\alpha \leqq \beta} E(Y_\alpha^2)/b_\alpha^2 \\ &\quad \leqq (4^{2r-1}/c^2)\{1/N^2 \textstyle\sum_{\alpha \leqq \beta, |\alpha| \leqq N} E(Y_\alpha^2) + \textstyle\sum_{\alpha \leqq \beta, |\alpha| > N} E(Y_\alpha^2)/|\alpha|^2\}\,, \end{aligned}$$

the fourth inequality being a consequence of Theorem 1.1. Since $\mathscr{A}$ is filtering to the right we deduce that

$$\begin{aligned} &P\{\sup_{|\alpha| > N} |T_\alpha|/|\alpha| \geqq 2c\} \\ &\quad \leqq (4^{2r-1}/c^2)\{1/N^2 \textstyle\sum_{|\alpha| \leqq N} E(Y_\alpha^2) + \textstyle\sum_{|\alpha| > N} E(Y_\alpha^2)/|\alpha|^2\} < \varepsilon\,. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows by a routine application of the Borel–Cantelli lemma that

$$P\{|T_\alpha|/|\alpha| \geqq 2c \ \text{ finitely often}\} = 1\,.$$

But since $E(Y_\alpha') \to_{|\alpha| \to \infty} 0$, one checks easily that

$$1/|\alpha|\, |\textstyle\sum_{\beta \leqq \alpha} E(Y_\beta')| \geqq \varepsilon \ \text{ finitely often} \qquad \text{for any } \varepsilon > 0\,;$$

thus we conclude that

$$P\{1/|\alpha|\, |\textstyle\sum_{\beta \leqq \alpha} Y_\beta'| \geqq \varepsilon \ \text{ finitely often}\} = 1\,.$$

By Lemma 2.1 it follows that given any $\varepsilon > 0$,

$$P\{|Z_\beta| \geqq \varepsilon \ \text{ finitely often}\} = 1\,. \qquad \square$$

EXAMPLE 2.3. For the complete lattice K^r, one verifies that $M(j) \sim j(\log j)^{r-1}$ for large j. Since $d(j) = o(j^\delta)$ for any $\delta > 0$ one has $\sum_{j=1} d(j)/j^2 < \infty$; thus if $E\{|X|(\log^+ |X|)^{r-1}\}$ one recovers the result of Dunford and Zygmund ([3], [13]; see also [9]), that $P\{|Z_\beta| > \varepsilon$ finitely often$\} = 1$ for any $\varepsilon > 0$.

EXAMPLE 2.4. Consider the "half-lattice" on K^2, i.e., $\{(n, m) \mid n \geqq m\}$. One checks that $M(j) = O(j \log j)$ for this set also; thus $E\{|X| \log^+ |X|\} < \infty$ implies $Z_\beta \mapsto 0$ a.s.

EXAMPLE 2.5. Let $\mathscr{B}$ be the sector of K^2 contained between the lines $y = cx$ and $y = (1/c)x$ for $c > 0$. At first glance one might expect this to behave like Example 2.4; but the presence of the boundary (the x-axis) in Example 2.4 makes a considerable difference. In fact, for $\mathscr{B}$ one has $M(j) \sim O(j)$; thus for any sector whatever of this form, existence of the expectation guarantees convergence of Z_β to zero a.s., whereas for the limiting case of the sectors (K^2 itself), an $X \log X$ moment is required. In light of the relation between martingale convergence and derivation, this behavior appears to be analogous to that observed by Jessen, Marcinkiewicz, and Zygmund [7] concerning the differentiability of multiple integrals, i.e. that the first moment is sufficient if one restricts the ratios of the sides in the rectangles over which the limit is taken.

Conspicuous by its absence in these examples is Example 2.2. For this one has $M(j) \sim O(j^2)$, which implies that $\sum_{j=1}^{\infty} d(j)/j^2$ is not convergent; Theorem 2.2 thus has nothing to say about this case. This is a rather annoying limitation; however, we shall see in Section 4 that weaker results for arrays of this type can be obtained by a different method.

3. Necessity of the condition $E\{M(|X|)\} < \infty$. For certain of the local lattices discussed in Section 2, and some other partially ordered sets as well, a converse to Theorem 2.2 can be obtained. Let $(\mathscr{A}, \leqq)$ be a denumerable, partially ordered set, such that

(a) $E_\alpha = \{\beta \in \mathscr{A} \colon \beta \leqq \alpha\}$ is finite for each $\alpha \in \mathscr{A}$.
(b) $\{\alpha \colon |\alpha| = j\}$ is finite for each j.

DEFINITION 3.1. $\mathscr{A}$ will be called *n-differentiable* if there exists a positive integer n such that for each $\alpha_1 \in \mathscr{A}$ we can find $\alpha_2, \cdots, \alpha_j$ $(j \leqq n)$ with $\alpha_i \leqq \alpha_1$ for $i = 2, \cdots, j$ and with the properties that

(i) $|\alpha_i| \geqq K(\alpha_1)$ for $i = 2, \cdots, j$ where $K(\alpha_1) \to_{|\alpha_1| \uparrow \infty} \infty$
(ii) $E_{\alpha_{2m}} \subset E_{\alpha_1} - E_{\alpha_2} + \cdots + E_{\alpha_{2m-1}}$ $m \leqq [j/2]$; $E_{\alpha_{2m-1}} \cap [E_{\alpha_1} - E_{\alpha_2} + \cdots - E_{\alpha_{2m}}] = \phi$ $m \leqq [j - 1/2]$
(iii) $\{\alpha_1\} = E_{\alpha_1} - E_{\alpha_2} + E_{\alpha_3} - E_{\alpha_4} \cdots - E_{\alpha_j}$.

(The operations $+$ and $-$ above are to be performed sequentially from left to right, e.g., $E_{\alpha_1} - E_{\alpha_2} + E_3 = [E_{\alpha_1} \cap E_{\alpha_2}^c] \cup E_{\alpha_3}$.)
All of the examples in Section 2 are differentiable; so, for example, is any tree.

THEOREM 3.1. *Let $\{X_\alpha\}$ be identically distributed as X. Let $\mathscr{A}$ satisfy conditions* (a) *and* (*b*) *above, and suppose that $\mathscr{A}$ is n-differentiable for some n. Then*

$$E\{M(|X|)\} = \infty \to P\{|Z_\beta| \geqq 1/n \text{ infinitely often}\} = 1\,.$$

PROOF. By Lemma 2.1, $E\{M(|X|)\} = \infty$ implies $\sum_\alpha P\{|X_\alpha| > |\alpha|\} = \infty$ which in turn implies by Borel-Cantelli that $P\{|X_\alpha| > |\alpha| \text{ infinitely often}\} = 1$. By the definition of n-differentiability, we have

$$X_{\alpha_1} = S_{\alpha_1} - S_{\alpha_2} + S_{\alpha_3} - \cdots - S_{\alpha_j}$$

so that $P\{|S_\alpha| \geqq |\alpha|/n \text{ infinitely often}\} = 1$. □

It should be noted at this point that, in contrast with the linear case, $P\{|Z_\alpha| > \varepsilon \text{ infinitely often}\} = 1$ is, a priori, compatible with the existence, for each ω in a set of full probability, of a β such that $|Z_\gamma| < \varepsilon$ for all $\gamma \geqq \beta$. Thus for a local lattice, for example, convergence of Z_β to zero in the usual sense of net convergence is weaker than the condition $P\{|Z_\beta| > \varepsilon \text{ finitely often}\} = 1$ for every $\varepsilon > 0$. The following proposition indicates that the two notions are equivalent when $\mathscr{A}$ is an integer lattice, i.e., a subset of K^r for some r, whenever the first moment of the X_α is finite. It is easy to construct examples to show that the two notions are not equivalent when the expectation does not exist.

PROPOSITION 3.1. *Let $\mathscr{A}$ be a subset of K^r for some r. Then the statements "$P\{|Z_\beta| > \varepsilon$ finitely often$\} = 1$ for each $\varepsilon > 0$" and "With probability 1, given $\varepsilon > 0$, there exists γ such that $\gamma \leqq \beta$ implies $|Z_\beta| < \varepsilon$" are equivalent.*

PROOF. It is clear that the first statement implies the second. Suppose the second true; we prove the case $r = 2$, leaving the more or less straightforward extension to $r > 2$ to the reader.

For a given ω, and a given $\varepsilon > 0$, fix $\gamma = (m, n)$ such that $\gamma \leqq \beta$ implies $|Z_\beta| < \varepsilon$. Fix $j < m$ and consider the Z_β on the column of K^2 with first coordinate equal to j. Since we can obviously do for rows anything that we can do for columns, it will suffice to show that on a set of full probability, $|Z_\beta| > \varepsilon$ only finitely often on this column.

To do this we enumerate the points (k, l) of $\mathscr{A}$ with $k \leqq j$ as follows: Starting with the first point (if any) on the first row, count left to right until j is reached, then move to $(0, 2)$ and count along the second row up to j, etc. Let N_α be the number given to the point α in this ordering, and note that on the line $k = j$ one has $N_\alpha = |\alpha|$. Since $E\{|X|\} < \infty$ we have $S_{N_\alpha}/N_\alpha \to_{N_\alpha \uparrow \infty} 0$ a.s.; it follows that on the line $k = j$ we can have $|Z_\alpha| > \varepsilon$ only finitely often a.s. □

We will say "$\lim_\beta \sup |Z_\beta| = \infty$ a.s." if with probability 1, given $M > 0$ and $\gamma \in \mathscr{A}$, there exists $\beta \geqq \gamma$ with $|Z_\beta| > M$.

COROLLARY TO THEOREM 3.1 AND PROPOSITION 3.1. Let $\mathscr{A}$ be a subset of K^r for some r. Suppose that $\mathscr{A}$ is differentiable and that $E\{M(|X|)\} = \infty$. Then if $M(x)$ varies dominatedly at infinity, $\lim\sup_\beta |Z_\beta| = \infty$ a.s.

REMARK. Theorem 3.1 and Proposition 3.1 illustrate clearly the difficulty of proving any general a.s. convergence theorems for martingales on directed sets, even very regular ones (at least for reversed martingales, the case which one would expect to be, if anything, easier). In particular one can easily construct a reversed martingale on a subset of K^2 which has bounded moments of order $1 + \delta$ for $\delta > 0$ but which does not converge a.s.

4. An analogue of a theorem of Hsu and Robbins. A theorem of Hsu and Robbins (Erdös [4]) states that for i.i.d. random variables with zero means,

$$E\{X^2\} < \infty \mapsto \textstyle\sum_{n=1}^{\infty} P\{|S_n| > n\varepsilon\} < \infty \qquad \text{for any} \quad \varepsilon < 0\,.$$

The converse is also true, a result due to Erdös. It is evident that the Hsu–Robbins theorem and the Borel–Cantelli lemma give the strong law of large numbers in the case when the second moment is finite.

We now derive an analogue of this result for partially ordered sets.

THEOREM 4.1 *Let* $\{X_\alpha\}$ *be identically distributed as* X. *Let* $(\mathscr{A}, \leqq)$ *be a denumerable partially ordered set satisfying conditions* (a) *and* (b) *of Section* 3. *If* $M(x)$ *varies regularly at infinity, then*

$$E\{|X|M(|X|)\} < \infty \mapsto \sum_\alpha P\{|S_\alpha| > \varepsilon|\alpha|\} < \infty \qquad \forall \varepsilon > 0 .$$

PROOF. The proof is modeled on that of Erdös [4] for the linearly ordered case. (For simplicity we take $\varepsilon = 1$.)

Suppose $2^i \leqq |\alpha| < 2^{i+1}$. We set

$$R_\alpha^{\ 1} = \{\omega \mid \max_{\gamma\leqq\alpha} |X_\gamma| > 2^{i-2}\}$$

$$R_\alpha^{\ 2} = \{\omega \mid |X_{\gamma_1}| > |\alpha|^\mu, |X_{\gamma_2}| > |\alpha|^\mu \text{ for at least two } \gamma_1 \leqq \alpha, \gamma_2 \leqq \alpha\}$$

where μ is a number to be determined, between $\frac{1}{2}$ and 1.

$$R_\alpha^{\ 3} = \{\omega \mid |S_\alpha^*| > 2^{i-2}\}$$

where

$$S_\alpha^* = \sum_{\gamma\leqq\alpha, |X_\gamma|\leqq|\alpha|^\mu} X_\gamma .$$

Since $\{|S_\alpha| > |\alpha|\} \subset R_\alpha^{\ 1} \cup R_\alpha^{\ 2} \cup R_\alpha^{\ 3}$ it will suffice to prove that $\sum_\alpha P(R_\alpha^{\ i}) < \infty$ for $i = 1, 2, 3$.

Letting $a_i = P\{|X| > 2^i\}$, if $E\{|X|M(|X|)\} < \infty$ we must have

$$\sum_{i=1}^\infty M(2^i)2^i(a_{i-1} - a_i) < \infty . \tag{4.1}$$

But $P(R_\alpha^{\ 1}) \leqq |\alpha|a_{i-2} < 2^{i+1}a_{i-2}$ so that

$$\sum_\alpha P(R_\alpha^{\ 1}) = \sum_{i=0}^\infty \sum_{2^i\leqq|\alpha|<2^{i+1}} P(R_\alpha^{\ 1}) \leqq \sum_{i=0}^\infty 2^{i+1}a_{i-2}[M(2^{i+1}) - M(2^i)]$$

and we need only show the convergence of the last series. But from (4.1) we easily deduce

$$\sum_{i=0}^\infty a_i 2^i[M(2^{i+1}) - M(2^i)] < \infty \tag{4.2}$$

and since M varies regularly at infinity this implies the desired convergence.

Now consider $R_\alpha^{\ 2}$. We have

$$P(R_\alpha^{\ 2}) \leqq \sum_{\gamma_1,\gamma_2\leqq\alpha} \{P[|X| > |\alpha|^\mu]\}^2 \leqq |\alpha|^2C\{|\alpha|^{2\mu}(M(|\alpha|^\mu))^2\}^{-1}$$

by Chebyshev's inequality. Let $M(x) = x^\rho L(x)$, where $L(x)$ is slowly varying at infinity; clearly $\rho \geqq 1$. We have

$$P(R_\alpha^{\ 2}) \leqq |\alpha|^{2-2\mu}[|\alpha|^{2\mu\rho}\{L(|\alpha|^\mu)\}^2]^{-1} = |\alpha|^{2-2\mu-2\rho\mu}/[L(|\alpha|^\mu)]^2$$

so that

$$\sum_\alpha P(R_\alpha^{\ 2}) \leqq \sum_{j=1}^\infty d(j)j^{2-2\mu-2\rho\mu}/[L(j^\mu)]^2 .$$

But one verifies easily by partial summation that the convergence of $\sum_{j=1} d(j)/j^\delta$ is equivalent to that of $\sum_{j=1} M(j)/j^{\delta+1}$; since L is slowly varying the convergence of $\sum_\alpha P(R_\alpha^{\ 2})$ will thus be assured provided that $\sum_{j=1}^\infty \{L(j)/[L(j^\mu)]^2\}j^{\rho+1-2\mu-2\rho\mu} < \infty$. Hence if μ is chosen such that

$$\rho + 2 < 2\mu(\rho + 1) \tag{4.3}$$

we will have $\sum_\alpha P(R_\alpha^{\ 2}) < \infty$ (since $L(j)/j^\varepsilon$ and $1/L(j)j^\varepsilon \mapsto 0$ for any $\varepsilon > 0$).

Finally we come to R_α^3. We define

$$Y_\alpha' = X_\alpha \qquad |X_\alpha| < |\alpha|^\mu$$
$$= 0 \qquad |X_\alpha| < |\alpha|^\mu$$
$$Y_\alpha = Y_\alpha' - E(Y_\alpha')$$

(note that $E(Y_\alpha') \mapsto 0$ as $|\alpha| \mapsto \infty$).
Let k be the smallest integer $\geqq \rho + 1$. Let $\varepsilon > 0$ be chosen small enough that no integer lies between $\rho + 1 - \varepsilon$ and $\rho + 1$. Then by definition of R_α^3 we have

$$\begin{aligned} P(R_\alpha^3) &\leqq P\{|S_\alpha^*| > |\alpha|/16\} \leqq E\{|S_\alpha^*|^{2k+2}\}[16/|\alpha|]^{2k+2} \\ &\leqq C[16/|\alpha|]^{2k+2}\{\textstyle\sum_{\gamma\leqq\alpha} E(Y_\gamma^{2k+2}) + \cdots \\ &\qquad + \textstyle\sum_{\gamma_i\leqq\alpha;i=1,2,\cdots,k+1} E(Y_{\gamma_1}^2)\cdots E(Y_{\gamma_{k+1}}^2)\} \end{aligned}$$

where C depends only on k (by the independence of the Y_γ, we need only consider sums in which each Y_γ appears to at least the second power). Using the fact that $E\{|X|^{\rho+1-\varepsilon}\} < \infty$ and the assumption that $\frac{1}{2} < \mu < 1$, it is a straightforward verification (which we defer until the end of the proof as Lemma 5.1) that each of the above sums of the form

$$\textstyle\sum_{p_1+p_2+\cdots+p_j=2k+2;\gamma_i\leqq\alpha;i=1,2,\cdots,j} E(Y_{\gamma_1}^{p_1})\cdots E(Y_{\gamma_j}^{p_j})$$

is bounded by

$$C_1[|\alpha|^\mu]^{2k+2-(\rho+1-\varepsilon)}|\alpha| + C_2\begin{cases}|\alpha|^{k+1} & \rho > 1\\ |\alpha|^{k+1+3\mu\varepsilon} & \rho = 1\end{cases}$$

where C_1 and C_2 are constants depending only on k.
Hence $P(R_\alpha^3)$ is dominated by

$$C\{|\alpha|^{(2k+1-\rho+\varepsilon)\mu-2k-1} + |\alpha|^{3\mu\varepsilon-(k+1)}\};$$

thus the convergence of $\sum_\alpha P(R_\alpha^3)$ will follow from that of

$$\textstyle\sum_{j=1}^\infty d(j)\{j^{(2k+1-\rho+\varepsilon)\mu-2k-1} + j^{3\mu\varepsilon-(k+1)}\}$$

or, (by the remark above), what is the same thing, the convergence of

$$\textstyle\sum_{j=1}^\infty L(j)\{j^{(2k+1-\rho+\varepsilon)\mu-2k-2+\rho} + j^{3\mu\varepsilon-(k+1)+\rho-1}\}.$$

The sum of the second terms is obviously convergent; the first series will converge provided that

$$(2k + 1 - \rho + \varepsilon)\mu - 2k - 2 + \rho < -1 \tag{4.4}$$

i.e., if (since ε can be taken arbitrarily small)

$$\mu < 1. \tag{4.5}$$

Hence from (4.3) and (4.5), the theorem will be proven if we choose μ to satisfy

$$(\rho + 2)/2(\rho + 1) < \mu < 1. \tag{4.6}$$

□

REMARK. Under more restrictive conditions on $M(x)$, the converse of Theorem 4.1 is true as well. For example, $L(x)$ monotonically increasing will suffice.

Thus for the lattice K^r the convergence of $\sum_\alpha P\{|S_\alpha| > \varepsilon|\alpha|\}$ is equivalent to the condition

$$E\{X^2(\log^+ |X|)^{r-1}\} < \infty .$$

Here is the result which completes the proof of Theorem 5.1.

LEMMA 4.1. *Under the hypotheses of Theorem* 5.1, *each sum*

$$\sum_{p_1+p_2+\cdots+p_j=2k-2,\gamma_i\leqq\alpha;i=1,2,\cdots,j} E(Y_{\gamma_1}^{p_1}) \cdots E(Y_{\gamma_j}^{p_j})$$

is bounded by

$$C_1[|\alpha|^\mu]^{2k+2-(\rho+1-\varepsilon)}|\alpha| + C_2 \begin{cases} |\alpha|^{k+1} & \rho > 1 \\ |\alpha|^{3\mu\varepsilon+k+1} & \rho = 1 . \end{cases}$$

PROOF. We may suppose that $p_1 \geqq p_2 \geqq p_3 \geqq \cdots \geqq p_j$; since each $p_i \geqq 2$, $j \leqq k+1$. There are three cases to be considered (when $\rho = 1$ the lemma is easy, so we will assume $\rho > 1$):

Case 1. *All the p_i are $< \rho + 1$.* In this case $E(Y_{\gamma_i}^{p_i}) \leqq E\{|X|^{p_i}\} < \infty$ so that the sum is $\leqq C_2|\alpha|^{k+1}$.

Case 2. $p_1 \geqq \rho + 1$, $p_i < \rho + 1$ *for* $i \geqq 2$. In this case $p_1 \geqq k$ and we thus have $j \leqq [2k+2-p_1/2]+1$. Then

$$E(Y_{\gamma_1}^{p_1}) \leqq E(Y_{\gamma_1}^{\rho+1-\varepsilon}Y_{\gamma_1}^{p_1-(\rho+1-\varepsilon)}) \leqq C(|\alpha|^\mu)^{p_1-(\rho+1-\varepsilon)}$$

so the sum in question is bounded by $(|\alpha|^\mu)^{p_1-(\rho+1-\varepsilon)}|\alpha|^{[k+1-p_1/2]+1}$ which is $\leqq C(|\alpha|^\mu)^{2k+2-(\rho+1-\varepsilon)}|\alpha|$ since $\mu > \frac{1}{2}$.

Case 3. $p_1 \geqq \rho + 1$, $p_2 \geqq \rho + 1$. Here we distinguish two subcases:

(a) $p_1 + p_2 = 2k + 2$. Then, reasoning as in Case 2, the sum is bounded by

$$C_1|\alpha|^2(|\alpha|^\mu)^{p_1-(\rho+1-\varepsilon)}(|\alpha|^\mu)^{p_2-(\rho+1-\varepsilon)} = C_1|\alpha|^2(|\alpha|^\mu)^{2k+2-2(\rho+1-\varepsilon)}$$

which is

$$\leqq C_1(|\alpha|^\mu)^{2k+2-(\rho+1-\varepsilon)}|\alpha|$$

provided that $\mu \geqq 1/(\rho+1-\varepsilon)$, i.e., provided that $\mu > \frac{1}{2}$.

(b) $p_1 + p_2 = 2k$. In this case the sum is

$$\leqq C_1|\alpha|^3(|\alpha|^\mu)^{p_1-(\rho+1-\varepsilon)}(|\alpha|^\mu)^{p_2-(\rho+1-\varepsilon)}$$

which is

$$\leqq C_1(|\alpha|^\mu)^{2k+2-(\rho+1-\varepsilon)}|\alpha|$$

whenever $\mu \geqq 2/(\rho+3-\varepsilon)$; i.e., whenever $\mu > \frac{1}{2}$. This completes the lemma. □

According to Theorem 4.1, $E\{|X|^3\}$ is sufficient to guarantee $P\{|Z_\beta| > \varepsilon$ finitely often$\} = 1$ in Example 2.2, whereas by Theorem 3.1 the condition $E(X^2) < \infty$ is necessary. (As far as ordinary net convergence is concerned, it is easy to show that $E\{|X| \log^+ |X|\} < \infty$ is sufficient to give $Z_\beta \to 0$ a.s.)

Theorem 4.1 has the virtue of considerable generality, but since it takes no account of the order properties of the set (beyond conditions (a) and (b)) one cannot expect it to yield sharp results in any particular case.

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Note added in proof. The result of the special case in Example 2.5 has also been obtained (in a different way) by P. Gabriel (note to appear in *C.R. Acad. Sci. Paris*). Gabriel also shows that for more general martingales in the "wedge" of Example 2.5, the condition of $X \log X$ integrability is necessary for a.s. convergence; the putative analogy with the results on differentiability of multiple integrals thus appears suspect.