The Annals of Probability
1974, Vol. 2, No. 5, 840-864

R-THEORY FOR MARKOV CHAINS ON A GENERAL STATE SPACE I: SOLIDARITY PROPERTIES AND R-RECURRENT CHAINS

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This paper develops, for a Markov chain $\{X_n\}$ on a general space $(\mathscr{X}, \mathscr{F})$ with n-step transition probabilities $P^n(x, A)$, $x \in \mathscr{X}$, $A \in \mathscr{F}$, a theory analogous to that of Vere-Jones for Markov chains on the integers. If the chain is ϕ-irreducible there is a partition $\mathscr{K}$ of $\mathscr{X}$ such that ϕ-almost all of the power series $G_z(x, A) = \sum_n P^n(x, A)z^n$ have a common radius of convergence R for A in any element of $\mathscr{K}$, and they all diverge (R-recurrence) or all converge (R-transience) for $z = R$. The R-recurrent case is then investigated, and it is shown that there exist essentially unique non-zero solutions Q, f to the R-subinvariant equations $Q \geqq RQP$ and $f \geqq RPf$, and that Q and f satisfy these inequalities with equality: a relationship between Q and f and first-entrance probabilities is also established. Further, if $\{X_n\}$ is aperiodic, $\lim_{n\to\infty} R^nP^n(x, A) = f(x)Q(A)/\int_{\mathscr{X}} f(y)Q(dy)$ for almost all $x \in \mathscr{X}$ and A in any element of a second partition.

The methods used are probabilistic and depend mainly on generating function techniques: it is pointed out that these techniques do not depend on the substochasticity of the transition probabilities, and hence the results are true in a much wider context.

1. Description of the process; Condition I. Let $\mathscr{X}$ be an arbitrary set, on which is defined a σ-field $\mathscr{F}$. We consider a Markov chain $\{X_n; n = 0, 1, \cdots\}$ taking values in $\mathscr{X}$, with stationary transition probabilities $P(x, A)$, $A \in \mathscr{F}$; for fixed $x \in \mathscr{X}$, $P(x, \cdot)$ is a probability measure on the σ-field $\mathscr{F}$, and for each fixed $A \in \mathscr{F}$, $P(\cdot, A)$ is a measurable function on $\mathscr{X}$. We denote the n-step transition probabilities of the chain $\{X_n\}$ by

$$P^n(x, A) = \Pr\{X_n \in A \mid X_0 = x\}, \qquad n \geqq 1;$$

these are defined iteratively by

$$P^n(x, A) = \int_{\mathscr{X}} P(x, dy)P^{n-1}(y, A). \tag{1.1}$$

We shall write $P^0(x, A) = \delta(x, A)$, where $\delta(x, A) = 1$ if $x \in A$ and $\delta(x, A) = 0$ if $x \notin A$.

Extensive use will be made of the idea of *taboo probabilities*; we write, for $A, B \in \mathscr{F}$,

$$_BP^n(x, A) = \Pr\{X_n \in A, Y_r \notin B, r = 1, \cdots, n-1 \mid X_0 = x\}.$$

Received January 3, 1973; revised January 15, 1974.

AMS 1970 *subject classifications.* Primary 60J05, 47D05; Secondary 60J35, 45N05, 45C05, 47A35.

Key words and phrases. R-theory, R-recurrence, Markov chains, invariant measures, invariant functions, equilibrium measures, limit probabilities, ratio limit theorems, positive operators.

These are defined analytically by

$$ {}_BP^n(x, A) = \int_{B^c} P(x, dy)\,{}_BP^{n-1}(y, A)\,, \tag{1.2} $$

where B^c denotes the complement of B in $\mathscr{X}$.

The purpose of this paper is firstly to investigate the properties of the power series

$$ G_z(x, A) = \sum_{n=1}^{\infty} P^n(x, A)z^n \tag{1.3} $$

for $x \in \mathscr{X}$ and $A \in \mathscr{F}$. Since $P^n(x, A) \leqq 1$, $G_z(x, A)$ exists and is finite at least for $0 \leqq |z| < 1$. In particular, in Section 2 we prove some solidarity theorems regarding the radii of convergence of these series. These theorems, and in many cases their proofs, are suggested by similar results in the case $\mathscr{X} = \mathbb{Z}$ (where $\mathbb{Z}$ is the set of nonnegative integers) proved by Vere-Jones (1962), which we shall refer to as VJ1.

Secondly, we prove existence results for r-subinvariant measures and functions for $\{X_n\}$ (defined in Section 3) which extend those of VJ1 and Vere-Jones (1967), which shall be referred to as VJ2. We show in Section 3 that results of Harris (1956) for $r = 1$ can be generalized: in a sequel we shall show that under certain conditions other results of Harris (1957), Veech (1963), Pruitt (1964) and Moy (1967) can also be generalized from the discrete state space case.

The fourth section contains work on first entrance analogues, and criteria for the finiteness of r-subinvariant measures in terms of these; the fifth R-positivity results; and the sixth indicates how the work extends to general nonnegative operators as VJ2 extends VJ1.

We write, for $n, j = 1, 2, \cdots$

$$ \begin{aligned} \bar{B}(n, j) = \{x \in \mathscr{X} \colon & P^n(x, B) \in ((j+1)^{-1}, j^{-1}], \\ & P^r(x, B) = 0, r = 1, \cdots, n-1\}\,; \end{aligned} \tag{1.4} $$

also put

$$ \bar{B} = \bigcup_{j,n=1}^{\infty} \bar{B}(n, j) \tag{1.5} $$

so that $\bar{B}$ is the set of points in $\mathscr{X}$ from which it is possible to reach B.

In order to prove solidarity results for Markov chains on a general state space, one needs an analogue of irreducibility, and in this we follow Orey (1971) and call the chain $\{X_n\}$ *ϕ-irreducible* if there exists a σ-finite measure ϕ, not identically zero, such that $\phi(B) > 0$ implies $\bar{B} = \mathscr{X}$.

LEMMA 1.1. *Suppose that $\{X_n\}$ is ϕ-irreducible for some ϕ; then there exists a measure M on $\mathscr{F}$ such that*

(i) *$\{X_n\}$ is M-irreducible;*
(ii) *if $B \in \mathscr{F}$ is such that $M(B) = 0$, then $M(\bar{B}) = 0$.*

PROOF. Take a fixed real α, $0 < \alpha < 1$. Since ϕ is σ-finite, there exists a partition $\mathscr{K} = (K(j))$ with $0 < \phi(K(j)) < \infty$ for each j. Define M by setting, for each $B \in \mathscr{F}$,

$$M(B) = \sum_j 2^{-j} \int_{K(j)} \phi(dx) G_\alpha(x, B)/\phi(K(j)) ; \tag{1.6}$$

since $M(\mathscr{X}) < \alpha/(1-\alpha)$, M is trivially σ-finite. Suppose $M(A) > 0$ for some $A \in \mathscr{F}$. Write

$$A_k = \{y : G_\alpha(y, A) > k^{-1}\} ;$$

using the obvious inequality

$$P^r(x, A) \geqq \int_{A_k} P^n(x, dy) P^{r-n}(y, A) , \qquad r > n ,$$

shows that, for $x \in \bar{A}_k(n, j)$,

$$G_\alpha(x, A) \geqq [k(j+1)]^{-1} . \tag{1.7}$$

But from (1.6), $M(A) > 0$ implies $\phi(A_k) > 0$ for some k, and since $\{X_n\}$ is ϕ-irreducible, $\bigcup_{n,j} \bar{A}_k(n, j) = \bar{A}_k = \mathscr{X}$, and so (1.7) shows $\{X_n\}$ is also M-irreducible.

If, on the other hand, $M(A) = 0$, then (1.6) shows that $\phi(\bar{A}) = 0$. Notice that $\bar{\bar{A}} \subseteq \bar{A}$, since certainly if one can reach $\bar{A}$ from x then one can reach A from x; hence $\phi(\bar{\bar{A}}) = 0$, and from (1.6), we must also have $M(\bar{A}) = 0$, which is (ii). □

We shall henceforth assume that M denotes a fixed, σ-finite, measure on $\mathscr{F}$, not identically zero and that $\{X_n\}$ satisfies

CONDITION I. (i) $\{X_n\}$ is M-irreducible
(ii) $M(B) = 0$ implies $M(\bar{B}) = 0$.

We know from Lemma 1.1 that it is enough to assume that $\{X_n\}$ is ϕ-irreducible for some ϕ for Condition I to hold. When in the sequel we speak of results holding for *almost all $x \in \mathscr{X}$, it will be understood that the exceptional set is of M-measure zero*, although we shall often emphasize this explicitly.

A related notion of "irreducibility" for general state space Markov chains is sometimes expressed by demanding that $\{X_n\}$ satisfy

CONDITION I′. The measures γ_x, defined by

$$\gamma_x(\cdot) = \sum_1^\infty 2^{-n} P^n(x, \cdot) = G_{\frac{1}{2}}(x, \cdot) , \tag{1.8}$$

are all equivalent (have the same null sets).

Condition I′ is used in, for example, Šidák (1967); it is somewhat stronger than Condition I, for if we take $\gamma_x = M$, $\{X_n\}$ is M-irreducible and, under Condition I′, $M(A) = 0$ implies not merely that $\bar{A}$ is M-null, but that $\bar{A}$ is empty. The following example provides a chain satisfying Condition I but not Condition I′: we shall employ this example more than once to illustrate that naturally occurring null sets need not be empty.

EXAMPLE 1. Let $\{Y_n\}$ be an irreducible (in the classical sense) Markov chain on the *positive* integers, and let $\{X_n\}$ be a Markov chain on $\mathcal{Z}$, with

$$\Pr\{X_n \in A \mid X_0 = x\} = \Pr\{Y_n \in A \mid Y_0 = x\}, \qquad x > 0;$$
$$\Pr\{X_n = 0 \mid X_{n-1} = 0\} = \alpha < 1;$$
$$\Pr\{X_n \in A \mid X_{n-1} = 0\} = \beta\delta(1, A), \qquad 0 < \beta \leqq 1 - \alpha.$$

Example 1 shows the essential difference between Condition I and the standard notion of irreducibility on the integers, for the former ensures, when $\mathscr{X} = \mathcal{Z}$, only that there is exactly one closed class of the chain: it does not ensure that this closed class exhausts the state space.

Finally, we define the collection of sets $\mathscr{F}^+$ by $A \in \mathscr{F}^+$ if and only if $A \in \mathscr{F}$ and $M(A) > 0$ ($\mathscr{F}^+$ will not, of course, be a σ-field); and for $A \in \mathscr{F}^+$, we shall write $\mathscr{F}_A$ for the collection of $\mathscr{F}$-measurable subsets of A, and $\mathscr{F}_A{}^+$ for those elements of $\mathscr{F}_A$ which have positive M-measure. If $\mathscr{K}$ is a partition of $\mathscr{X}$, then we also write $\mathscr{F}_{\mathscr{K}}$ for the union of the σ-fields $\mathscr{F}_K$, $K \in \mathscr{K}$, and $\mathscr{F}_{\mathscr{K}}{}^+ = \bigcup \mathscr{F}_K{}^+$.

2. Solidarity results for the series $G_z(x, A)$. In VJ1, Vere-Jones proved that if $\mathscr{X} = \mathcal{Z}$, and if R_{ij} is the radius of convergence of the series $G_z(i, \{j\})$, then all the R_{ij} have a common value R when the chain is irreducible (in the classical sense), and that either all the series $G_R(i, \{j\})$ are convergent or they are all divergent. The result which we shall prove in this section is

THEOREM 1. *Suppose $\{X_n\}$ is as in Section 1, and M satisfies Condition I. Then there exists a real number $R \geqq 1$, a partition $\mathscr{K}$ of $\mathscr{X}$ and an M-null set N such that, for $x \notin N$ and $A \in \mathscr{F}_{\mathscr{K}}{}^+$, R is the radius of convergence of the series $G_z(x, A)$. Either $G_R(x, A) < \infty$ for every $x \notin N$ and $A \in F_{\mathscr{K}}{}^+$, or $G_R(x, A) = \infty$ for every x and A. If $R = 1$, then in fact $N = \emptyset$; if $R > 1$, then N may be nonempty.*

The proof of this theorem will occupy the bulk of this section, and we give it in several propositions.

PROPOSITION 2.1. *Let $r > 0$ be fixed. Either $G_r(x, A) = \infty$ for every $x \in \mathscr{X}$ and $A \in \mathscr{F}^+$, or there is a partition $\mathscr{K}_r$ and a null set N_r such that for $x \notin N_r$, and $A \in \mathscr{F}^+_{\mathscr{K}_r}$, $G_r(x, A) < \infty$.*

PROOF. Suppose for some $\zeta \in \mathscr{X}$ and $B \in \mathscr{F}^+$, $G_r(\zeta, B) < \infty$. We show that the set $N_r = \{y: G_r(y, B) = \infty\}$ and the collection of sets $\mathscr{K}_r = \{\bar{B}(n, j), n, j = 1, 2, \cdots\}$ (which is a partition since $M(B) > 0$) satisfy the proposition. For any $n, m \geqq 0$ and any $\theta < 1$,

$$\theta^m r^{n+m} P^{n+m}(\zeta, B) \geqq \int_{N_r} (\theta r)^m P^m(\zeta, dy) r^n P^n(y, B);$$

summing over n and m gives

$$\infty > (1 - \theta)^{-1} G_r(\zeta, B) \geqq \int_{N_r} G_{r\theta}(\zeta, dy) G_r(y, B).$$

By the definition of N_r, this means $G_{r\theta}(\zeta, N_r) = 0$, and from Condition I, $M(N_r) = 0$.

Secondly, for $m > n \geqq 1, j \geqq 1$, we have

$$r^m P^m(x, B) \geqq \int_{\bar{B}(n,j)} r^{m-n} P^{m-n}(x, dy) r^n P^n(y, B) \geqq r^n(j+1)^{-1} r^{m-n} P^{m-n}(x, \bar{B}(n,j));$$

summing over m gives

$$G_r(x, B) \geqq r^n(j+1)^{-1} G_r(x, \bar{B}(n,j)),$$

and so for all $x \notin N_r$, $G_r(x, \bar{B}(n,j)) < \infty$. □

Adopting the nomenclature of Vere-Jones (VJ1) from the countable state space, we shall call the chain $\{X_n\}$ *r-transient* if there exists $\zeta \in \mathscr{X}$ and $A \in \mathscr{F}^+$ such that $G_r(\zeta, A) < \infty$. We shall call a set $A \in \mathscr{F}^+$ an *r-transient set* if, for some one (and hence, from Proposition 2.1, almost all) $x \in \mathscr{X}$, $G_r(x, A) < \infty$. From Proposition 2.1, if $\{X_n\}$ is r-transient, then $\mathscr{X}$ can be partitioned into r-transient sets. Note that, if $\{X_n\}$ is r-transient, then $\{X_n\}$ is α-transient for each $\alpha \leqq r$. We can thus define R by

$$R = \sup\{r : \{X_n\} \text{ is } r\text{-transient}\}, \tag{2.1}$$

and we have that $\{X_n\}$ is r-transient for all $r < R$; by stochasticity, $R \geqq 1$. We say that $\{X_n\}$ is *R-recurrent* if it is not R-transient, and call a set A an *R-recurrent* set if $\{X_n\}$ is R-recurrent and A is r-transient for all $r < R$.

It should be noted that if $R = 1$, this definition of 1-recurrence is somewhat weaker than that of *ϕ-recurrence* given in Orey (1971), page 4; ϕ-recurrence demands that for every $A \in \mathscr{F}^+$, ${}_AG_1(x, A) \equiv 1$, $x \in \mathscr{X}$. Some connections between the two notions of recurrence are given in Section 2 of Jain and Jamison (1967).

PROPOSITION 2.2. *Suppose $\{X_n\}$ is R-recurrent. Then there exists a partition $\mathscr{K}_R$ of $\mathscr{X}$ into R-recurrent sets.*

PROOF. Choose some sequence $\{r_n\}$ of real numbers with $r_n \uparrow R$, and write $N_R = \bigcup_n N_{r_n}$, where N_{r_n} is as in Proposition 2.1. Then for fixed $\zeta \in N_R^c$, $G_{r_n}(\zeta, \cdot)$ is a σ-finite measure on $\mathscr{X}$ for each n, and it follows from a result of Kingman ((1967) page 73) that there is a partition $\mathscr{K}_R = (K_R(j))$ of $\mathscr{X}$ such that $G_{r_n}(\zeta, K_R(j)) < \infty$ for each n, j. Thus $\mathscr{K}_R$ is a partition of $\{X_n\}$ into R-recurrent sets. □

PROPOSITION 2.3. *Suppose $\{X_n\}$ is r-transient. Then if $r = 1$, there exists a partition $\mathscr{K} = (K(j))$ such that $G_1(x, K(j)) < \infty$ for every $x \in \mathscr{X}$ and all j; but if $r > 1$, then the null set N on which $G_r(x, A) = \infty$, for all $A \in \mathscr{F}^+$ need not be empty.*

PROOF. Suppose $r = 1$, and that for some $\zeta \in \mathscr{X}$ and $A \in \mathscr{F}^+$, $G_1(\zeta, A) < \infty$. Let $A_n = \{x \in \mathscr{X} : G_1(x, A) < n\}$; from Proposition 2.1, for large enough n, $A_n \cap A \in \mathscr{F}^+$. For such an n, write $B = A_n \cap A$. We have, decomposing over the first entrance to B, and using the taboo probability notation introduced in (1.2),

$$P^m(x, B) = \sum_{k=1}^{m-1} \int_B {}_BP^k(x, dy) P^{m-k}(y, B) + {}_BP^m(x, B). \tag{2.2}$$

Writing the generating functions for taboo probabilities as

$$ {}_BG_z(x, A) = \textstyle\sum_{m=1}^{\infty} {}_BP^m(x, A)z^m , \tag{2.3} $$

for any sets $A, B \in \mathscr{F}$, we have, summing both sides of (2.2) over m,

$$ \begin{aligned} G_1(x, B) &= \textstyle\int_B {}_BG_1(x, dy)G_1(y, B) + {}_BG_1(x, B) \\ &\leqq (n+1)_BG_1(x, B) , \end{aligned} \tag{2.4} $$

since by definition $G_1(y, B) \leqq G_1(y, A) \leqq n$ when $y \in B \subseteqq A_n$. But

$$ {}_BG_1(x, B) = \Pr\{\text{the chain } \{X_n\} \text{ ever enters } B \mid X_0 = x\} \leqq 1 , $$

and so from (2.4), $G_1(x, B) \leqq n + 1$ for every $x \in \mathscr{X}$. As in Proposition 2.1, this means that $G_1(x, \bar{B}(n, j))$ is finite for each n, j and *every* x. Hence the first statement of the proposition is proved.

To show the second, in Example 1 put $\alpha = c^{-1}$ and $\beta = 1 - c^{-1}$, where c is chosen with $1 < c < r$: since

$$ \begin{aligned} P^n(0, 1) &\geqq \Pr\{X_n = 1, X_s = 0, s = 1, \cdots, n-1 \mid X_0 = 0\} \\ &= (1 - c^{-1})c^{-n+1} , \end{aligned} $$

the series $G_z(0, 1)$ diverges for any real $z > c$, and so in particular for $z = r$. □

PROOF OF THEOREM 1. Define R by (2.1); then $R \geqq 1$, since $P^n(x, A) \leqq 1$ for every n, $x \in \mathscr{X}$, $A \in \mathscr{F}$. If $\{X_n\}$ is R-transient, we may choose $\mathscr{K} = \mathscr{K}_R$ and $N = N_R$ as in Proposition 2.1, and if X_n is R-recurrent as in Proposition 2.2; this then gives R as the radius of convergence of each of the power series $G_z(x, A)$, $x \notin N_R$, $A \in \mathscr{F}^+_{\mathscr{K}_R}$; for if one of these power series had radius of convergence $\alpha > R$, then $G_\beta(x, A) < \infty$ for $R \leqq \beta < \alpha$, and this would contradict the definition of R; whilst the nonnegativity of $P^n(x, A)$ ensures that the radius of convergence of such a series cannot be less than R, since A is an r-transient set, $r < R$. Proposition 2.1 also proves the penultimate, and Proposition 2.3 the ultimate statement of the theorem. □

Throughout the remainder of this paper we shall use R to denote the common radius of convergence shared by the series $G_z(x, A)$, and call R^{-1} the *convergence norm* of $\{X_n\}$ (cf. VJ2).

Finally in this section, we look at the M-absolutely continuous and M-singular parts of $P^n(x, \cdot)$; that is, write

$$ P^n(x, B) = \textstyle\int_B p^n(x, y)M(dy) + P_s^{\,n}(x, B) , \tag{2.5} $$

where for each x, $p^n(x, \cdot)$ is an $\mathscr{F}$-measurable function and $P_s^{\,n}(x, \cdot)$ is concentrated on a set S_x with $M(S_x) = 0$.

Write $g_z(x, y) = \sum_1^\infty p^n(x, y)z^n$, and let R_{xy} be the radius of convergence of $g_z(x, y)$. Perhaps the exact analogue of Vere-Jones' result is

THEOREM 2. *There exists a real number $R \geqq 1$, an M-null set N, and, for each $x \in N^c$, an M-null set N_x, such that*

(i) *for* $x \in N^c$, $y \in N_x^c$, $R_{xy} = R$.

(ii) *either* $g_R(x, y) < \infty$ *for all* $x \in N^c$ *and* $y \in N_x^c$, *or* $g_R(x, y)$ *diverges for all such* x, y; *in the former case* $\{X_n\}$ *is R-transient, in the latter R-recurrent.*

The choice of version of the densities may determine the null sets N_x, *but not the radius of convergence R or the null set N.*

PROOF. Let R, N and $\mathscr{K}$ be as in Theorem 1. Since $G_r(x, K(j)) \geqq \int_{K(j)} g_r(x, y)M(dy)$ from (2.5), there is a null set $N_x(j) \subseteq K(j)$ such that $R_{xy} \geqq R$ for $x \notin N$, $y \notin N_x(j)$. Let $L_x(n) = \{y \in \mathscr{X}: R_{xy} > R + 1/n\}$, and suppose $M(L_x(m)) > 0$ for some $m < \infty$; write $L_x'(k) = \{y \in L_x(m): g_{R+1/m}(x, y) \leqq k\}$. Since $L_x'(k) \uparrow L_x(m)$ as $k \to \infty$, we can find h such that $M(L_x'(h)) > 0$.

Finally, write $H_x = L_x'(h)\backslash S_x$; we have, from (2.5) and the above,

$$G_{R+1/m}(x, H_x) = \int_{H_x} g_{R+1/m}(x, y)M(dy) \leqq hM(H_x) .$$

This contradicts the definition of R since it implies $\{X_n\}$ is $(R + 1/m)$-transient, and so $L_x(n)$ is M-null for each n. Writing $N_x' = [\bigcup_j N_x(j)] \cup [\bigcup_n L_x(n)]$ leads to (i); a similar argument shows that there exists a set N_x'' for $x \in N^c$ such that (ii) holds. Writing $N_x = N_x' \cup N_x''$ then gives the theorem. □

One might hope to replace the null sets N_x in this theorem with a single M-null set, N_0 say. In general, except when $\mathscr{X}$ is countable, this cannot be done; if, for example, M allots zero measure to every singleton $\{x\}$, then by choosing a version of the density with $p^n(x, x) = \alpha^{-n}$ ($\alpha \neq R$), we can have $R_{xx} = \alpha \neq R$ for every $x \in \mathscr{X}$, and so $\bigcup N_x = \mathscr{X}$.

The solidarity results which we have proved are related, when $r = 1$, to those of Theorem 2 of Jain and Jamison (1967) if $\mathscr{F}$ is *separable* (countably generated), and to those of Sidák (1967). The latter shows that, if Q is a σ-finite 1-subinvariant measure, then (under Condition I′) either all sets of finite Q-measure are 1-transient, or all such sets are 1-recurrent. We shall show in Part II that this result can be derived from ours under Condition I, and further that if $r \leqq R$, and Q is an r-subinvariant measure (cf. (3.1) below) then all sets of finite Q-measure are r-transient sets (if $\{X_n\}$ is r-transient) or r-recurrent sets (if $\{X_n\}$ is r-recurrent). This identification of r-transient and r-recurrent sets is useful, for no solidarity properties exist for general transition probabilities $P^n(x, A)$, where A is an arbitrary member of $\mathscr{F}$; for we always have $\mathscr{X} \in \mathscr{F}$, and obviously $G_1(x, \mathscr{X}) = \infty$ for all $x \in \mathscr{X}$.

3. *R*-invariant measures and functions for *R*-recurrent chains. In this section we consider the inequalities

$$Q(A) \geqq r \int_{\mathscr{X}} Q(dx)P(x, A) , \qquad \text{all } A \in \mathscr{F} \tag{3.1}$$

$$f(x) \geqq r \int_{\mathscr{X}} P(x, dy)f(y) \qquad \text{almost all } x \in \mathscr{X} \tag{3.2}$$

and the corresponding equalities

$$Q(A) = r \int_{\mathscr{X}} Q(dx)P(x, A) \qquad \text{all } A \in \mathscr{F} \tag{3.3}$$

$$f(x) = r \int_{\mathscr{X}} P(x, dy)f(y) \qquad \text{almost all } x \in \mathscr{X}. \tag{3.4}$$

A σ-finite measure Q, *not* identically zero, satisfying (3.1) is called an *r-subinvariant measure* for $\{X_n\}$, and *r-invariant* if it also satisfies (3.3); whilst a non-negative measurable function f with $M\{y : f(y) > 0\} > 0$, satisfying (3.2) is called an *r-subinvariant function* for $\{X_n\}$, and r-invariant if it satisfies (3.4). If f is an r-subinvariant function, we write

$$N_f = H_f \cup \bar{H}_f , \tag{3.5}$$

where H_f is the null set on which (3.2) fails: from Condition I, $M(N_f) = 0$ and from (3.2), $f(x) > 0$ on N_f^c. In the countable case, (3.1)—(3.4) are studied extensively in VJ2.

PROPOSITION 3.1. *When $r > R$, $\{X_n\}$ admits no r-subinvariant measure or function.*

PROOF. Suppose Q satisfies (3.1) for $r > R$, and choose β such that $r > \beta > R$. Iterating (3.1) gives

$$Q(A) \geqq r^n \int_{\mathscr{X}} Q(dx)P^n(x, A) , \tag{3.6}$$

and multiplying each side of (3.6) by β^n and summing over n gives

$$\int_{\mathscr{X}} Q(dx)G_\beta(x, A) \leqq Q(A) \sum (\beta/r)^n < \infty$$

for some A with $0 < Q(A) < \infty$. But $\beta > R$ implies $G_\beta(x, A) = \infty$ for every $x \in \mathscr{X}$ (Theorem 1). So $Q \equiv 0$, which we have specifically excluded. A similar argument shows that no r-subinvariant function can exist when $r > R$. □

The next lemma, although simply proved, is a powerful tool for the study of r-subinvariant measures and functions (cf. VJ2 in the countable state case).

LEMMA 3.1. (i) *If f is r-subinvariant, $r \leqq R$, then*

$$f(x) \geqq \int_A {}_AG_r(x, dy)f(y)$$

for almost all $x \in \mathscr{X}$ and every $A \in \mathscr{F}^+$.

(ii) *If Q is r-subinvariant, then*

$$Q(B) \geqq \int_A Q(dy)_AG_r(y, B)$$

for every $B \in \mathscr{F}$ and $A \in \mathscr{F}^+$.

PROOF. From (3.2), defining N_f by (3.5),

$$f(x) \geqq r \int_{\mathscr{X}} P(x, dy)f(y) \geqq r \int_A P(x, dy)f(y) , \qquad x \notin N_f .$$

Assuming inductively that, for $y \notin N_f$

$$f(y) \geqq \int_A \sum_1^n r^m {}_AP^m(y, dw)f(w) ,$$

we have for $x \notin N_f$ (since $\bar{N}_f \subseteq N_f$)

$$\begin{aligned} f(x) &\geqq r \textstyle\int_{\mathscr{X}} P(x, dy) f(y) \\ &\geqq r \textstyle\int_{A^c} P(x, dy) \int_A \sum_1^n r^m {}_A P^m(y, dw) f(w) + r \int_A P(x, dy) f(y) \\ &= \textstyle\int_A \sum_1^{n+1} r^m {}_A P^m(x, dw) f(w) . \end{aligned}$$

Letting $n \to \infty$ in the induction proves (i); (ii) is proved similarly. □

Using the notation of Proposition 2.1, we have the following result:

PROPOSITION 3.2. *If $\{X_n\}$ is r-transient, each of the σ-finite measures $G_r(x, \cdot)$, $x \notin N_r$, is r-subinvariant for $\{X_n\}$, and each of the functions $G_r(\cdot, A)$, $A \in \mathscr{F}^+_{\mathscr{X}_r}$, is r-subinvariant for $\{X_n\}$.*

PROOF. For the r-subinvariant measure result, note

$$\begin{aligned} G_r(x, A) &= rP(x, A) + \textstyle\sum_{n=2}^\infty r^n P^n(x, A) \\ &= rP(x, A) + r \textstyle\int_{\mathscr{X}} \sum_{n=1}^\infty r^n P^n(x, dy) P(y, A) \\ &\geqq r \textstyle\int_{\mathscr{X}} G_r(x, dy) P(y, A) . \end{aligned}$$

A similar breakup of $P^n(x, A) = \int_{\mathscr{X}} P(x, dy) P^{n-1}(y, A)$ proves the r-subinvariant function result. □

A more detailed investigation of r-subinvariant measures for r-transient chains is carried out in a sequel to this paper. We shall now go on to present results for R-invariant measures and functions for R-recurrent chains: these parallel and extend those of VJ1 and VJ2 for $\mathscr{X} = \mathbb{Z}$, and those of Harris (1956) for 1-invariant measures for ϕ-recurrent chains on a general state space.

The method of proof is, firstly, to establish an existence result for R-invariant functions, and then to show that one can use this to utilise Harris' result for ϕ-recurrent chains to prove an existence result for R-invariant measures for R-recurrent chains.

For the remainder of this section, we shall assume not only that R^{-1} is the convergence norm of $\{X_n\}$, but also that *$\{X_n\}$ is R-recurrent.*

Suppose that f is some R-subinvariant function for $\{X_n\}$, and define N_f by (3.5): write $\mathscr{X}(f) = \mathscr{X} \backslash N_f$, $\mathscr{F}(f) = \mathscr{F}_{\mathscr{X}(f)}$. Then we can define a Markov chain $\{X_n(f)\}$ on $(\mathscr{X}(f), \mathscr{F}(f))$ by the (possibly defective) transition law

$$P_f(x, A) = R \textstyle\int_A P(x, dy) f(y)/f(x) , \tag{3.7}$$

$x \in \mathscr{X}(f)$, $A \in \mathscr{F}(f)$. Since, by construction, $P^r(x, N_f) = 0$, $x \notin N_f$, $r \geqq 1$ we have as the n-step transition probabilities of $\{X_n(f)\}$,

$$P_f^n(x, A) = R^n \textstyle\int_A P^n(x, dy) f(y)/f(x) , \qquad x \in \mathscr{X}(f),\ A \in \mathscr{F}(f) ,$$

and consequently

$$G_{(f)}(x, A) = \textstyle\sum_{n=1}^\infty P_f^n(x, A) = \int_A G_R(x, dy) f(y)/f(x) , \qquad x \in \mathscr{X}(f) . \tag{3.8}$$

Since $\{X_n\}$ is R-recurrent, this implies that $G_{(f)}(x, A)$ diverges for all $x \in \mathscr{X}(f)$ and $A \in \mathscr{F}(f)^+$, and so $\{X_n(f)\}$ must be 1-recurrent.

Now a 1-recurrent chain must have $P(x, \mathscr{X}) = 1$ for almost all x: supposing otherwise, we can find $A \in \mathscr{F}^+$ with $P(y, \mathscr{X}) \leqq \delta < 1$ for all $y \in A$, and using a last exit decomposition, we have, for all $r < 1$ and every x

$$\begin{aligned} G_r(x, A) &= {}_AG_r(x, A) + \int_A G_r(x, dy)\,{}_AG_r(y, A) \\ &\leqq 1 + G_r(x, A)\delta \end{aligned}$$

which implies $G_r(x, A) \leqq 1/(1-\delta)$; hence by monotone convergence $G_1(x, A) < \infty$, which contradicts the recurrence assumption. (It is, though, possible that $P(x, \mathscr{X}) < 1$ on a null-set, as Example 1 with $\{Y_n\}$ 1-recurrent and $\alpha + \beta < 1$ shows.)

But from (3.7), $P_f(x, \mathscr{X}(f)) = 1$ for almost all x proves

PROPOSITION 3.3. *If f is R-subinvariant and $\{X_n\}$ is R-recurrent, f is R-invariant.*

Using this proposition, together with Lemma 3.1, we can prove

PROPOSITION 3.4. *Suppose f is R-subinvariant for $\{X_n\}$. Then f is unique, in the sense that, if g is also R-subinvariant for $\{X_n\}$, $f(x) = cg(x)$ for some constant $c > 0$, except perhaps for x on a set of measure zero; and for any $A \in \mathscr{F}^+$, f satisfies the equation*

$$f(x) = \int_A {}_AG_R(x, dy)f(y) \tag{3.9}$$

for almost every $x \in \mathscr{X}$.

PROOF. (i) Suppose that f is an R-subinvariant (and so from Proposition 3.3, R-invariant) function, and A a set in $\mathscr{F}^+$. Write

$$f_A(y) = \int_A {}_AG_R(y, dw)f(w)\,;$$

from Condition I, $f_A(y) > 0$ for all y, and from Lemma 3.1 (i), $f(y) \geqq f_A(y)$ for $y \notin N_f$. We have, for $x \notin N_f$,

$$\begin{aligned} &R \int_{\mathscr{X}} P(x, dy)f_A(y) \\ &\quad = R \int_{A^c} P(x, dy)\Big[\int_A \textstyle\sum_1^\infty R^n\,{}_AP^n(y, dw)f(w)\Big] + R \int_A P(x, dw)f_A(w) \\ &\quad = \int_A \textstyle\sum_2^\infty R^n\,{}_AP^n(x, dw)f(w) + R \int_A P(x, dw)f_A(w) \\ &\quad \leqq f_A(x) \end{aligned} \tag{3.10}$$

and so $f_A(x)$ is R-subinvariant. From Proposition 3.3, f_A must thus be R-invariant, and so from (3.10), $f_A(w) = f(w)$ for almost all $w \in A$. If $w \in A \backslash N_f$ is such that this equality holds, R-invariance and Lemma 3.1 (i) then give, for any n,

$$f(w) = R^n \int_{\mathscr{X}} P^n(w, dy)f(y) \geqq R^n \int_{\mathscr{X}} P^n(w, dy)f_A(y) = f_A(w)\,.$$

Consequently $f(y) = f_A(y)$ for almost all $x \in \mathscr{X}$, which is (3.9).

(ii) Now let f, g be any two R-invariant functions for $\{X_n\}$, and define N_f, N_g as in (3.5); write $H = [N_f \cup N_g]^c$. For $x \in H$, (3.4) holds for both f and g. Choose $A \in \mathscr{F}_H^{\,+}$ such that

$$\inf_{y \in A} f(y) = \delta_f > 0\,, \qquad \infty > \eta_g = \sup_{y \in A} g(y)\,;$$

writing $d = \delta_f/\eta_g > 0$, we have that, for every $y \in A$, $f(y) \geqq dg(y)$. From (i) above, there exists an M-null set N with $\bar{N} \subseteqq N$ such that, for $x \in N^c$,

$$f(x) = \int_A {}_AG_R(x, dy)f(y) \geqq d \int_A {}_AG_R(x, dy)g(y) = dg(x) . \tag{3.11}$$

Since $\bar{N} \subseteqq N$, we have, writing $B = A \backslash N$, that for each $x \in N^c$, $E \in \mathscr{F}$,

$$ {}_AG_R(x, E) = {}_BG_R(x, E\backslash N) . \tag{3.12}$$

Define $c \geqq d$ as $c = \inf_{x \in B} [f(x)/g(x)]$; from (3.11) and (3.12), $x \in N^c$ implies

$$f(x) = \int_B {}_BG_R(x, dy)f(y) \geqq c \int_B {}_BG_R(x, dy)g(y) = cg(x) . \tag{3.13}$$

Let ε be small and positive; by definition we can find $\zeta \in B$ such that $[f(\zeta)/g(\zeta)] < c + \varepsilon$, and so

$$cg(\zeta) > f(\zeta) - \varepsilon g(\zeta) \geqq f(\zeta) - \varepsilon\eta_g . \tag{3.14}$$

Since $\zeta \in H$, R-invariance of f and g, (3.13) and (3.14) imply, for each n,

$$\begin{aligned} f(\zeta) &= R^n \int_{\mathscr{X}} P^n(\zeta, dy)f(y) \\ &= R^n \int_{N^c} P^n(\zeta, dy)f(y) \\ &\geqq cR^n \int_{N^c} P^n(\zeta, dy)g(y) \\ &= cg(\zeta) \\ &\geqq f(\zeta) - \varepsilon\eta_g . \end{aligned} \tag{3.15}$$

Since ε is arbitrary, (3.15) shows that

$$M\{y \in N^c : f(y) > cg(y)\} = 0 ,$$

and so, except for y in some null set K,

$$f(y) = cg(y) . \qquad \square$$

LEMMA 3.2 (see Orey (1971) page 10). *Suppose that $\mathscr{F}$ is countably generated, and let $p^n(x, y)$ be the density of $P^n(x, \cdot)$ with respect to M (cf. (2.5): if $\mathscr{F}$ is countably generated the density $p^n(x, y)$ can be chosen measurable with respect to both variables).*

Then for every set $A \in \mathscr{F}^+$, there is a set $C \in \mathscr{F}_A{}^+$, an integer n and a real number $\delta > 0$ such that

$$\inf_{(x,y) \in C \times C} p^n(x, y) = \delta . \tag{3.16}$$

THEOREM 3. *If $\{X_n\}$ is R-recurrent and satisfies Condition I, then there is a unique R-subinvariant function f for $\{X_n\}$ (in the sense that, if f and g are both R-subinvariant, $f(x) = cg(x)$ for some $c > 0$ except perhaps for x in some M-null set); the function* *f is strictly R-invariant, and satisfies, for each* $A \in \mathscr{F}^+$,

$$f(x) = \int_A {}_A G_R(x, dy) f(y) \tag{3.17}$$

for almost every $x \in \mathscr{X}$.

PROOF. We only need to prove that some R-subinvariant function f exists: the strict R-invariance then follows from Proposition 3.3 whilst the uniqueness, and (3.17), follow from Proposition 3.4.

Suppose first that $\mathscr{F}$ is countably generated. Let $\{r_k\}$ be a sequence of real numbers with $r_k \uparrow R$ as $k \to \infty$, and let A be some R-recurrent set (whose existence is guaranteed by Proposition 2.2); let C, n and δ be such that $C \in \mathscr{F}_A^+$, and (3.16) holds.

For $m > n$, we have, for every $x \in C$,

$$\begin{aligned} P^m(x, C) &\geqq \int_C P^n(x, dy) P^{m-n}(y, C) \\ &\geqq \int_C p^n(x, y) M(dy) P^{m-n}(y, C) \\ &\geqq \delta \int_C M(dy) P^{m-n}(y, C) , \end{aligned} \tag{3.18}$$

and, so defining f_k by

$$f_k(x) = G_{r_k}(x, C) / \int_C M(dy) G_{r_k}(y, C) , \tag{3.19}$$

for those x for which the right-hand side of (3.19) is finite (that is almost all x, by Theorem 1), and zero elsewhere, we have that for every k and almost every $\zeta \in C$,

$$f_k(\zeta) > \delta . \tag{3.20}$$

Define f by

$$f(x) = \liminf_{k \to \infty} f_k(x) ; \tag{3.21}$$

from (3.20), $f(\zeta) > \delta$ for almost every $\zeta \in C$. Moreover, since by Fatou's lemma $1 = \int_C M(dy) f_k(y) \geqq \int_C M(dy) f(y)$, f is finite for almost every $y \in C$.

Now using Proposition 3.2, we see that $f_k(x)$ defined by (3.19) is r_k-subinvariant for $\{X_n\}$; hence, again by Fatou, for almost all $x \in \mathscr{X}$

$$\begin{aligned} f(x) &= \liminf_{k \to \infty} f_k(x) \\ &\geqq \liminf_{k \to \infty} r_k \int_{\mathscr{X}} P(x, dy) f_k(y) \\ &\geqq R \int_{\mathscr{X}} P(x, dy) f(y) . \end{aligned} \tag{3.22}$$

Since, for some $\zeta \in C$, $f(\zeta) < \infty$, (3.22) shows that $f(y)$ is finite for almost every $x \in \mathscr{X}$, and so f defined by (3.21) is R-subinvariant for $\{X_n\}$.

Now suppose that $\mathscr{F}$ is arbitrary, and let A be an R-recurrent set. Again using a result in Orey ((1971) page 7), we can find a countably generated σ-field $\mathscr{F}_0$ such that, for each $B \in \mathscr{F}_0$, $P(\cdot, B)$ is measurable with respect to $\mathscr{F}_0$, and $A \in \mathscr{F}_0$. Thus to the Markov chain $\{X_n\}$ on $(\mathscr{X}, \mathscr{F}_0)$ we can apply the first part of the proof to find an $\mathscr{F}_0$-measurable R-subinvariant function f, which, since $\mathscr{F}_0 \subseteqq \mathscr{F}$, is clearly also $\mathscr{F}$-measurable; and so the theorem holds in general. □

We now use Theorem 3 to prove the analogous result for R-invariant measures.

LEMMA 3.3. *Suppose* $\{X_n\}$ *is* ϕ*-recurrent and satisfies Condition* I. *Then there is a unique* (*up to constant multiples*) 1*-subinvariant measure* Q_1 *for* $\{X_n\}$, *which is equivalent to* M, *and strictly* 1*-invariant for* $\{X_n\}$.

PROOF. The bulk of the proof is again in Orey ((1971) Theorem 7.2), where it is shown that

(i) there exists a set $D \in \mathscr{F}^+$ (called a D-set; for a definition see Orey, (1971) page 29) and a probability measure Q_D on $\mathscr{F}_D$ such that

$$Q_D(B) = \int_D Q_D(dy)_D G_1(y, B)\,, \qquad B \in \mathscr{F}_D\,, \tag{3.23}$$

and Q_D is the unique probability measure satisfying (3.23);

(ii) the measure Q_1 defined by

$$Q_1(B) = \int_D Q_D(dy)_D G_1(y, B)\,, \qquad B \in \mathscr{F}$$

is 1-invariant for $\{X_n\}$, and is the unique 1-invariant measure satisfying $Q(D) = 1$;

(iii) M is absolutely continuous with respect to Q_1.

It remains to show that no 1-subinvariant measure other than Q_1 exists for $\{X_n\}$, and that Q_1 is absolutely continuous with respect to M.

Suppose that U is 1-subinvariant for $\{X_n\}$, and that D is a D-set with $U(D) < \infty$. Then $U(A)/U(D) = Q_D(A)$, $A \in \mathscr{F}_D$; for Q_D is the unique solution of (3.23), but because of Lemma 3.1, for any $A \in \mathscr{F}_D$

$$\begin{aligned} U(D) &= U(A) + U(D \backslash A) \\ &\geqq \int_D U(dy)_D G_1(y, A) + \int_D U(dy)_D G_1(y, D\backslash A) \\ &= \int_D U(dy)_D G_1(y, D) \\ &= U(D) \end{aligned} \tag{3.24}$$

(the last equality since ${}_D G_1(y, D) \equiv 1$, $y \in D$, when D is a D-set) and (3.24) implies $U(\cdot)/U(D)$ is a solution of (3.23). Finally, another application of Lemma 3.1 for any $A \in \mathscr{F}$ gives

$$\begin{aligned} U(A)/U(D) &\geqq [U(D)]^{-1} \int_D U(dy)_D G_1(y, A) \\ &= \int_D Q_D(dy)_D G_1(y, A) \\ &= Q_1(A)\,; \end{aligned} \tag{3.25}$$

since U is 1-subinvariant and Q_1 is 1-invariant, (3.25) gives, for any n,

$$\begin{aligned} 1 = U(D)/U(D) &\geqq [U(D)]^{-1} \int_{\mathscr{X}} U(dy)P^n(y, D) \\ &\geqq \int_{\mathscr{X}} Q_1(dy)P^n(y, D) \\ &= Q_1(D) = 1\,. \end{aligned} \tag{3.26}$$

Thus the central inequality in (3.26) must be an equality, and (since $M(D) > 0$), $U(A)/U(D) = Q_1(A)$ for every $A \in \mathscr{F}$.

That $Q_1 \equiv M$ follows from this result as in Šidák (1967), Theorem 6. □

Statements (i)—(iii) in the proof of the lemma were first proved for separable $\mathscr{X}$ by Harris (1956), and the separability condition removed by Jamison and Orey (1967). Šidák (1967), under the stronger Condition I′, proved that the unique 1-invariant measure is the unique 1-subinvariant measure, and used this to show that it is equivalent to M. In fact, under Condition I′, if Q_r is r-invariant for $\{X_n\}$ for *any* $r \leqq R$ (whether $\{X_n\}$ is r-recurrent or not), then $Q_r(A) = 0$ implies $M(A) = 0$, and also $M(A) = 0$ implies $P(y, A) = 0$ for *all* $x \in \mathscr{X}$, and so

$$Q_r(A) = r \int_{\mathscr{X}} Q_r(dy)P(y, A) = 0 .$$

Hence under Condition I′ *all* r-invariant measures, $r \leqq R$, are equivalent to M: this is not necessarily true under Condition I, as we show in Part II.

We cannot apply Lemma 3.3 directly to $\{X_n(f)\}$, which is 1-recurrent rather than ϕ-recurrent, and we need

LEMMA 3.4. *Suppose $\{X_n\}$ is R-recurrent, and let f be an R-invariant function for $\{X_n\}$. Put, for $A \in \mathscr{F}^+$,*

$$\Delta_1(f, A) = \{y : f(y) \neq \int_A {}_AG_R(y, d\omega)f(\omega)\} ,$$

and put $\Delta(f, A) = \Delta_1(f, A) \cup \overline{\Delta_1(f, A)}$. If $\mathscr{F}$ is separable, then there exists $\Delta \in \mathscr{F}$ with $\bar{\Delta} \subseteqq \Delta$ and $M(\Delta) = 0$, such that $\Delta(f, A) \subseteqq \Delta$ for every $A \in \mathscr{F}^+$. Further, if $\{X_n(\Delta)\}$ is the Markov chain defined on $\mathscr{X}_\Delta = \mathscr{X} \backslash \Delta$ by

$$P_\Delta(x, A) = R \int_A P(x, dy)f(y)/f(x) ,$$

for $x \in \mathscr{X}_\Delta$, $A \in \mathscr{F}$ with $A \subseteqq \Delta^c$, then $\{X_n(\Delta)\}$ is ϕ-recurrent.

PROOF. Define $\{X_n(f)\}$ by (3.7): we know that $\{X_n(f)\}$ is 1-recurrent. From Theorem 2 of Jain and Jamison (1967), if $\mathscr{F}$ is separable there exists $H \subseteqq \mathscr{F}(f)$ with $\bar{H} \subseteqq H$, $M(H) = 0$, such that $\{X_n(f)\}$ restricted to $\mathscr{X}(f)\backslash H$ is ϕ-recurrent: if we take $\Delta = H \cup N_f$, this says that $\{X_n(\Delta)\}$ is ϕ-recurrent. The definition of $\{P_\Delta(x, A)\}$ together with that of ϕ-recurrence thus implies that for every $A \in \mathscr{F}^+$ with $A \subseteqq \Delta^c$ and every $x \in \Delta^c$,

$$1 = \sum_{n=1}^{\infty} {}_AP_\Delta{}^n(x, A) = \int_A \sum_{n=1}^{\infty} R^n {}_AP^n(x, d\omega)f(\omega)/f(x) ,$$

and since, if $x \in \Delta^c$, $\int_{A \cap \Delta} {}_AG_R(x, d\omega)f(\omega) = 0$ for any $A \in \mathscr{F}$, it follows that $\Delta_1(f, A) \subseteqq \Delta$. Finally, since $\overline{\Delta_1(f, A)}$ is then contained in $\bar{\Delta} \subseteqq \Delta$, we have that $\Delta(f, A) \subseteqq \Delta$. □

An example due to Blackwell (1945) shows that the 'global' null set Δ containining all the null sets $\Delta(f, A)$, $A \in \mathscr{F}^+$, may not exist when $\mathscr{F}$ is not separable.

From Theorem 3 and Lemmas 3.3 and 3.4 we can prove

THEOREM 4. *If $\{X_n\}$ is R-recurrent, there exists a unique R-subinvariant measure Q for $\{X_n\}$: Q is R-invariant, equivalent to M on $\mathscr{F}$, and satisfies, for each $B \in \mathscr{F}^+$*

$$Q(A) = \int_B Q(dy)_BG_R(y, A) , \qquad A \in \mathscr{F} . \tag{3.27}$$

PROOF. Note first that if Q is some R-subinvariant measure for $\{X_n\}$, and N is some M-null set with $\bar{N} \subseteq N$, and if for every $A \in \mathscr{F}_{N^c}$

$$Q(A) = R \int_{N^c} Q(dy)P(y, A),$$

then $Q(N) = 0$ and Q is R-invariant for $\{X_n\}$.

Assume first that $\mathscr{F}$ is separable, and define Δ and $\{X_n(\Delta)\}$ as in Lemma 3.4. Assume that some R-subinvariant measure Q for $\{X_n\}$ exists; since f is positive on $\mathscr{X}_\Delta$, R-subinvariance for Q implies

$$\begin{aligned} \int_A Q(dy)f(y) &\geqq R \int_{\mathscr{X}_\Delta} Q(dx) \int_A P(x, dy)f(y) \\ &= \int_{\mathscr{X}_\Delta} [Q(dx)f(x)]P_\Delta(x, A), \qquad A \in \mathscr{F}_{\Delta^c}. \end{aligned} \tag{3.28}$$

Since $\{X_n(\Delta)\}$ is ϕ-recurrent, we know from Lemma 3.3 that there is a unique 1-subinvariant measure Q_1 for $\{X_n(\Delta)\}$; thus from (3.28) we have

$$Q_1(A) = \int_A Q(dy)f(y), \qquad A \in \mathscr{F}_{\Delta^c}. \tag{3.29}$$

From the first remark of this proof and the properties of Q_1, it follows that

(i) Q is strictly R-invariant for $\{X_n\}$;

(ii) Q is the unique R-subinvariant measure for $\{X_n\}$;

(iii) Q is equivalent to M on $\mathscr{F}_{\Delta^c}$, and hence, since $Q(\Delta) = M(\Delta) = 0$, on $\mathscr{F}$.

Secondly we must show that there does exist an R-subinvariant measure for $\{X_n\}$. Since Q_1 is 1-invariant for $\{X_n(\Delta)\}$ and $[f(x)]^{-1} < \infty$, $x \in \mathscr{X}_\Delta$, we have for $A \in \mathscr{F}_{\Delta^c}$

$$\int_A Q_1(dy)[f(y)]^{-1} = \int_{\mathscr{X}_\Delta} Q_1(dx) \int_A P_\Delta(x, dy)[f(y)]^{-1}. \tag{3.30}$$

Write $q_1(x)$ for the density of Q_1 with respect to M, and define the (measurable) function q by setting

$$q(x) = q_1(x)/f(x).$$

Define the measure Q by setting $Q(\Delta) = 0$, and, for $A \in \mathscr{F}_{\Delta^c}$, $Q(A) = \int_A q(x)M(dx)$. With these definitions, Q_1 and Q satisfy (3.29), and so for $A \in \mathscr{F}_{\Delta^c}$, (3.30) shows

$$\begin{aligned} Q(A) = \int_A Q(dy)f(y)[f(y)]^{-1} &= \int_{\mathscr{X}_\Delta} [Q(dx)f(x)] \int_A P_\Delta(x, dy)[f(y)]^{-1} \\ &= R \int_{\mathscr{X}_\Delta} Q(dx)P(x, A) \\ &= R \int_{\mathscr{X}} Q(dx)P(x, A); \end{aligned} \tag{3.31}$$

hence, since both $Q(\Delta)$ and $\int_{\mathscr{X}} Q(dx)P(x, \Delta)$ are zero, Q is R-invariant for $\{X_n\}$.

The separability assumption is necessary to enable us to use Lemma 3.4. It can be removed by the use of admissible sub-σ-fields of $\mathscr{F}$, exactly as in the final section of the proof of Theorem 7.2 of Orey (1971), and it follows that a unique R-invariant measure Q for $\{X_n\}$ exists.

Finally, we prove that Q above satisfies (3.27). From Lemma 3.1 (ii), for any $B \in \mathscr{F}^+$,

$$Q(A) \geqq \int_B Q(dy)\,{}_BG_R(y, A), \qquad A \in \mathscr{F};$$

fix $B \in \mathscr{F}^+$ and write $Q_B(A) = \int_B Q(dy)_B G_R(y, A)$, $A \in \mathscr{F}$. Since

$$\begin{aligned} & R \int_{\mathscr{X}} Q_B(dy)P(y, A) \\ & = R \int_{B^c} [\int_B Q(dx) \sum_1^\infty R^n{}_B P^n(x, dy)]P(y, A) + R \int_B Q_B(dy)P(y, A) \\ & = \int_B Q(dy) \sum_2^\infty R^n{}_B P^n(x, A) + R \int_B Q_B(dy)P(y, A) \\ & \leqq Q_B(A) , \end{aligned} \tag{3.32}$$

Q_B is R-subinvariant for $\{X_n\}$, and hence from (i) above, Q_B is R-invariant and must be some multiple of Q. However, if $Q_B(A) = (1 - \varepsilon)Q(A)$ for some $\varepsilon > 0$, the inequality in (3.32) would remain, for some A, a strict inequality; hence $Q_B = Q$, and the theorem is proved. □

When $R = 1$, Theorem 4 (except for $M \gg Q$ and (3.27)) is given by the Corollary to Theorem 2 of Jain and Jamison (1967), and we could have applied their result directly to $\{X_n(f)\}$, rather than Lemma 3.3 to $\{X_n(\Delta)\}$, to prove the theorem. Their proof that Q is the unique 1-subinvariant measure is rather more complicated than that using Lemma 3.1 given in Lemma 3.3.

The proof of Theorem 4 needs only Condition I, but it relies heavily on the equivalent result for ϕ-recurrent chains, given in Lemma 3.3. It is in fact possible to emulate the proof of Theorem 3 to derive Theorem 4, but only (as far as I can see) under the following irreducibility condition

CONDITION I″. There exists a σ-finite measure M, not identically zero, on $\mathscr{F}$ such that

(i) $M(A) > 0$ implies $G_{\frac{1}{2}}(x, A) > 0$, for all $x \in X$

(ii) there exists an M-null set N_M with $\bar{N}_M \subseteqq N_M$ such that, for each $x \notin N_M$, $G_{\frac{1}{2}}(x, \cdot)$ is absolutely continuous with respect to M.

Condition I″ is weaker than Condition I′, which demands $N_M = \emptyset$, but stronger than Condition I. It ensures that the 'reversed' chain, with transition probabilities

$$P_q(x, A) = R \int_A q(y)p(y, x)M(dy)/q(x)$$

is 1-recurrent and hence stochastic for almost all x: the proof of Theorem 3 is then easily imitated. This gives a new proof of Harris's result which avoids any need to introduce uniformly ϕ-recurrent chains and 'processes on A'.

4. First passage analogues and finiteness results. When $X = Z$, the following dichotomy (well-known for $r = 1$) is given in VJ1:

(a) if $\{X_n\}$ is r-transient, $r \leqq R$, then ${}_iG_r(i, i) < 1$, $i \in Z$

(b) if $\{X_n\}$ is R-recurrent, then ${}_iG_R(i, i) = 1$, $i \in Z$.

In this section we shall begin by investigating the analogues of these results for general $\mathscr{X}$. We first prove the r-transient result.

PROPOSITION 4.1. *Let $\{X_n\}$ be r-transient $r \leqq R$. Then there is a partition $\mathscr{K}_r = (K_r(j))$ of $\mathscr{X}$, such that*

(i) $M(K_r(0)) = 0$

(ii) *if* $A \in \mathscr{F}_{K_r(j)}$ *for any* $j \neq 0$,

$$ {}_AG_r(x, A) = \sum_1^\infty {}_AP^n(x, A)r^n < 1 \tag{4.1}$$

for every $x \in A$.

PROOF. Let $\mathscr{K} = (K(j))$ be a partition of $\mathscr{X}$ into r-transient sets. For fixed j, write, for $n = 1, 2, \cdots$

$$B_n(j) = \{y \in K(j): G_r(y, K(j)) \in [(n-1)/2, n/2)\}\,; \tag{4.2}$$

since $K(j)$ is r-transient, there is a set $B_0(j) \subseteq K(j)$ with $M(B_0(j)) = 0$ such that

$$\bigcup_{n=1}^\infty B_n(j) = K(j)\backslash B_0(j)\,.$$

Thus $\{B_n(j)\}$ is a partition of $K(j)$; if we write $K_r(0) = \bigcup_j B_0(j)$, and denote by $\mathscr{K}_r$ the collection of sets $\{K_r(0), B_n(j), n = 1, 2, \cdots, j = 0, 1, \cdots\}$, then $\mathscr{K}_r$ is certainly a partition of $\mathscr{X}$ into r-transient sets such that (i) holds. To show that (ii) also holds, let B be any set in $\mathscr{K}_r$ other than $K_r(0)$, and let K be the element of $\mathscr{K}$ such that $B \subseteq K$.

The first entrance decomposition, for any $A \in \mathscr{F}_B$, and $\zeta \in A$, gives

$$\begin{aligned} G_r(\zeta, K) &= {}_AG_r(\zeta, K) + \textstyle\int_A {}_AG_r(\zeta, dy)G_r(y, K) \\ &\geqq {}_AG_r(\zeta, K) + \inf_{y\in A} G_r(y, K)\,{}_AG_r(\zeta, A) \\ &\geqq {}_AG_r(\zeta, A)[1 + \inf_{y\in A} G_r(y, K)]\,. \end{aligned} \tag{4.3}$$

By construction (4.2), we have (since $A \subseteq B$), that for any $\zeta \in A$,

$$\inf_{y\in A} G_r(y, K) \geqq G_r(\zeta, K) - \tfrac{1}{2}\,; \tag{4.4}$$

putting (4.4) into (4.3) shows that (4.1) holds. □

When $X = Z$, (a) can be used to prove (b), since there is a finest possible partition of the integers. However, the partition $\mathscr{K}_r$ in Proposition 4.1 depends in its construction upon r (via (4.2)), and consequently from Proposition 4.1 we cannot assert the existence of a partition $\mathscr{K} = (K(j))$, independent of r, such that ${}_AG_r(x, A) \leqq 1$, $A \in \mathscr{F}_{\mathscr{K}}{}^+$, for every $r \leqq R$, $x \in A$, when $\{X_n\}$ is R-recurrent and $R > 1$ (it is clearly true that such a partition exists when $R = 1$, from probabilistic considerations).

In fact, as the following results show, the existence of such a partition is not in general plausible: this is one of the few instances where results do not seem to carry over from the countable case.

PROPOSITION 4.2. *Suppose* $\{X_n\}$ *is R-recurrent, and let f be the unique R-invariant function for* $\{X_n\}$. *Then a set* $A \in \mathscr{F}^+$ *satisfies*

$${}_AG_R(x, A) \leqq 1 \tag{4.5}$$

for almost every $x \in A$, *if and only if f is constant almost everywhere on A and then we have equality in* (4.5) *for almost every* $x \in A$.

If (4.5) *holds for some* $A \in \mathscr{F}^+$, *and* $\mathscr{F}$ *is separable, then there is a set* A_0 *with*

$M(A_0) = 0$, *such that for every set* $B \in \mathscr{F}^+_{A \backslash A_0}$,

$$ {}_BG_R(x, B) = 1 \qquad \textit{for all} \quad x \in B . $$

PROOF. From Theorem 3, for any $A \in \mathscr{F}^+$, there exists $\Delta(f, A)$ with $M(\Delta(f, A)) = 0$ such that

$$ f(x) = \int_A {}_AG_R(x, dy)f(y) , \qquad x \notin \Delta(f, A) . \tag{4.6} $$

Suppose first that for some given A, there is a subset $A_1 \in \mathscr{F}_A$ with $M(A_1) = 0$ and $f(x) = c$, $x \in A \backslash A_1$. From Condition I, we can find $A_0 \subseteqq A$ with $M(A_0) = 0$ such that $\bar{A}_0 \subseteqq A_0$, $\Delta(f, A) \cap A \subseteqq A_0$, and $f(x) = c$ when $x \in A \backslash A_0$.

Since $\bar{A}_0 \subseteqq A_0$, writing $A \backslash A_0 = \tilde{A}$,

$$ {}_AG_R(x, A) = {}_{\tilde{A}}G_R(x, \tilde{A}) , \qquad x \in \tilde{A} . \tag{4.7} $$

Also, for $x \in \tilde{A}$, (4.6) holds, and implies

$$ {}_AG_R(x, A) = 1 , \qquad x \in \tilde{A} . $$

If $\mathscr{F}$ is separable, then A_0 can be chosen so that $\Delta(f, B) \subseteqq A_0$ for every $B \in \mathscr{F}_A{}^+$, from Lemma 3.4; it then follows from (4.6) (with $x \in B$ and B in place of A) that for all $B \in \mathscr{F}_A{}^+$,

$$ {}_BG_R(x, B) = 1 \qquad \text{for all} \quad x \in B . \tag{4.8} $$

Secondly, suppose (4.5) holds for some set $A \in \mathscr{F}^+$. From Theorem 4, the unique R-invariant measure Q for $\{X_n\}$ then satisfies

$$ Q(A) = \int_A Q(dy)_AG_R(y, A) \leqq \int_A Q(dy) , $$

and so, except for y in some set A_0 with $Q(A_0) = 0$, ${}_AG_R(y, A) = 1$. Write $\tilde{A} = A \backslash (A_0 \cup \bar{A}_0)$; as before, (4.7) holds so that ${}_{\tilde{A}}G_R(y, \tilde{A}) = 1$, $y \in \tilde{A}$.

Putting $f_A(y) = {}_{\tilde{A}}G_R(y, \tilde{A})$, $y \in \mathscr{X}$, it is easy to show (cf. 3.10) that f_A is R-invariant for $\{X_n\}$, and by uniqueness, $f = cf_A$ for some real $c > 0$. Hence f is constant almost everywhere on $\tilde{A}$, that is, almost everywhere in A. It then follows from the first part of the proof that for $B \in \mathscr{F}_{\tilde{A}}{}^+$, (4.8) holds when $\mathscr{F}$ is separable. □

COROLLARY 1. *If* $\{X_n\}$ *is R-recurrent, and* $\mathscr{F}$ *is separable, there is a partition* $\mathscr{K} = (K(j))$ *such that*

$$ {}_AG_R(x, A) \leqq 1 , \qquad \textit{for almost all} \quad x \in A , \quad \textit{and all} \quad A \in \mathscr{F}_{\mathscr{K}}{}^+ \tag{4.9} $$

if and only if there is a null set N such that $f(x)$ *takes on only a countable number of values for* $x \in X \backslash N$. *If* (4.9) *holds, there is a null set* N_1 *such that the partition* $\mathscr{K}' = (N_1, K(1)\backslash N_1, K(2)\backslash N_1, \cdots)$ *satisfies*

$$ {}_AG_R(x, A) = 1 , \qquad x \in A,\ A \in \mathscr{F}^+_{\mathscr{K}'} . $$

COROLLARY 2. *There is at least one R-recurrent set* $A \in \mathscr{F}^+$ *such that* (4.5) *fails to hold for almost all* $x \in A$, *if* $R > 1$.

PROOF. If not, then f must then be constant almost everywhere on $\mathscr{X}$; but f is R-invariant, so for almost every x,

$$c = f(x) = R \int_{\mathscr{X}} P(x, dy)f(y) = Rc ,$$

and $R = 1$. □

COROLLARY 3. *If* $\mathscr{X} = \mathbb{Z}$, *and* $\{X_n\}$ *is R-recurrent*, $R > 1$, *at least one of the pairs* $\{k, j\}$, $j = 0, 1, \cdots$, *fails to satisfy* (4.5) *for each* $k = 0, 1, \cdots$.

The second part of this section will be devoted to finding various finiteness criteria for r-subinvariant measures which generalize those in VJ2.

We shall call a positive-valued measurable function h an *r-superinvariant function* for $\{X_n\}$ if

$$h(x) \leqq r \int_{\mathscr{X}} P(x, dy)f(y) , \tag{4.10}$$

for almost $x \in \mathscr{X}$. Such functions are studied for $\mathscr{X} = \mathbb{Z}$ in VJ2.

We reserve a proof of the first result to the sequel to this paper, where it is easily proved when we have more information on the structure of r-subinvariant measures: however, it is stated here since it naturally begins the sequence of results we wish to prove.

PROPOSITION 4.3. *If U is an r-subinvariant measure for* $\{X_n\}$ *and h is an r-superinvariant function, then if* $\{X_n\}$ *is r-transient,*

$$\int_{\mathscr{X}} h(x)U(dx)$$

is divergent.

For the remainder of this section, *we shall again assume that* $\{X_n\}$ *is R-recurrent*, and that Q and f denote the unique R-invariant measure and function for $\{X_n\}$.

PROPOSITION 4.4. *If h is an R-superinvariant function for* X_n, *then*

$$\int_{\mathscr{X}} h(x)Q(dx) < \infty$$

implies that h is R-invariant, and hence $h = f$.

PROOF. R-superinvariance of h gives, since $Q \sim M$,

$$\begin{aligned} \int_{\mathscr{X}} Q(dx)h(x) &\leqq R \int_{\mathscr{X}} Q(dx) \int_{\mathscr{X}} P(x, dy)h(y) \\ &= \int_{\mathscr{X}} [R \int_{\mathscr{X}} Q(dx)P(x, dy)]h(y) \\ &= \int_{\mathscr{X}} Q(dy)h(y) , \end{aligned} \tag{4.11}$$

so that if the right-hand side of (4.11) converges, we must have equality throughout (4.11); hence h is R-invariant, as claimed. □

Notice that, since $h(x) \equiv 1$ is r-superinvariant for $r \geqq 1$, but not r-invariant for $r > 1$, these two propositions imply that a finite r-subinvariant measure can only exist when $\{X_n\}$ is 1-recurrent.

We now derive a criterion for the finiteness of

$$\int_{\mathscr{X}} Q(dx)f(x) \tag{4.12}$$

in terms of

$$(4.13)\qquad {}_AG_r'(x, A) = \lim_{\varepsilon\downarrow 0}[{}_AG_r(x, A) - {}_AG_{r-\varepsilon}(x, A)]/\varepsilon = \sum_{n=1}^{\infty} nr^{n-1}{}_AP^n(x, A)$$

for $x \in \mathscr{X}$, $r \leqq R$ and $A \in \mathscr{F}^+$.

THEOREM 5. *If $\{X_n\}$ is R-recurrent, then*

$$(4.14)\qquad \int_{\mathscr{X}} Q(dy)f(y) < \infty$$

if and only if, for some one (and then every) $B \in \mathscr{F}^+$

$$(4.15)\qquad \int_B Q(dx) \int_B {}_BG_R'(x, dw)f(w) < \infty .$$

PROOF. For any set $B \in \mathscr{F}^+$, and each $n > 0$ and $r \leqq R$, we have the identity

$$\begin{aligned}\int_{\mathscr{X}} \sum_{k=1}^{n-1} {}_BP^{n-k}(x, dy)r^{n-k} \int_B {}_BP^k(y, dw)f(w)r^k &= \sum_{k=1}^{n-1} \int_B r^n{}_BP^n(x, dw)f(w) \\ &\quad + \int_B \sum_{k=1}^{n-1} {}_BP^{n-k}(x, dy)r^{n-k} \int_B {}_BP^k(y, dw)r^kf(w) .\end{aligned}$$

Summing this identity with n gives, formally,

$$(4.16)\qquad \begin{aligned}\int_{\mathscr{X}} {}_BG_r(x, dy)[\textstyle\int_B {}_BG_r(y, dw)f(w)] &= \sum_{n=1}^{\infty} (n-1)[\textstyle\int_B {}_BP^n(x, dw)f(w)]r^n \\ &\quad + \textstyle\int_B {}_BG_r(x, dy)[\int_B {}_BG_r(y, dw)f(w)] \\ &= r \textstyle\int_B {}_BG_r'(x, dw)f(w) \\ &\quad + [\textstyle\int_B {}_BG_r(x, dy) \int_B {}_BG_r(y, dw)f(w) - \int_B {}_BG_r(x, dw)f(w)] .\end{aligned}$$

(Note that, if B is such that $\sup_{x\in B} f(x) < \infty$, (3.9) shows that for $r < R$, every term in (4.16) is finite for almost all $x \in \mathscr{X}$.) Letting $r \uparrow R$ in (4.16) gives us, for almost all $x \in \mathscr{X}$,

$$(4.17)\qquad \int_{\mathscr{X}} {}_BG_R(x, dy) \int_B {}_BG_R(y, dw)f(w) = R \int_B {}_BG_R'(x, dw)f(w)$$

(where one, but then both, sides of (4.17) may be infinite). Suppose (4.14) holds; then from Theorems 3 and 4,

$$(4.18)\qquad \begin{aligned}\infty > \int_{\mathscr{X}} Q(dy)f(y) &= \int_{\mathscr{X}} \int_B Q(dx){}_BG_R(x, dy) \int_B {}_BG_R(y, dw)f(w) \\ &= R \int_B Q(dx) \int_B {}_BG_R'(x, dw)f(w) ,\end{aligned}$$

and so (4.15) holds for every $B \in \mathscr{F}^+$.

If, on the other hand, (4.15) holds for some $B \in \mathscr{F}^+$, then (4.18) read backwards shows that (4.14) holds, and hence, from the preceding paragraph, that (4.15) holds for every $B \in \mathscr{F}^+$. □

COROLLARY 4. *If $\{X_n\}$ is R-recurrent, then if* (4.14) *holds,*

$$(4.19)\qquad \int_B Q(dx){}_BG_R'(x, B) < \infty$$

for every $B \in \mathscr{F}^+$ such that $\inf_{x\in B} f(x) > 0$.

In particular (4.19) *holds for every $B \in \mathscr{F}^+$ such that* (4.8) *holds.*

5. R-positivity results. In this section we investigate another dichotomy. In the countable case, VJ1 shows that aperiodic R-recurrent chains can be classified as either

(a) *R-null*; that is, as $n \to \infty$,

$$R^n P^n(i,j) \to 0\,, \qquad \text{all } i,j \in \mathscr{Z};\ \text{or}$$

(b) *R-positive*; that is, as $n \to \infty$

$$R^n P^n(i,j) \to \lambda_{ij} > 0\,, \qquad \text{all } i,j \in \mathscr{Z}\,. \tag{5.1}$$

Theorem 6 gives the analogue of these results for chains on a general state space. For a definition of the *period* of such a Markov chain, see Orey (1971) pages 13–15.

THEOREM 6. *If $\{X_n\}$ is R-recurrent and aperiodic, then there exists a partition $\mathscr{K}$ such that for each $A \in \mathscr{F}_{\mathscr{K}}{}^+$, as $n \to \infty$*

$$P^n(x, A)R^n \to \pi(x, A)\,, \qquad x \notin N(f, A) \tag{5.2}$$

where $M(N(f, A))$ and

$$\pi(x, A) = f(x)Q(A)/\textstyle\int_{\mathscr{X}} f(y)Q(dy)\,. \tag{5.3}$$

(Here f and Q are the unique R-invariant function and measure for $\{X_n\}$.) If $\mathscr{F}$ is separable, there exists a null set Δ with $N(f, A) \subseteq \Delta$ for every $A \in \mathscr{F}_{\mathscr{K}}{}^+$.

If $\{X_n\}$ is periodic with period d, then there exists a partition $\mathscr{K}$ such that, if $A \in \mathscr{F}_{\mathscr{K}}{}^+$, there is a null set $N(f, A)$, and

$$\lim_{n\to\infty} \frac{1}{d} \textstyle\sum_{k=0}^{d-1} P^{n+k}(x, A)R^{n+k} = f(x)Q(A)/\int_{\mathscr{X}} f(y)Q(dy) \tag{5.4}$$

for $x \notin N(f, A)$: if $\mathscr{F}$ is separable, $N(f, A)$ can be chosen independent of A.

PROOF. Define $\Delta(f, A)$ as in Lemma 3.4, so that $\Delta(f, \mathscr{X}) \supseteq \{y : f(y) \neq \int_{\mathscr{X}} P(y, dx)f(x)\}$. Let $\mathscr{K} = (K_j)$, $j = 0, 1, \cdots$ be a partition of $\mathscr{X}$ with $K_0 = \Delta(f, \mathscr{X})$ and with the properties that

(i) $\int_{K_j} f(y)Q(dy) < \infty$, for every $j \geqq 1$; and
(ii) $\inf_{y \in K_j} f(y) > 0$ for every $j \geqq 1$.

Let Θ be a fixed subset of $\mathscr{F}_{\mathscr{K}}{}^+$. From Proposition 1.3 of Orey (1971), there exists an admissible σ-field $\mathscr{F}_1 \subseteq \mathscr{F}$ with $\Theta \in \mathscr{F}$, and $K_j \in \mathscr{F}_1$ for all j. Hence from Lemma 3.4, we can find an M-null set $\Delta \in \mathscr{F}_1$ with $\bar{\Delta} \subseteq \Delta$ such that the chain $\{X_n(\Delta)\}$ defined as in Lemma 3.4 on $\mathscr{X}_\Delta = \mathscr{X}\backslash\Delta$ is ϕ-recurrent. From Lemma 3.3, there is a unique 1-invariant measure Q_Δ for $\{X_n(\Delta)\}$, which then satisfies (cf. (3.31))

$$Q_\Delta(B) = \textstyle\int_B Q(dy)f(y)\,, \qquad B \in \mathscr{F}_1\,. \tag{5.5}$$

From (i), $Q_\Delta(B) < \infty$ if $B \in \mathscr{F}_{\mathscr{K}} \cap \mathscr{F}_1$.

Let K be any element of $\mathscr{K}$ other than K_0. For every $B \in \mathscr{F}_{\mathscr{K}} \cap \mathscr{F}_1$, it follows from Orey (1971), page 30–35, that

$$\lim_{n\to\infty} P_\Delta{}^n(x, B) = Q_\Delta(B)/Q_\Delta(\mathscr{X}), \qquad x \in \mathscr{X}_\Delta,$$

and hence that for any such B,

$$(5.6) \qquad \lim_{n\to\infty} \int_B P_\Delta{}^n(x, d\omega)g(\omega) = \int_B Q_\Delta(d\omega)g(\omega)/Q_\Delta(\mathscr{X}), \qquad x \in \mathscr{X}_\Delta,$$

where g is any bounded measurable function on K (cf. Gänssler, (1971).) By construction, $[f(y)]^{-1}$ is such a function: using the definition of P_Δ, and (5.5), we find that setting $g = f^{-1}$ in (5.6) gives

$$\lim_{n\to\infty} R^n P^n(x, B) = f(y)Q(B)/\int_{\mathscr{X}} f(\omega)Q(d\omega)$$

for $x \in \mathscr{X}_\Delta$, $B \in \mathscr{F}_K \cap \mathscr{F}_1$.

In particular, since $\Theta \in \mathscr{F}_K \cap \mathscr{F}_1$ for some $K \neq K_0$, we see that (5.2) and (5.3) hold with $N(f, \Theta) = \Delta$. In general, Δ depends on $\mathscr{F}_1$ and hence on Θ. However, if $\mathscr{F}$ itself is separable, then $\mathscr{F}$ can be taken as $\mathscr{F}_1$, and $N(f, \Theta)$ can be chosen independent of Θ from Lemma 3.4.

The proof in the periodic case is similar. □

It is clear that the limits in (5.2) are identically zero if $\int_{\mathscr{X}} Q(d\omega)f(\omega) = \infty$, and all positive if $\int_{\mathscr{X}} Q(d\omega)f(\omega) < \infty$, since Q and M are equivalent. In the former case we call $\{X_n\}$ *R-null*, in the latter *R-positive*.

The condition $\int_{\mathscr{X}} f(x)Q(dx) < \infty$, which determines R-positivity, has been investigated already in Section 4. Combining the results of Theorem 6 and Section 4 gives the following important criteria for R-positivity:

THEOREM 7. *The following conditions are equivalent*:

(i) $\{X_n\}$ *is R-positive*;

(ii) *there exists an R-subinvariant measure U and an R-superinvariant function h such that*

$$\int_{\mathscr{X}} h(y)U(dy) < \infty;$$

(iii) $\{X_n\}$ *is R-recurrent, and for some one* (*and then for all*) $A \in \mathscr{F}^+$,

$$\int_A Q(dy) \int_A {}_AG_R'(y, dw)f(w) < \infty,$$

where ${}_AG_r'(y, \cdot)$ *is defined by* (4.13).

PROOF. If (i) holds, Theorem 6 shows (ii) is true with $U = Q$, $h = f$. If (ii) holds from Proposition 4.4, $U = Q$ and $h = f$, whence (i) is true from Theorem 6 and (iii) from Theorem 5. If (iii) holds, Theorem 5 shows (ii) holds for $U = Q$ and $h = f$. □

The final R-positivity result we give is of particular interest when $\{X_n\}$ is 1-recurrent.

PROPOSITION 5.1. *Let* $\{X_n\}$ *be R-recurrent and aperiodic, and let* $A \in \mathscr{F}^+$ *be such that* $Q(A) = 1$. *Suppose f is constant on A. Then for almost every* $x \in A$,

$$(5.7) \qquad \lim_{n\to\infty} R^n P^n(x, A) = [R \int_A Q(dy)\,{}_AG_R'(y, A)]^{-1}.$$

PROOF. From Theorem 6,

$$\lim_{n\to\infty} R^n P^n(x, A) = \pi(x, A) = f(x)Q(A)/\int_{\mathscr{X}} f(y)Q(dy) ;$$

and therefore from (4.18) it follows that

$$\pi(x, A) = f(x)Q(A)/R \int_A Q(dy) \int_A {}_AG_R'(y, dw)f(w) . \tag{5.8}$$

If $Q(A) = 1$ and $f(x) = c$ on A, then for $x \in A$ the right-hand side of (5.8) is in fact the right-hand side of (5.7), and the proposition holds. □

COROLLARY 5. *If $\{X_n\}$ is 1-recurrent and aperiodic,*

$$\lim_{n\to\infty} P^n(x, A) = [\int_A Q_A(dy)_A G_1'(y, A)]^{-1} ,$$

where Q_A is the unique 1-invariant probability measure for the transition law ${}_AG_1(y, \cdot)$ on $(A, \mathscr{F}_A)$.

6. Semigroups of nonnegative operators. One of Vere–Jones' main achievements was to point out in VJ2 that the results for Markov chain transition matrices, proved in VJ1, held for semigroups of nonnegative matrices which were not substochastic.

The same is true in general: the context we require is the following. Let $(\mathscr{X}, \mathscr{F})$ be a space equipped with σ-field, and let the collection $\{T^n(x, A), x \in \mathscr{X}, A \in \mathscr{F}, n = 0, 1, 2, \cdots\}$ be a *semigroup of nonnegative operators*; that is,

(i) for each $x \in \mathscr{X}$, $T^n(x, \cdot)$ is a σ-finite measure on $\mathscr{F}$
(ii) for each $A \in \mathscr{F}$, $T^n(\cdot, A)$ is a nonnegative, measurable function on $\mathscr{X}$
(iii) for any $n, m \geqq 0$, $x \in \mathscr{X}$, $A \in \mathscr{F}$,

$$T^{n+m}(x, A) = \int_{\mathscr{X}} T^n(x, dy)T^m(y, A) .$$

The reader will verify that all the results of Section 2, except those which allot special place to the value $R = 1$, depend solely on the semigroup property of the transition functions, rather than their stochasticity, and hence that a convergence norm $R^{-1} \geqq 0$ can be defined for a semigroup of nonnegative operators. If $R > 0$, then we can find an r-subinvariant function for some r with $0 < r < R$ (as, for example, in Proposition 3.2), and form a substochastic kernel by the transformation

$$P(x, A) = r \int_A T(x, dy)f(y)/f(x) ,$$

already used to such effect: translating the results of the previous sections for P into results for $\{T^n\}$ gives

THEOREM 8. *If $\{T^n(x, A), x \in \mathscr{X}, A \in \mathscr{F}\}$ is a semigroup of nonnegative operators which satisfy*

(iv) *There exists a σ-finite measure ϕ on $\mathscr{F}$ such that $\phi(A) > 0$ implies that for each $x \in \mathscr{X}$, at least one of the elements $T^n(x, A)$, $n = 1, 2, \cdots$ is positive, (that is, $\{T^n(x, A\}$ is ϕ-irreducible), then there is a real number $R \geqq 0$, a partition*

$\mathscr{K}$ and a ϕ-null set N such that for $x \notin N$, $A \in \mathscr{F}_{\mathscr{K}}$, $\phi(A) > 0$, the power series $\sum_{n=1}^{\infty} T^n(x, A)z^n$ has radius of convergence R.

If $R > 0$, then the results of Sections 1–5, except those which allot the case $R = 1$ special place (specifically, the fact that $R \geqq 1$ and the empty null set for $R = 1$ in Theorem 1 and the corollary to Proposition 5.1) are all true (with T^n in place of P^n throughout).

Semigroups of Markov chain transition functions can be set in the theory of positive contraction operators (cf. Foguel (1969)). In this paper we have used probabilistic rather than operator-theoretic methods; however, Theorem 8 shows that the contraction assumption usually employed can be replaced by the much weaker assumption that the convergence norm is finite, and many of the familiar Markov chain properties will carry through to their *R*-theoretic analogues.

Acknowledgments. I would like to thank Professor D. G. Kendall, Professor J. F. C. Kingman and Professor G. E. H. Reuter for their comments on earlier drafts of this paper, and to express my gratitude to the Statistical Laboratory of Cambridge University for the time I spent there. That part of this work done in Cambridge was supported by an Australian National University Travelling Scholarship.

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