*The Annals of Probability*
1974, Vol. 2, No. 6, 999–1026

# SOJOURNS AND EXTREMES OF GAUSSIAN PROCESSES[1]

BY SIMEON M. BERMAN

*New York University*

Let $X(t)$, $0 \leqq t \leqq 1$, be a real Gaussian process with mean 0 and continuous sample functions. For $u > 0$, form the process $u(X(t) - u)$. In this paper two related problems are studied. (i) Let $G$ be a nonnegative measurable function, and put $L = \int_0^1 G(u(X(t) - u))\, dt$. For certain classes of processes $X$ and functions $G$, we find, for $u \to \infty$, the limiting conditional distribution of $L$ given that it is positive. (ii) For the same class of processes $X$, we find the asymptotic form of $P(\max_{[0,1]} X(t) > u)$ for $u \to \infty$. Finally, these results are extended to the process with the "moving barrier," $X(t) - f(t)$, where $f$ is a continuous function.

**1. Introduction.** Let $X(t)$, $0 \leqq t \leqq 1$, be a real Gaussian process with mean 0 and continuous sample functions. For $u > 0$, form the process $u(X(t) - u)$. For large $u$ we call this a "high level" process derived from $X$. Let $G$ be a nonnegative measurable function, and put $L = \int_0^1 G(u(X(t) - u))\, dt$. In this paper we study the limiting distribution of $L$ for $u \to \infty$; in particular

(i) the general conditions on $X$ and $G$ under which there exists $v = v(u)$ such that the conditional distribution $P(vL \leqq x \mid vL > 0)$ converges, and

(ii) the forms of the limiting distribution.

Finally we consider the related problem of determining the asymptotic form of $P(\max_{[0,1]} X(t) > u)$ for $u \to \infty$.

Here we unify and generalize all our previous results in this area [1]—[5]. There $G$ was limited to be the indicator of the positive axis, where $L$ represents the time spent above $u$. Here $G$ may be a function of a general type.

In most of our previous work the calculations were done under the assumption of stationarity. But we did not really use the full force of this hypothesis; indeed, stationarity is mostly about long run behavior of processes, while we were interested in local behavior. Here we use a weaker hypothesis which seems just right for the problem: local stationarity. This arose in a natural way in [4], in connection with processes with stationary increments. Local stationarity means that the process is approximately stationary in the neighborhood of each point; it is formally defined in Section 8. It appears to the writer that many of the known properties of the local behavior of stationary processes are actually valid also under the more general assumption of local stationarity.

Received July 5, 1973; revised February 24, 1974.

[1] This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation, Grant NSF-GP-28651.

*AMS* 1970 *subject classifications*. Primary 60G10, 60G15, 60G17; Secondary 60F99.

*Key words and phrases*. Gaussian process, level barrier, moving barrier, sample function maximum, weak compactness, local stationarity, regular variation.

In the previous papers we started with specific assumptions on $r$, and then deduced from these the properties needed to prove our limit theorems. But now these auxiliary properties appear to be of interest themselves. Here we discuss these properties in more detail, and show that they hold for broad classes of Gaussian processes, those satisfying some or all of Assumptions 1–5 below. In particular we give

(i) two variations of the well-known Fernique Inequality which are sharper than the original in the cases of interest to us;

(ii) a new criterion for weak compactness of measures on function space induced by continuous Gaussian processes; and

(iii) a general asymptotic relation between the distribution of $L$ and the distribution of the maximum functional.

We point out an important notational difference between this paper and our recent one [5]: The function $F(x)$ introduced in [5], formulas (6.11) and (6.12), is equivalent to a multiple of the function $F(0) - F(x)$, where $F$ is defined in Section 6 below. (See Section 12.)

I thank the referee for the care he took in reading this paper and for the comments which helped me to improve it.

**2. Two inequalities for the maximum of a Gaussian process.** Let $X(t)$, $0 \leqq t \leqq 1$, be a real separable Gaussian stochastic process with mean 0. Define the function $q(t)$ as

$$q^2(t) = \sup_{|s'-s| \leqq t} E(X(s') - X(s))^2 , \qquad 0 \leqq t \leqq 1 . \tag{2.1}$$

We assume:

ASSUMPTION 1. $\int_1^\infty q(e^{-x^2})\, dx < \infty$.

This condition was first formulated by Fernique, who stated that it is sufficient for the continuity of the sample functions [8]. (See also [7] Theorem 7.1.) Furthermore, he derived an upper bound on the tail of the distribution of $\max_{[0,1]} |X(t)|$. This is known as the Fernique Inequality. The continuity of the sample functions under the above assumption was proved in a different way by Garsia, Rodemich, and Rumsey [9].

In this section we will derive sharper forms of this tail inequality in two cases: first when the process has a fixed zero, and second, when the variance is constant.

Throughout this paper $\phi(x)$ stands for the standard normal density function:

$$\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} .$$

Put

$$\Psi(u) = \int_u^\infty \phi(x)\, dx .$$

It is well known that

$$\Psi(u) \leqq \phi(u)/u , \qquad \text{for } u > 0 , \tag{2.2}$$

and that

$$\Psi(u) \sim \phi(u)/u , \qquad \text{for } u \to \infty . \tag{2.3}$$

We will also use another simple fact about the normal density. It follows from the form of $\phi$ that

(2.4) $$\phi(u + x/u) = \phi(u)e^{-x}e^{-x^2/2u^2};$$

therefore,

(2.5) $$\phi(u + x/u) \leqq \phi(u)e^{-x}.$$

Define

(2.6) $$Q(t) = \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}} - 1} \int_1^\infty q(te^{-x^2})\,dx, \qquad 0 \leqq t \leqq 1.$$

This is well defined under Assumption 1, and is a non-decreasing function of $t$.

Our first result is a bound for the displacement of $X$:

THEOREM 2.1. *There exists a positive constant $K$ such that*

(2.7) $$P(\sup_{[a,b]} X(t) - X(a) > u) \leqq K\Psi\left(\frac{u}{Q(b-a)}\right)$$

*for all subintervals $[a, b]$ of $[0, 1]$ and all $u > KQ(b - a)$.*

PROOF. We modify the proof of the Fernique Inequality given by Marcus [11]. Suppose first that $a = 0$ and $b = 1$. Let $n$ be a fixed positive integer greater than 1, and define the sequence

$$c(p) = n^{2^p}, \qquad p \geqq 1.$$

For each $t$ in $[0, 1]$ and each $p \geqq 1$, choose the integer $k(p)$ so that $0 \leqq t - k(p)/c(p) < 1/c(p)$, and choose $k(0) = 0$. Then $X(t) - X(0)$ may be represented as the absolutely convergent series

(2.8) $$X(t) - X(0) = \sum_{p\geq 0}\left[X\left(\frac{k(p+1)}{c(p+1)}\right) - X\left(\frac{k(p)}{c(p)}\right)\right].$$

The almost sure absolute convergence follows from Assumption 1; see [11].

If $X(t) - X(0) > u$ for some $t$, then, for some $k = 0, 1, \cdots, c(p) - 1$, some $q = 0, 1, \cdots, c(p) - 1$, and some $p \geqq 0$, it would follow that

(2.9) $$X\left(\frac{k}{c(p)} + \frac{q}{c(p+1)}\right) - X\left(\frac{k}{c(p)}\right) > \frac{uq(1/c(p))(\log c(p))^{\frac{1}{2}}}{\sum_{m\geq 0} q(1/c(m))(\log c(m))^{\frac{1}{2}}}.$$

To verify this we first note that the series in the denominator on the right-hand side of (2.9) is convergent; indeed, by replacing the terms by dominating integrals, we find that the series is at most $Q(1) < \infty$. Next, if the alternative to (2.9) held for every $k$, $q$ and $p$, then, by (2.8), $X(t) - X(0)$ would not exceed $u$ for any $t$.

Now we replace the series in the denominator in (2.9) by the larger quantity $Q(1)$, and sum the probabilities of the events (2.9) over $k$, $q$ and $p$. The sum is bounded above by

(2.10) $$\sum_{p\geq 1} c^2(p)\Psi\left(\frac{u(\log c(p))^{\frac{1}{2}}}{Q(1)}\right).$$

Next we consider the case of an arbitrary subinterval $[a, b]$ of $[0, 1]$. Put $Y(t) = X(a + t(b - a))$, for $0 \leqq t \leqq 1$; and let $\bar{q}$ and $\bar{Q}$ be the corresponding functions for the $Y$-process. Then $\bar{q}(t) \leqq q(t(b - a))$, and so

$$\bar{Q}(t) \leqq Q(t(b - a)) . \tag{2.11}$$

Now apply the bound (2.10) to the $Y$-process, but with $\bar{Q}$ in place of $Q$. It then follows, by (2.11), that the probability in (2.7) is at most equal to the series (2.10) with $Q(b - a)$ in place of $Q(1)$.

In order to complete the proof of the theorem, it suffices to show that

$$\sum_{p \geq 1} c^2(p) \Psi \left(\frac{u(\log c(p))^{\frac{1}{2}}}{Q(b - a)}\right) \Big/ \Psi \left(\frac{u}{Q(b - a)}\right)$$

is bounded above by a constant for all $u$, $b$ and $a$ such that $u/Q(b - a)$ is sufficiently large. This is done in a direct way by means of (2.2) and (2.3). □

Now we will derive a bound for the distribution of $\sup_{[a,b]} X(t)$ in the case of a process with constant variance. For simplicity we take it to be equal to 1. First we get a bound on a conditional distribution of the maximum.

LEMMA. 2.1. *Suppose that*

$$EX^2(t) = 1 , \qquad \text{for } 0 \leqq t \leqq 1 . \tag{2.12}$$

*Let $d$ be a positive number such that*

$$q(d) < 1 . \tag{2.13}$$

*Then*

$$P(\sup_{[a,b]} X(t) > u \mid X(a) = u + y/u) \leqq K\Psi \left(\frac{-y}{2uQ(b - a)}\right), \tag{2.14}$$

*for all $u > 0$ and $y < 0$, and all subintervals $[a, b]$ of $[0, 1]$ such that*

$$b - a < d , \qquad -y > 2uKQ(b - a) . \tag{2.15}$$

PROOF. Put $E^*X(t) = E(X(t) \mid X(a) = u + y/u)$; then by the elementary formula for the conditional mean, we have $E^*X(t) = (u + y/u)EX(t)X(a)$. The condition (2.12) implies that

$$q^2(t - a) \geqq 2(1 - EX(t)X(a)) , \qquad \text{for } a \leqq t \leqq b ;$$

this and the conditions (2.12) and (2.13) imply that $E^*X(t)$ satisfies

$$E^*X(t) \leqq u + y/2u , \qquad \text{for } a \leqq t \leqq b .$$

Therefore, if $\sup_{[a,b]} X(t) > u$, then

$$\sup_{[a,b]} u(X(t) - E^*X(t)) > -y/2 . \tag{2.16}$$

Put $Y(t) = u(X(t) - E^*X(t))$, $a \leqq t \leqq b$. Condition the process by $Y(a) = 0$, that is, $X(a) = u + y/u$. The conditional probability in (2.14) is then at most equal to

$$P(\sup_{[a,b]} Y(t) - Y(a) > -y/2 \mid Y(a) = 0) . \tag{2.17}$$

We wish to apply Theorem 2.1 to this probability. All we need is the correct "$Q$-function" on the right-hand side of (2.7). So we find a bound on the "$q$-function" of the $Y$-process.

By the constancy of the variance of the conditional normal distribution, $\mathrm{Var}[Y(s) - Y(s') \mid Y(a)]$ may be calculated under $Y(a) = 0$, or, equivalently, $X(a) = u + y/u$. By definition, this conditional variance is

$$\mathrm{Var}\{u[X(s) - E^*X(s)] - u[X(s') - E^*X(s')] \mid X(a) = u + y/u\}.$$

In evaluating this, we treat $E^*X(s)$ and $E^*X(s')$ as fixed—not random variables—because they are conditional expectations given $X(a)$. Hence, these may be ignored in the calculation of the variance:

$$\mathrm{Var}\{u[X(s) - X(s')] \mid X(a) = u + y/u\}.$$

This is at most equal to $\mathrm{Var}\{u[X(s) - X(s')]\}$ because the conditional variance never exceeds the corresponding unconditional variance. (This argument was also used in [2], top of page 69.) It follows that $uq$ is a bound for the "$q$-function" of the $Y$-process; therefore, by (2.6), $uQ$ is a bound for the "$Q$-function." We conclude that a bound for (2.17) may be obtained by replacing $Q$ and $u$ in (2.7) by $uQ$ and $-y/2$, respectively. □

COROLLARY TO LEMMA 2.1.

$$P(\sup_{[a,b]} X(t) > u,\ X(a) \leqq u - M/u) \leqq [\phi(u)/u]K \int_M^\infty \Psi\left(\frac{y}{2uQ(b-a)}\right) e^y \, dy, \tag{2.18}$$

*for all* $u > 0$ *and* $M > 0$, *and all intervals* $[a, b]$ *such that* $b - a \leqq d$ *and* $M > 2uKQ(b - a)$.

PROOF. Write the probability in (2.18) as the integral of the conditional probability given $X(a) = y$, times the density $\phi(y)$. Then change the variable of integration from $y$ to $y' = u(u - y)$, and apply the inequality (2.5). The resulting integral is at most equal to

$$\frac{\phi(u)}{u} \int_M^\infty P(\sup_{[a,b]} X(t) > u \mid X(a) = u - y'/u) e^{y'} \, dy'.$$

Now apply (2.14) to the integrand. □

Our main result is now stated:

THEOREM 2.2. *If* (2.12) *and* (2.13) *hold, then*

$$P(\sup_{[a,b]} X(t) > u) \leqq \frac{\phi(u)}{u} \left\{ \frac{e^M}{1 - Mu^{-2}} + K \int_M^\infty \Psi\left(\frac{y}{2uQ(b-a)}\right) e^y \, dy \right\} \tag{2.19}$$

*for all* $u > 0$ *and* $M > 0$ *and intervals* $[a, b]$ *such that*

$$b - a \leqq d \quad \text{and} \quad 2uKQ(b - a) \leqq M < u^2. \tag{2.20}$$

PROOF. The event $\{\sup_{[a,b]} X(t) > u\}$ is decomposed into its intersections with $\{X(a) > u - M/u\}$ and $\{X(a) \leqq u - M/u\}$. The former event has probability $\Psi(u - M/u)$, which, by the last inequality in (2.20), and by (2.2) and (2.4), is at most equal to

$$\frac{\phi(u)}{u} \cdot \frac{e^M}{1 - Mu^{-2}} .$$

The intersection with the second event has a probability satisfying (2.18). The right-hand side of (2.19) is the sum of these probabilities. □

**3. Application to the distribution of the maximum at a high level.** Now we study the asymptotic form of the bound (2.19) for $u \to \infty$. If the interval $[a, b]$ consists of just a single point $t$, then $\sup_{[a,b]} X(t)$ is equal to $X(t)$; hence, by (2.4), the bound for the distribution of the maximum is simply $\phi(u)/u$. For any interval $[a, b]$ the coefficient of $\phi(u)/u$ on the right-hand side of (2.19) represents an additional factor in the bound due to the nondegeneracy of the interval. As $u$ increases, the $\Psi$-integral becomes infinite, and so the coefficient of $\phi(u)/u$ tends to $\infty$ with $u$. The following theorem tells how quickly the coefficient increases. For simplicity we take the interval to be $[0, 1]$.

THEOREM 3.1. *Let $M$ be an arbitrary positive number, and let $d$ satisfy* (2.13). *For every $u$ sufficiently large, let $v = v(u)$ be so large that*

$$uQ(1/v) \leqq M/2K \quad \text{and} \quad v - 1 > 1/d . \tag{3.1}$$

*Then, under the assumption* (2.12),

$$P(\sup_{[0,1]} X(t) > u) \leqq \frac{v\phi(u)}{u} \left\{ \frac{e^M}{1 - Mu^{-2}} + K \int_M^\infty \Psi(yK/M) e^y \, dy \right\} . \tag{3.2}$$

PROOF. Decompose $[0, 1]$ into $[v]$ intervals $I$, each of length $[v]^{-1}$. Then $\{\sup_{[0,1]} X(t) > u\}$ implies $\{\sup_I X(t) > u\}$ for at least one $I$. Apply Theorem 2.2 to each of these intervals: the condition (3.1) implies the hypothesis (2.20). By Boole's inequality the right-hand side of (3.2) is a bound for the sum of the probabilities for the various intervals. □

The advantage of the bound on the right-hand side of (3.2) is that the coefficient of $v\phi(u)/u$ converges to a finite limit as $u \to \infty$:

$$P(\sup_{[0,1]} X(t) > u) = O(v\phi(u)/u) . \tag{3.3}$$

(A special case of this appeared in [2].) The estimate (3.3) is sharper in simple examples than the estimate obtainable from the original Fernique Inequality [8]. If $q(t) \leqq ct^\alpha$, for $t$ near 0, where $0 < \alpha \leqq 2$, then $v$ can be chosen to be $O(u^{2/\alpha})$, and so the right hand member of (3.3) is $O(u^{2/\alpha - 1}\phi(u)/u)$. However the Fernique Inequality yields only

$$P(\sup_{[0,1]} X(t) > u) = O\left(u^{-1}\phi\left(\frac{u}{1 + Q(1)}\right)\right) ,$$

which is a much larger bound for $u \to \infty$.

**4. Application to weak compactness.** The Fernique Inequality was stated for the absolute maximum $\sup|X(t)|$, and our inequalities for $\sup X(t)$. However, these estimates are simply related by the double inequality

$$P(\sup X(t) > u) \leqq P(\sup|X(t)| > u) \leqq 2P(\sup X(t) > u) .$$

Indeed, the first inequality is clear from $|X| \geqq X$; and the second from the equivalence of the processes $X(t)$ and $-X(t)$ when the mean is 0.

It follows that (2.7) can be put in the form

$$P(\sup_{[a,b]} |X(t) - X(a)| > u) \leqq 2K\Psi\left(\frac{u}{Q(b-a)}\right) . \tag{4.1}$$

This can be used to state criteria for the weak compactness of families of measures induced by Gaussian processes.

THEOREM 4.1. *Let* $\{X_\gamma(t), 0 \leqq t \leqq 1\}$, *where* $\gamma$ *runs over some index set, be a family of Gaussian processes with mean* 0 *and continuous sample functions, and such that*

$$\sup_\gamma EX_\gamma^2(0) < \infty . \tag{4.2}$$

*Let* $Q_\gamma(t)$ *be the "Q-function"* (2.6) *for the process* $X_\gamma$. *If*

$$\lim_{h\downarrow 0} \sup_\gamma Q_\gamma(h)(\log h^{-1})^{\frac{1}{2}} = 0 , \tag{4.3}$$

*then the family of measures induced on* $C[0, 1]$ *by the family* $(X_\gamma)$ *is weakly compact.*

PROOF. The condition (4.2) implies

$$\lim_{a\to\infty} \sup_\gamma P(|X_\gamma(0)| > a) = 0 .$$

Under (4.3) and by virtue of (2.2), it follows that

$$\lim_{h\to 0} h^{-1}\Psi(\varepsilon/\sup_\gamma Q_\gamma(h)) = 0 , \qquad \text{for } \varepsilon > 0 . \tag{4.4}$$

According to the inequality (4.1), the relation (4.4) implies

$$\lim_{h\to 0} \sup_{\gamma,t} h^{-1}P(\max_{t\leqq s\leqq t+h} |X_\gamma(s) - X_\gamma(t)| > h) = 0 .$$

Therefore a well-known criterion for weak compactness implies the conclusion of our theorem. (See the monograph of Billingsley [6] page 56.)

In the next section we will apply Theorem 4.1 to a family of "high level conditioned processes" obtained from a given Gaussian process $X$.

**5. Weak compactness of the family of high level conditioned processes.** Throughout this section let $v = v(u)$, $u \geqq 1$, be a non-decreasing function which tends to $\infty$ with $u$ and satisfies

$$\sup_{u\geqq 1} uQ(1/v) < \infty . \tag{5.1}$$

Such a function certainly exists under Assumption 1; indeed, let $v(u)$ be a solution of the equation

$$uQ(1/v) = Q(1) . \tag{5.2}$$

(The existence of a solution is ensured by the monotonicity and continuity of $q$ (and of $Q$).) Put

$$Q^*(t) = \sup_{u \geqq 1} uQ(t/v)\,, \qquad 0 \leqq t \leqq 1\,. \tag{5.3}$$

For $u \geqq 1$, and fixed $t$, $0 \leqq t \leqq 1$, form the process

$$X^*(s) = u(X(t + s/v) - u)\,, \qquad -tv \leqq s \leqq v(1 - t)\,, \tag{5.4}$$

and condition it by $X^*(0)$. This process arises in a natural way in the analysis of the high level excursions of $X$ (see [2]—[5]). Note that the process $X^*$ depends on two indices, $t$ and $u$: $X^*(s) = X^*_{t,u}(s)$, $0 \leqq t \leqq 1$, $u \geqq 1$; however, for convenience, the indices have been suppressed.

We now assume for $Q^*$ a version of the condition (4.3). In the course of the proof of Theorem 5.1 below we will show that $Q^*$ dominates the "$Q$-function" of the process $X^*_{t,u}$ for all $t$ and $u$.

ASSUMPTION 2. There exists a function $v$ satisfying (5.1) such that

$$(Q^*(h))^2 \log 1/h \to 0 \qquad \text{for } h \downarrow 0\,.$$

(Assumption 1 merely states the finiteness of $Q(1)$, but Assumption 2 imposes a rate of convergence on $Q(t)$ for $t \to 0$.)

THEOREM 5.1. *For a closed bounded interval on the real line, $I$, contained in $[-tv, v(1 - t)]$, let $P_{t,u}$ be the measure on $C(I)$ induced by the centered process*

$$X^*(s) - E(X^*(s) \mid X^*(0) = y)\,, \qquad s \in I\,, \tag{5.5}$$

*conditioned by $X^*(0) = y$, for fixed $y$. Under Assumptions 1 and 2 on $X$, the family $(P_{t,u}\colon 0 \leqq t \leqq 1, u \geqq 1)$ is weakly compact over $C(I)$.*

PROOF. We apply Theorem 4.1 with $I$ in place of $[0, 1]$. The conditioned process (5.5) is Gaussian with mean 0. The condition (4.2) is satisfied for this process because $\operatorname{Var}(X^*(0) \mid X^*(0)) = 0$. The condition (4.3) is also satisfied: As at the end of the proof of Lemma 2.1, we find that the "$q$-function" of this conditional process is dominated by the function $uq(t/v)$:

$$\begin{aligned}\operatorname{Var}(X^*(s) - X^*(s') \mid X^*(0)) &\leqq u^2 \operatorname{Var}(X(t + s/v) - X(t + s'/v)) \\ &\leqq u^2 q^2(|s - s'|/v)\,.\end{aligned}$$

Therefore, its "$Q$-function" is dominated by $Q^*$ for all $t$ and $u$; thus, Assumption 2 implies the condition (4.3). □

The proof of Theorem 5.1 depended on the fact that the centered conditional process has mean 0, so that Theorem 4.1 can be directly used. But we will need a version for the uncentered process $X^*$. For this purpose we state:

ASSUMPTION 3. There exists $u_0 > 0$ such that the family of functions of the variable $s$, $E(X^*(s) \mid X^*(0) = y)$, $s \in I$, forms a totally bounded subset of $C(I)$ for $u \geqq u_0$ and $0 \leqq t \leqq 1$.

THEOREM 5.2. *Under Assumptions* 1–3 *the measures* $(P^*_{t,u})$, $u \geqq u_0$, $0 \leqq t \leqq 1$, *induced by the conditioned processes* $X^*(s)$, $s \in I$, *are weakly compact over* $C(I)$.

PROOF. Write $X^*(s)$ as the sum of the centered process (5.5) and the centering function $E(X^*(s) \mid X^*(0) = y)$. By Theorem 5.1, the measures of the centered process are weakly compact. Under Assumption 3 the centering functions are contained in a compact subset of $C(I)$. It follows from the linearity of weak convergence that the measures induced by the sum $X^*$ are also weakly compact. □

Assumptions 2 and 3 hold whenever $q(t) = O(t^\alpha)$ for $t \downarrow 0$, for some $\alpha > 0$ (see Section 8); and whenever $q(t) = O(|\log t|^{-\alpha})$ for some $\alpha > 1$ (see Section 13). Our last result of this section is:

LEMMA 5.1. *Under* (2.12) *the process* $X^*(s)$, *conditioned by* $X^*(0) = y$, *is equivalent to the process* $X^*(s) + yEX(t)X(t + s/v)$, *conditioned by* $X^*(0) = 0$.

PROOF. As in the proof of Theorem 5.2, write $X^*(s)$ as the sum of the centered process (5.5) and the centering function. The conditional finite-dimensional distributions of the centered process, given $X^*(0) = y$, do not depend on $y$; indeed, by the well-known property of the multivariate normal distribution, the conditional covariance matrix does not depend on the conditioning value. Therefore, the conditioned process $X^*(s)$, given $X^*(0) = y$, is the same as the conditioned process $X^*(s) - E(X^*(s) \mid X^*(0) = 0) + E(X^*(s) \mid X^*(0) = y)$, given $X^*(0) = 0$. The statement of the lemma is now a direct consequence of the elementary formula for the conditional expectations of $X^*(s)$, given $X^*(0) = 0$, $y$, respectively.

**6. Sojourn times of Gaussian processes.** Let $Y(t)$, $0 \leqq t \leqq 1$, be a separable measurable stochastic process with Borel sample functions, and $G(y)$ a nonnegative measurable function. Then the integral $L = \int_0^1 G(Y(t))\,dt$ is called a "sojourn time" of $Y$; indeed, when $G$ is the indicator of a set, then $L$ represents the time spent by $Y$ in the set. Throughout this paper we assume that $G$ satisfies

$$0 < \int_{-\infty}^{\infty} G(y)e^{-cy}\,dy < \infty\,, \qquad \text{for all } c > 0\,. \tag{6.1}$$

This means that $G$ may increase—but not too quickly—for $y \to \infty$, and that $G$ must tend quickly to 0 for $y \to -\infty$.

For $u \geqq 1$, take the process $Y(t)$ to be $u(X(t) - u)$, $0 \leqq t \leqq 1$, where $X$ is Gaussian, so that

$$L = \int_0^1 G(u(X(t) - u))\,dt\,. \tag{6.2}$$

If $G$ is the indicator of the positive axis, then $L$ represents the time spent above the level $u$.

Suppose that $EX(t) = 0$ and $EX^2(t) = 1$ for all $t$. By Fubini's theorem we get a simple formula for $EL$:

$$EL = \int_{-\infty}^{\infty} G(y)\phi(u + y/u)\,dy/u\,. \tag{6.3}$$

It follows from (2.4) that

(6.4) $$EL \sim \phi(u)/u \int_{-\infty}^{\infty} G(y)e^{-y}\,dy\,, \qquad \text{for } u \to \infty\,.$$

We also have a formula for the conditional expectation. For any fixed $t$ and $y$, and with $r = EX(s)X(t)$:

(6.5) $$E(L \mid X(t) = u + y/u) = \int_0^1 \int_{-\infty}^{\infty} G(x)\phi\left(\frac{x - ry + u^2(1-r)}{u(1-r^2)^{\frac{1}{2}}}\right) dx \, \frac{ds}{u(1-r^2)^{\frac{1}{2}}}\,.$$

This follows from Fubini's theorem (like (6.3)) and the definition of the conditional normal density.

Now we state an integral identity for the distribution of $L$. It is a more general version of the one in [5], where $G$ was the indicator of the positive axis.

LEMMA 6.1. *Let $I\{\cdots\}$ be the indicator of the event $\{\cdots\}$; and put*

$$L_t = \int_0^t G(u(X(s) - u))\,ds\,.$$

*Then for all* $0 \leqq A < B \leqq \infty$:

(6.6) $$\int_A^B P(L > x)\,dx = \int_0^1 E[I\{A < L_t \leqq B\}G(u(X(t) - u))]\,dt\,.$$

PROOF. It suffices to prove the result for $A = 0$ and all $B > 0$; furthermore, it suffices to prove

$$\int_0^B I\{L > x\}\,dx = \int_0^1 I\{0 < L_t \leqq B\}G(u(X(t) - u))\,dt\,, \qquad \text{for } B > 0\,,$$

almost surely, and then take expectations and apply Fubini's Theorem. We claim that each side of the above equation is equal to $\min(L, B)$. This is easy to see for the left-hand side. As for the right-hand side, suppose first that $L \leqq B$. Then $L_t \leqq B$ for $0 \leqq t \leqq 1$, and the integral is equal to $\int_0^1 G\,dt$, or $L$. Next, suppose that $L > B$. Put $t^* = \sup(t\colon L_t = B)$; then the integral is $\int_0^{t^*} G\,dt = L_{t^*} = B$. □

Let $v = v(u)$ be an increasing function of $u$. One of our concerns is determining the limit

(6.7) $$\lim_{u\to\infty} \frac{P(vL > x)}{E(vL)}\,, \qquad \text{for } x > 0\,.$$

For $u \geqq 1$, put

(6.8) $$F_u(A) = \frac{\int_A^\infty P(vL > x)\,dx}{E(vL)}\,;$$

then $F_u$ is non-increasing, and $F_u(0) = 1$ and $F_u(\infty) = 0$ because

$$\int_0^\infty P(vL > x)\,dx = E(vL)\,.$$

The family of functions $(F_u)_{u\geqq 1}$ is uniformly absolutely continuous on every half-line $A \geqq A_0 > 0$:

$$F_u(A) - F_u(A + h) = \int_A^{A+h} \frac{P(vL > x)}{E(vL)}\,dx$$

$$\leqq \int_A^{A+h} \frac{dx}{x} \quad \text{(by Chebyshev inequality)} \quad \leqq h/A_0\,.$$

Now we apply the identity (6.6) to the numerator in (6.8) to get an alternate expression for $F_u$. Put $B = \infty$ and replace $L$ by $vL$ on the left hand side of (6.6); then

$$F_u(A) = (EL)^{-1} \int_0^1 E[I\{vL_t > A\}G(u(X(t) - u))]\, dt ,$$

and the latter is equal to

$$(EL)^{-1} \int_0^1 \int_{-\infty}^{\infty} P\{vL_t > A \mid u(X(t) - u) = y\} G(y)\phi(u + y/u)u^{-1}\, dy\, dt .$$

(The constant $v$ in $E(vL)$ is absorbed by the change of variable in (6.8).)

Let $P$ stand for the probability in the integrand in the integral displayed above; then, by (2.4) and (6.3),

$$F_u(A) = \frac{\int_0^1 \int_{-\infty}^{\infty} P \cdot G(y) \exp(-y - y^2/2u^2)\, dy\, dt}{\int_{-\infty}^{\infty} G(y) \exp(-y - y^2/2u^2)\, dy} .$$

The denominator converges to $\int_{-\infty}^{\infty} G(y)e^{-y}\, dy$ for $u \to \infty$. Write the integrand in the numerator as

$$P \cdot G(y)e^{-y} + P \cdot G(y)(e^{-y^2/2u^2} - 1)e^{-y} .$$

The double integral of the second term converges to 0 for $u \to \infty$. We conclude that

$$(6.9) \qquad F_u(A) = \frac{\int_0^1 \int_{-\infty}^{\infty} P\{vL_t > A \mid u(X(t) - u) = y\} G(y)e^{-y}\, dy\, dt}{\int_{-\infty}^{\infty} G(y)e^{-y}\, dy} + o(1) ,$$

for $u \to \infty$ .

In the following theorem we show that the existence of the limit of $F_u$ is tied to the existence of the limit of the conditional distribution of $vL_t$, and that the limit of $F_u$ is a mixture of the conditional limits.

THEOREM 6.1. *If*

$$(6.10) \qquad \mathscr{B}(A; t, y) = \lim_{u\to\infty} P\{vL_t > A \mid u(X(t) - u) = y\}$$

*exists for almost all* $A > 0$, $0 \leqq t \leqq 1$, *and* $-\infty < y < \infty$, *then so does*

$$(6.11) \qquad F(A) = \lim_{u\to\infty} F_u(A) , \qquad \textit{for all } A > 0 .$$

*$F$ is then given by*

$$(6.12) \qquad F(A) = \frac{\int_0^1 \int_{-\infty}^{\infty} \mathscr{B}(A; t, y)G(y)e^{-y}\, dy\, dt}{\int_{-\infty}^{\infty} G(y)e^{-y}\, dy} .$$

*$F$ is absolutely continuous, and its Radon–Nikodym derivative $F'$ satisfies*

$$(6.13) \qquad \lim_{u\to\infty} \frac{P(vL > x)}{E(vL)} = -F'(x) ,$$

*at all points of continuity* $x > 0$.

PROOF. The convergence (6.11) for almost all $A > 0$ is a consequence of (6.9) and Fubini's theorem. The convergence for all $A > 0$ then follows from the uniform absolute continuity of $F_u$. The form (6.12) of the limit is a consequence of (6.9) and (6.10). The absolute continuity of $F$ follows from the uniform absolute continuity of $F_u$.

For the proof of (6.13) note, by (6.8), that $F_u(A)$ may be expressed as an integral with integrand $P(vL > x)/E(vL)$. By Chebyshev's inequality, the latter is dominated by $x^{-1}$. The integrand is a monotone function; therefore, by weak compactness, it has a weak subsequential limit $W(x)$. By dominated convergence it follows that

$$F(A) - F(B) = \int_A^B W(x)\,dx\,, \qquad \text{for every } 0 \leqq A < B < \infty\,;$$

therefore, $F'(x) = -W(x)$ for almost all $x$. We conclude that $W$ is the only weak limit, and (6.13) follows. □

According to Theorem 6.1 we can find the limit (6.7) by first finding the limit (6.10). Our only hypotheses were the facts that the mean is 0 and the variance 1; we made no use of Assumptions 1–3. Now we will have to make two more assumptions in order to develop a method for finding the limit (6.10). The earlier set of assumptions, involving the $q$-function, put an upper bound on the variance of the increments of the process. In the assumptions below we put a lower bound on the variance. By analogy with the definition of the $q$-function, we define a "$p$-function":

$$p^2(t) = \inf_{|s'-s|\geqq t} E(X(s') - X(s))^2\,, \qquad 0 \leqq t \leqq 1\,. \tag{6.14}$$

The next assumption will be stated in two forms. The first is simpler, and is called

ASSUMPTION 4a. $p(s) > 0$ for $s > 0$.

Now we record this in an equivalent form which is more convenient for later reference:

ASSUMPTION 4. There exists a function $v = v(u)$ for $u \geqq 1$, such that $v(u) \to \infty$, for $u \to \infty$, and $\inf_{u\geqq 1, v\geqq s} up(s/v) > 0$, for $s > 0$.

Since the form 4 obviously implies 4a, it suffices to show that 4a implies 4. The monotonicity and continuity of $p$ imply that for every $u$ and $s$ there exists $v = v(u, s)$ such that $up(s/v(u, s)) = p(s)$. Then $\lim_{u\to\infty} v(u, s) = \infty$, for fixed $s$, and thus

$$v(u) = \inf\,(v(u, s): v(u, s) \geqq s)$$

is well defined for all sufficiently large $u$. It follows that $v(u) \leqq v(u, s)$ for all large $u$, for each $s$, and so

$$up(s/v(u)) \geqq up(s/v(u, s)) = p(s)\,.$$

It also follows that for each fixed $s$, $v(u) \geqq s$ for all large $u$; hence, from the previous double inequality,

$$\inf_{v\geqq s} up(s/v(u)) \geqq p(s)\,, \qquad \text{for large } u\,.$$

This is sufficient for the statement of Assumption 4.

For arbitrary $\delta$, $0 < \delta < 1$, we define

$$p_\delta(s) = \inf_{u \geqq 1, \delta v \geqq s} up(s/v)\,, \qquad s > 0\,. \tag{6.15}$$

Under Assumption 4 this is positive for $s > 0$. Our next assumption is that no two values of the process are completely negatively correlated.

ASSUMPTION 5. $\inf_{s,s'}$ correlation $(X(s), X(s')) > -1$.

The purpose of the next lemma is to show that it is only the values $X(s)$ for $s$ relatively close to $t$ which significantly affect the conditional distribution of $vL_t$ given $u(X(t) - u) = y$. Here we estimate the contribution of the values $X(s)$ for $s$ significantly far from $t$.

LEMMA 6.2. *Let $X(t)$, $0 \leqq t \leqq 1$, be Gaussian with mean* 0 *and variance* 1. *Then under Assumptions* 4 *and* 5 *there exists a positive number $J$, depending only on $y$, such that*

$$\begin{aligned} &E(vL_{t-d/v} \mid X(t) = u + y/u) \\ &\quad \leqq J\left\{\int_d^\infty \phi(\tfrac{1}{2}p_\delta(s))\,\frac{ds}{\tfrac{1}{2}p_\delta(s)} + (t-\delta)v\,\frac{\phi(\tfrac{1}{2}up(\delta))}{\tfrac{1}{2}up(\delta)}\right\}, \end{aligned} \tag{6.16}$$

*for all $0 < t \leqq 1$, $0 < \delta < t$, $u \geqq 1$ and $d > 0$ such that $\delta v > d$.*

PROOF. The expected value in (6.16) is given by $v$ times the double integral in (6.5), with $t - d/v$ in place of 1 as the upper limit of integration. By simple algebra:

$$\frac{x - ry + u^2(1-r)}{u(1-r^2)^{\frac{1}{2}}} = \frac{x - ry}{u(1-r^2)^{\frac{1}{2}}} + u\left(\frac{1-r}{1+r}\right)^{\frac{1}{2}};$$

thus, as in (2.4), we find the $\phi$-kernel in (6.5) to be

$$\phi\left(u\left(\frac{1-r}{1+r}\right)^{\frac{1}{2}}\right)\exp\left(-\frac{x-ry}{1+r}\right)\exp\left(-\frac{1}{2}\frac{(x-ry)^2}{u^2(1-r^2)}\right).$$

Put $c = \inf EX(s)X(s')$; then the product above is at most equal to

$$\phi(u((1-r)/2)^{\frac{1}{2}})\exp\left(\frac{|y|}{1+c}\right)\exp\left[-\min\left(\frac{x}{2}, \frac{x}{1+c}\right)\right].$$

(Note that $-1 < c < 1$ under Assumption 5.) Substitute this bound for the $\phi$-kernel in the representation of the conditional expectation, and integrate over $x$. Then, from (6.5), we obtain the bound

$$Jv\int_0^{t-d/v} \phi(u((1-r)/2)^{\frac{1}{2}})\,\frac{ds}{u((1-r)/2)^{\frac{1}{2}}}\,, \tag{6.17}$$

where

$$J = \exp\left(\frac{|y|}{1+c}\right)\left\{\int_0^\infty G(x)e^{-x/2}\,dx + \int_{-\infty}^0 G(x)\exp\left(-\frac{x}{1+c}\right)dx\right\}.$$

According to (6.2), $J$ is finite.

It follows from the definition (6.14) of $p$ and the fact that $EX^2(s) \equiv 1$ that $1 - r = 1 - EX(s)X(t) \geqq \frac{1}{2}p^2(|s - t|)$; therefore, the expression (6.17) cannot but increase if $(1 - r)/2$ is replaced by $\frac{1}{4}p^2(|s - t|)$. Change the variable in the resulting integral from $s$ to $v(t - s)$; then we get this bound for (6.17):

$$J \int_d^{tv} \phi\left(\frac{u}{2} p\left(\frac{s}{v}\right)\right) \frac{ds}{(u/2)p(s/v)} . \tag{6.18}$$

Split the domain of integration at the point $s = \delta v$, to obtain two integrals. These are bounded by the corresponding terms on the right-hand side of (6.16); indeed, this follows from the definitions (6.14) and (6.15) of $p$ and $p_\delta$. □

The significance of the inequality (6.16) is as follows. If the integral in (6.16) is finite, then it can be made arbitrarily small by choosing $d$ large. Furthermore, we will show that the second term on the right-hand side tends to 0 as $u \to \infty$ for a large class of Gaussian processes. The resulting inequality means that if $X(t)$ is "high," then the expected sojourn time above a high level over an interval separated from $t$ by $d/v$ units is small if $d$ is large.

**7. On the relation between the maximum and the high level sojourn time.** The major result of this section is about the relation between the event that $X(t)$ surpasses $u$ at some point $t$ in $[0, 1]$, and the event that it spends "relatively little time" above $u$. As a continuous function, $X$ must spend positive time above $u$ if it surpasses $u$. The main theorem of this section estimates "how much" time it must spend above $u$. Such estimates were given in special cases in [2] and [5]. Throughout this section we hold to Assumptions 1–3, but we do not need Assumptions 4 and 5.

We take $G$ as a function satisfying (6.1); however, in this particular section we put additional restrictions on $G$:

*$G(y)$ is piecewise continuous, and $G(y) > 0$ for all $y$ in some nondegenerate interval $(0, \tau)$.*

Since $X$ is continuous the integral $L$ in (6.2) is positive whenever $X(0) \leqq u$ and $\max_{[0,1]} X(t) > u$.

THEOREM 7.1. *If $X$ has mean 0 and variance 1 then, under Assumptions 1–3,*

$$\lim_{\varepsilon \to \infty} \limsup_{u \to \infty} \frac{P(\max_{[0,1]} X(t) > u,\, vL \leqq \varepsilon)}{v\phi(u)/u} = 0 . \tag{7.1}$$

PROOF. In estimating the probability in (7.1), we refine the method of proof of Theorem 3.1. Split $[0, 1]$ into $[v + 1]$ disjoint intervals $I$, each of length $1/[v + 1]$. If $\max_{[0,1]} X(t) > u$, then the maximum exceeds $u$ either in the first interval $I$, or else for the *first* time in some interval $I$ after the first:

$$\{\max_{[0,1]} X(t) > u\} \subset \{\max_{[0,1/[v+1]]} X(t) > u\} \cup \bigcup_I \{X(a) \leqq u,\, \max_I X(t) > u\} , \tag{7.2}$$

where $a$ is the left endpoint of $I$. According to Theorem 2.2 and the property

(5.1) of $v$, the event $\{\max_{[0,1/[v+1]]} X(t) > u\}$ has probability of the order $\phi(u)/u$ for $u \to \infty$. Then it follows from (7.2) that (7.1) is implied by

$$(7.3) \qquad \lim_{\varepsilon\to 0} \limsup_{u\to\infty} \sup_{0\leqq t\leqq 1} \frac{P(X(t) \leqq u, \max_{[t,t+1/v]} X(s) > u, vL \leqq \varepsilon)}{\phi(u)/u} = 0 .$$

Next we claim that the inequality $X(t) \leqq u$ in (7.3) may be replaced by the double inequality $u - M/u \leqq X(t) \leqq u$ for large $M > 0$. Indeed, by (5.1) and the corollary to Lemma 2.1, the ratio

$$\frac{P(X(t) \leqq u - M/u, \max_{[t,t+1/v]} X(s) > u)}{\phi(u)/u}$$

tends to 0 as $M \to \infty$ uniformly in $u$ and $t$. Thus in (7.3) it suffices to estimate

$$\frac{P(u - M/u \leqq X(t) \leqq u, \max_{[t,t+1/v]} X(s) > u, vL \leqq \varepsilon)}{\phi(u)/u} .$$

Next we write the probability above as the integral of the conditional probability given $X(t)$; we recall the definition of the process $X^*$ in (5.4); and we note that $L \leqq \int_t^{t+1/v} G(u(X(s) - u))\, ds$. Then, as in the proof of Lemma 2.1, the ratio displayed above is at most equal to

$$\int_{-M}^{0} P\{\max_{[0,1]} X^*(s) > 0, \int_0^1 G(X^*(s))\, ds \leqq \varepsilon \mid X^*(0) = y\} e^{-y}\, dy .$$

By Lemma 5.1, this is equal to

$$(7.4) \qquad \int_{-M}^{0} P\{\max_{[0,1]} (X^*(s) + yR) > 0, \int_0^1 G(X^*(s) + yR)\, ds \leqq \varepsilon \mid X^*(0) = 0\} e^{-y}\, dy ,$$

where $R = EX(t + s/v)X(t)$.

In order to prove (7.1) it now suffices to estimate the bound (7.4), and to show that if $\{u\}$ is an arbitrary sequence tending to infinity, and $\{t\}$ is an arbitrary sequence in [0, 1], then the corresponding sequence (7.4) has a subsequence which converges to a limit $L(\varepsilon)$ for all $\varepsilon > 0$, and is such that $L(\varepsilon) \to 0$ for $\varepsilon \to 0$. Let $\{u, t\}$ be a double sequence constructed from such a $u$-sequence and $t$-sequence. Then, by weak compactness (Theorem 5.1) we can find a subsequence for which the process $X^*(s)$, conditioned by $X^*(0) = 0$, converges weakly over $C[0, 1]$ to a limiting process $U(s)$ for which $U(0) = 0$. It follows that the joint distribution of the functionals

$$\max_{[0,1]} (X^*(s) + yR) \qquad \text{and} \qquad \int_0^1 G(X^*(s) + yR)\, ds$$

converges weakly to the corresponding joint distribution of the functionals of the process $U(s) + y$. (Note that Assumption 1 implies $R \to 1$ uniformly.) Therefore, the conditional probability in (7.4) converges to

$$P\{\max_{[0,1]} U(s) > -y, \int_0^1 G(U(s) + y)\, ds \leqq \varepsilon\}$$

except possibly for a countable set of $y$'s and $\varepsilon$'s. Therefore, the integral (7.4) converges to the limit

$$\int_{-M}^{0} P\{\max_{[0,1]} U(s) > -y, \int_0^1 G(U(s) + y)\, ds \leqq \varepsilon\} e^{-y}\, dy .$$

This tends to 0 for $\varepsilon \to 0$. Indeed, by bounded convergence, the limit is obtained by putting $\varepsilon = 0$. Then the integrand is equal to 0 because $U(s)$ is a continuous function such that $U(0) = 0$. □

**8. Locally stationary Gaussian processes.** We will now describe a general class of Gaussian processes which satisfy Assumptions 1–4. Then we will use the results above and additional properties of these processes to prove limit theorems for the sojourn times and maximum functional. We suppose again that

$$EX(t) \equiv 0, \qquad EX^2(t) \equiv 1 . \tag{8.1}$$

We introduce the property of *local stationarity*. Suppose there exists a continuous function $H(t)$ such that $H(t) > 0$ for all $0 \leqq t \leqq 1$; and a continuous monotone function $K(s)$ with $K(0) = 0$ and $K(s) > 0$ for $s > 0$ such that

$$\lim_{s\to 0} \frac{E(X(t + s) - X(t))^2}{2K(|s|)} = H(t) , \qquad \textit{uniformly in } \ 0 \leqq t \leqq 1 . \tag{8.2}$$

Then $X$ is called locally stationary. In particular, when $X$ is stationary in the usual sense, then the numerator in (8.2) is a function of $|s|$ alone, and $H$ is a constant.

The interval $[0, 1]$ in the definition of local stationarity was chosen for simplicity; the definition can be extended to any compact interval $I$. There is a large class of locally stationary processes which are not strictly stationary. Let $Y(t)$ have stationary Gaussian increments, with $EY(t) = 0$ and $EY^2(t) = \sigma^2(t)$, $t \geqq 0$, where $\sigma^2(t)$ is positive for $t > 0$, and continuous, and $\sigma^2(0) = 0$. If $(\sigma(t + h) - \sigma(t))/\sigma(h)$ converges for $h \to 0$ uniformly in $t \in I$, where $I$ is a compact interval bounded away from 0, then the process $X(t) = Y(t)/\sigma(t)$ is locally stationary on $I$. This is implicit in [4], Section 1. Sufficient conditions on $\sigma$ are also given there.

Conditions (8.1) and (8.2) guarantee the continuity in $(s, t)$ of $EX(s)X(t)$.

Our previous work on high level excursions of stationary Gaussian processes depended on hypotheses of local regularity of the covariance function. Now we make a corresponding assumption for the locally stationary process, namely, that $K$ is regularly varying of index $\alpha$, for some $0 < \alpha \leqq 2$:

$$\lim_{s\downarrow 0} K(st)/K(s) = t^{\alpha} , \qquad \text{for } \ t \geqq 0 . \tag{8.3}$$

This condition, together with (8.2), implies that $E(X(t + s) - X(t))^2$ is uniformly regularly varying.

For convenience, we recall certain results on functions of regular variation:

$$\lim_{s\downarrow 0} K(s) s^{-\alpha-\varepsilon} = \infty , \qquad \text{for every } \ \varepsilon > 0 . \tag{8.4}$$

$$\lim_{s\downarrow 0} K(s) s^{-\alpha+\varepsilon} = 0 , \qquad \text{for every } \ \varepsilon > 0 . \tag{8.5}$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\tfrac{1}{2}s^{\alpha+\varepsilon} \leqq K(sh)/K(h) \leqq 2s^{\alpha-\varepsilon} \tag{8.6}$$

for all $0 < s \leqq 1$ and $h > 0$ such that $sh < \delta$.

These are basic results of Karamata [10]. His results also show that the convergence (8.3) is uniform on compact intervals in the open set $t > 0$. This was extended in [5] to the closed set $t \geqq 0$:

(8.7) The convergence (8.3) is uniform on compact $t$-intervals of $[0, \infty]$.

A major part of our analysis depends on the construction of a function $v = v(u)$ for $u > 0$. For sufficiently large $u > 0$, let $v$ be the largest solution of the equation

$$u^2K(1/v) = 1\,. \tag{8.8}$$

$v$ certainly exists because $K(s)$ is monotonic, continuous, and tends to 0 with $s$. As a function of $u$, $v$ is monotonic and tends to $\infty$ with $u$. It can then be extended arbitrarily to all positive $u$ in such a way as to preserve monotonicity; however, our interest is in only values of $v$ for large $u$.

As an immediate consequence of (8.2), (8.3) and (8.7) we obtain:

$$\lim_{u\to\infty} u^2E(X(t + s/v) - X(t))^2 = 2H(t)|s|^\alpha \tag{8.9}$$

*uniformly in* $0 \leqq t \leqq 1$, *and on compact s-intervals.*

We also recall from [5] an asymptotic relation between $u$ and $v$:

$$\lim_{u\to\infty} v^{\alpha-\varepsilon}u^{-2} = 0\,, \qquad \text{for every } \varepsilon > 0\,. \tag{8.10}$$

THEOREM 8.1. *If* (8.1), (8.2), *and* (8.3), *and Assumption* 4a *hold, then so do Assumptions* 1–4; *in particular* 2, 3 *and* 4 *hold when v is chosen according to* (8.8).

PROOF. According to the definition (2.1) and the condition (8.2), we have

$$q^2(t) \leqq \text{constant} \cdot K(t)\,. \tag{8.11}$$

Assumption 1 now follows from this and (8.5).

If $v$ is defined by (8.8), then it follows from (8.6) and (8.11) that

$$u^2q^2(v^{-1}e^{-z^2}) \leqq \text{constant}\,\frac{K(v^{-1}e^{-z^2})}{K(v^{-1})} \leqq \text{constant} \cdot e^{-z^2(\alpha-\varepsilon)} \tag{8.12}$$

for all sufficiently large $u$ and all $z \geqq 1$; therefore, (5.1) holds. If $Q^*$ is defined as in (5.3), then, from (8.12), it follows that $\lim_{s\downarrow 0} Q^*(s)s^{-\frac{1}{2}(\alpha-\varepsilon)} = 0$ for $\varepsilon > 0$, so that Assumption 2 holds.

By the elementary formula for conditional expectation we find (see (5.4))

$$E(X^*(s)\,|\,X^*(0) = y) = u^2(EX(t + s/v)X(t) - 1) + yEX(t + s/v)X(t)\,.$$

The latter is equal to

$$-\tfrac{1}{2}u^2E(X(t + s/v) - X(t))^2 - \tfrac{1}{2}yE(X(t + s/v) - X(t))^2 + y\,,$$

which, by (8.9), converges uniformly in $s$ (in compact sets) and $t \in [0, 1]$ to $-H(t)|s|^\alpha + y$ for $u \to \infty$. Therefore, Assumption 3 is valid.

Let $p(t)$ be defined as in (6.14), and $p_\delta(s)$ as in (6.15), with $v$ as in (8.8); then $up(s/v) \geqq \min(p_\delta(s), up(\delta))$. It follows from the left-hand inequality in (8.6) that

$$p_\delta^2(s) \geqq \tfrac{1}{2}s^{\alpha+\varepsilon} \qquad \text{for sufficiently small } \delta\,, \tag{8.13}$$

and, from Assumption 4a, that $p(\delta) > 0$. It follows that the statement of Assumption 4 holds with $v$ given in (8.8). □

We will show that the process $X^*$, defined in (5.4), converges weakly to a limiting process for $u \to \infty$. Let $U(s)$, $-\infty < s < \infty$, be a Gaussian process with

$$EU(s) \equiv 0\,, \qquad EU(s)U(t) = |s|^\alpha + |t|^\alpha - |t-s|^\alpha\,. \tag{8.14}$$

(See [2] page 67.) For each $t$ in $[0, 1]$, and with $H(t)$ in (8.2), let $W_t(s)$ be the process

$$W_t(s) = (H(t))^{\frac{1}{2}}U(s) - H(t)|s|^\alpha\,, \qquad -\infty < s < \infty\,. \tag{8.15}$$

THEOREM 8.2. *Under* (8.1), (8.2) and (8.3), *the process* $X^*(s)$, $s \in I$, *defined in* (5.4), *with* $v$ *as in* (8.8), *conditioned by* $X^*(0) = y$, *converges weakly to the process* $W_t(s) + y$, *for* $u \to \infty$, *for any closed bounded interval* $I$ *containing the origin, for any fixed* $t$ *in* $[0, 1]$, *and any* $y$.

PROOF. If $u$ is large, then so is $v$, and then $I$ is contained in $[-vt, v(1-t)]$. By Theorem 5.2, the measures induced by $X^*$ are then weakly compact over $C(I)$. Therefore, to complete the proof, it suffices to show the convergence of the finite dimensional distributions to those of $W_t(s) + y$.

According to the calculation in the proof of Theorem 8.1, we have

$$E(X^*(s) \mid X^*(0) = y) \to -H(t)|s|^\alpha + y\,.$$

By the elementary formula for conditional variance, we have

$$\begin{aligned}\operatorname{Var}&(X^*(s) - X^*(s') \mid X^*(0))\\ &= u^2E(X(t+s/v) - X(t+s'/v))^2\\ &\qquad - u^2[EX(t)X(t+s/v) - EX(t)X(t+s'/v)]^2\,,\end{aligned}$$

and this, by (8.9) and the relation

$$\begin{aligned}EX(t)&X(t+s/v) - EX(t)X(t+s'/v)\\ &= \tfrac{1}{2}E(X(t+s'/v) - X(t))^2 - \tfrac{1}{2}E(X(t+s/v) - X(t))^2\,,\end{aligned}$$

converges to $2H(t)|s|^\alpha$. These limiting moments agree with those of the process $W_t(s) + y$. □

**9. A limit theorem for the sojourn time above a high level.** In this section we show the existence of the limit (6.7) under the conditions of Section 8, and determine the explicit form of the limit. According to Theorem 6.1, it is enough to find the function $\mathscr{B}$ in (6.10).

THEOREM 9.1. *Let $X$ satisfy* (8.1), (8.2) *and* (8.3), *and Assumptions* 4a *and* 5; *and let $v$ be the function defined by* (8.8). *If $G$ is piecewise continuous, then the limit* (6.10) *exists and is equal to*

$$\mathscr{B}(A; t, y) = P\{\textstyle\int_0^\infty G(W_t(s) + y)\, ds > A\}, \tag{9.1}$$

*where $W_t(s)$ is defined by* (8.15).

PROOF. For arbitrary fixed $d < vt$, write

$$vL_t = v(L_t - L_{t-d/v}) + vL_{t-d/v}. \tag{9.2}$$

The first term on the right-hand side of (9.2) is, by definition, equal to $\int_{-d}^0 G(X^*(s))\, ds$. According to Theorem 8.2, its conditional distribution, given $X^*(0) = y$, converges to that of $\int_{-d}^0 G(W_t(s) + y)\, ds$. By the equivalence of the processes $W_t(s)$ and $W_t(-s)$, the latter integral has the same distribution as

$$\textstyle\int_0^d G(W_t(s) + y)\, ds. \tag{9.3}$$

Next we show that the contribution of the last term in (9.2) to the limiting distribution is negligible if $d$ is large. From this we will conclude that the limiting distribution of $vL_t$ is that of (9.3) with $d = \infty$.

Apply (6.16), and then to it the estimate (8.13):

$$E(vL_{t-d/v} \mid X(t) = u + y/u) \leqq J \int_d^\infty \phi\left(\frac{1}{2(2)^{\frac12}} s^{(\alpha+\varepsilon)/2}\right) 2(2)^{\frac12} s^{-(\alpha+\varepsilon)/2}\, ds$$
$$+ J \frac{2(t-\delta)v}{(2\pi)^{\frac12} u p(\delta)} \exp(-u^2 p^2(\delta)/8).$$

By (8.10), the last term above tends to 0 as $u \to \infty$; therefore,

$$\limsup_{u\to\infty} E(vL_{t-d/v} \mid X(t) = u + y/u)$$
$$\leqq J \int_d^\infty \phi\left(\frac{1}{2(2)^{\frac12}} s^{(\alpha+\varepsilon)/2}\right) 2(2)^{\frac12} s^{-(\alpha+\varepsilon)/2}\, ds.$$

The latter tends to 0 as $d \to \infty$. The proof is complete. □

According to Theorem 9.1 $\mathscr{B}$ exists and is given by (9.1). The result is stated for a process defined on the interval [0, 1]. However it can be extended to a process on an arbitrary interval $[a, b]$. For example, suppose $a = 0$ and $b = T$. Let $H(t)$, $0 \leqq t \leqq T$, be the function in the condition (8.2) for local stationarity. Put $Y(t) = X(t/T)$; then $Y$ is defined on [0, 1] and is locally stationary with "$H$-function" $H(t/T)$ and "$K$-function" $K(s/T)$. Thus the only alteration in the limit (9.1) is that $t$ is replaced by $t/T$.

When $X$ is stationary, $H(t)$ is a constant and so $W_t(s)$ is independent of $t$. $F(A)$ is then the ratio of single integrals in (6.12). A special case with $G$ the indicator of the positive axis first appeared in [2]. A particular class of locally stationary processes was considered in [4], and a limit of the type (6.12) obtained, again with $G$ the indicator of the positive axis.

We would like to show that the function $F$ given by (6.11), where the kernel $\mathscr{B}$ is in (9.1), is of positive variation. Otherwise, the limit (6.13) would be equal to 0.

THEOREM 9.2. *$F$ is of positive variation if*

(i) $\alpha < 2$, *or*

(ii) $\alpha = 2$ *and* $\int_{-1}^{1} G(x+y)|\log|x||\,dx < \infty$, *on a $y$-set of positive measure in the support of $G$.*

PROOF. Assume the contrary, that $F$ is constant. Then, as a monotonic function of $A$, $\mathscr{B}$ would be constant for all $y$ in the support of $G$ and all $t$ in $[0, 1]$. But this would imply

$$\int_0^\infty G(W_t(s)+y)\,ds = 0 \quad \text{almost surely} \tag{9.4}$$

or

$$\int_0^\infty G(W_t(s)+y)\,ds = \infty \quad \text{almost surely.} \tag{9.5}$$

The expected value of the above integral is

$$\int_{-\infty}^{\infty}\int_0^\infty G(x+y)\phi\left(\frac{x+H(t)s^\alpha}{(2H(t)s^\alpha)^{\frac{1}{2}}}\right)(2H(t)s^\alpha)^{-\frac{1}{2}}\,ds\,dx\,.$$

By a change of variable of integration we can remove the factor $H(t)$ and write this integral as a positive multiple of

$$\int_{-\infty}^{\infty} G(x+y)e^{-\frac{1}{2}x}\{\int_0^\infty \exp(-x^2/4s^\alpha - s^\alpha/4)s^{-\alpha/2}\,ds\}\,dx\,. \tag{9.6}$$

Under (6.1), $G$ is positive on a set of positive measure, and so (9.6) is positive; therefore (9.4) is impossible.

If $\alpha < 2$, then the inner integral in (9.6) is dominated by $\int_0^\infty \exp(-s^\alpha/4)s^{-\alpha/2}\,ds$; thus, by (6.1), the double integral (9.6) is finite; therefore, (9.5) is impossible.

If $\alpha = 2$, then the inner integral in (9.6) is a decreasing function of $x^2$ which is of the order $|\log|x||$ for $x^2 \to 0$. Therefore, the double integral (9.6) is finite under case (ii), and so (9.5) is impossible.

**10. The tail of the distribution of the maximum.** In this section we derive the exact asymptotic form of $P(\max_{[0,1]} X(t) > u)$ for $u \to \infty$. Theorem 3.1 implies

$$\limsup_{u\to\infty} \frac{P(\max_{[0,1]} X(t) > u)}{v\phi(u)/u} < \infty\,. \tag{10.1}$$

Now we show that under the conditions of Theorem 9.1 the ratio above actually converges to a positive limit.

THEOREM 10.1. *Let $X$ satisfy* (8.1), (8.2) *and* (8.3), *and Assumptions* 4a *and* 5; *and let $v$ be defined by* (8.8). *If $G$ is a function satisfying the conditions of Theorem* 7.1, *and also the condition*

$$G(y) = 0\,, \qquad \textit{for } y > 0\,, \tag{10.2}$$

*and F is given by* (6.12), *then*

$$-F'(0) = \lim_{x \downarrow 0} x^{-1}(F(0) - F(x)) \tag{10.3}$$

*exists and is positive and finite; furthermore,*

$$\lim_{u\to\infty} \frac{P(\max_{[0,1]} X(t) > u)}{v\phi(u)/u} = -F'(0) \int_{-\infty}^{\infty} G(y)e^{-y}\, dy . \tag{10.4}$$

PROOF. If $G$ satisfies the conditions of Theorem 7.1, and also satisfies (10.2), then $L > 0$ if and only if $\max_{[0,1]} X(t) > u$. The point of the theorem is to extend the relation (6.13) to $x = 0$, and so conclude that

$$P(\max_{[0,1]} X(t) > u) \sim -F'(0)E(vL) .$$

The above conditions on $G$ certainly imply those of Theorems 9.1 and 9.2; therefore, according to these theorems and Theorem 6.1, $F(x)$ exists, is of positive variation, and has the derivative $F'(x)$ satisfying (6.13). We write $x^{-1}(F(0) - F(x))$ as $x^{-1} \int_0^x -F'(y)\, dy$. By (6.13), $-F'$ is non-increasing, so that its right-hand limit exists at 0. On the other hand, it also follows that the limit of $x^{-1}(F(0) - F(x))$ for $x \downarrow 0$ exists, and the two limits are equal. We call this common limit $-F'(0)$; this proves (10.3).

The limit (10.3) is positive; indeed, if it were 0, then as a monotone function, $-F'(x)$ would vanish for all $x$, and $F$ would be constant. But this would contradict Theorem 9.2.

By the same reasoning as in the proof of Theorem 8.1 of [5], we find that

$$-F'(0) \leqq \liminf_{u\to\infty} [E(vL)]^{-1}P(\max_{[0,1]} X(t) > u) \tag{10.5}$$

and

$$\limsup_{u\to\infty} \frac{P(\max_{[0,1]} X(t) > u)}{E(vL)} \leqq -F'(0) + \lim_{\varepsilon\to 0} \limsup_{u\to\infty} \frac{P(\max_{[0,1]} X(t) > u,\ vL \leqq \varepsilon)}{E(vL)} . \tag{10.6}$$

By (6.4) $E(vL)$ may be replaced by a constant multiple of $v\phi(u)/u$. The finiteness of $-F'(0)$ now follows from (10.1) and (10.5); and, by Theorem 7.1, the last member of (10.6) is equal to 0. The relation (10.4) follows. □

It follows from Theorems 9.1 and 10.1 that

$$\lim_{u\to\infty} \frac{P(vL > x)}{P(vL > 0)} = \frac{F'(x)}{F'(0)} , \qquad x > 0 .$$

This means that the conditional distribution of $vL$, given that it is positive, converges:

$$\lim_{u\to\infty} P(vL \leqq x \mid vL > 0) = 1 - F'(x)/F'(0) .$$

If the interval [0, 1] is replaced by [0, $T$], then (10.4) takes the form

$$\lim_{u\to\infty} \frac{P(\max_{[0,T]} X(t) > u)}{Tv\phi(u)/u} = -F'(0) \int_{-\infty}^{\infty} G(y)e^{-y}\, dy .$$

**11. Sojourn times for the moving barrier.** Let $f(t)$, $0 \leqq t \leqq 1$, be a continuous function. For arbitrary $u > 0$, consider the integral

$$L = \int_0^1 G(u(X(t) - u - f(t)))\, dt\,. \tag{11.1}$$

We assume again that $G$ satisfies (6.1). If $G$ is the indicator of the positive axis, then $L$ represents the time spent above the curve $x = u + f(t)$ for $0 \leqq t \leqq 1$; thus, we call $L$ a "sojourn time above a moving barrier." As in Section 6, we also define

$$L_t = \int_0^t G(u(X(s) - u - f(s)))\, ds\,, \qquad 0 \leqq t \leqq 1\,. \tag{11.2}$$

The following expectation formulas correspond to (6.3) and (6.5), respectively:

$$EL_t = \int_{-\infty}^{\infty} G(y)\{\int_0^t \phi(u + y/u + f(s))\, ds\}\, dy/u\,; \tag{11.3}$$

$$\begin{aligned} &E(L_t \mid X(t) = u + y/u) \\ &\quad = \int_0^t \int_{-\infty}^{\infty} G(x)\phi\left(\frac{x - f(s) - ry + u^2(1-r)}{u(1-r^2)^{\frac{1}{2}}}\right) dx\, \frac{ds}{u(1-r^2)^{\frac{1}{2}}}\,. \end{aligned} \tag{11.4}$$

As in [5] we assume that $f(t) > 0$ for $t > 0$, and that $f(0) = 0$; and we note also that the calculations to follow can be extended to any $f$ with a unique minimum on $[0, 1]$. For fixed $\beta > 0$, we assume that $f$ is regularly varying of index $\beta$:

$$\lim_{t\to 0} f(ts)/f(t) = s^\beta\,, \qquad \text{for } s \geqq 0\,. \tag{11.5}$$

By analogy with the definition of the function $v$ in (8.8), we define the monotonic function $w = w(u)$, for all large $u$, as the largest solution of the equation

$$uf(1/w) = 1\,. \tag{11.6}$$

It follows from (11.5) and (11.6) that

$$\lim_{u\to\infty} uf(t/w) = t^\beta\,, \qquad \text{for } t > 0\,. \tag{11.7}$$

Now we derive an asymptotic formula analogous to (6.4).

LEMMA 11.1. *If $X$ has mean 0 and variance 1, then*

$$EL_t \sim \frac{\phi(u)}{wu}\, \Gamma(1 + 1/\beta) \int_{-\infty}^{\infty} G(y)e^{-y}\, dy\,,$$

*where $\Gamma$ is the Gamma function. The right-hand side of the above relation is independent of $t$.*

PROOF. Write the argument of the function $\phi$ in (11.3) as $u + (y + uf(s))/u$ and apply (2.4); then insert the resulting expression in the integrand and change the inner variable of integration from $s$ to $sw$:

$$\frac{\phi(u)}{uw} \int_{-\infty}^{\infty} G(y)e^{-y}\{\int_0^{tw} \exp[-uf(s/w) - (y + uf(s/w))^2/2u^2]\, ds\}\, dy\,.$$

By (11.7) we expect the inner integral to converge to

$$\int_0^\infty e^{-s^\beta}\,ds\,, \qquad \text{or} \qquad \Gamma(1+1/\beta)\,.$$

The convergence can be justified by the reasoning used in the proof of Theorem 9.1: The integrand is dominated by $\exp(-\frac{1}{2}s^{\beta+\varepsilon})$ for $s \leqq w\delta$; and the integral from $w\delta$ to $wt$ is of the order $w\exp(-u^2\cdot\text{constant})$, which tends to 0. □

Let $z = z(u)$ be a monotone function of $u$. By analogy with (6.8) we define

$$F_u(A) = \frac{\int_A^\infty P(zL > x)\,dx}{E(zL)}\,. \tag{11.8}$$

The main difference between sojourns for the level and the moving barriers is that the sojourn above the level barrier is likely to take place at any point of the interval, but that for the moving barrier only at the beginning of the interval. Indeed, in the latter case, the growth of the barrier for $t > 0$ makes it improbable for a sojourn to occur too far from the origin. This is reflected in the assertion of Lemma 11.1: $EL_t$ is asymptotically the same for any $t > 0$. We apply this result and show that the contribution of $t$-values to the sojourn time integral comes asymptotically only from those values which are of the order $1/w$. More exactly, we will show that the computation of the limit of $F_u$ may be done by replacing $L$ by $L_{d/w}$ in (11.8), passing to the limit over $u$, and then letting $d \to \infty$.

LEMMA 11.2.

$$\lim_{d\to\infty}\limsup_{u\to\infty}\sup_{A>0}\left|F_u(A) - \frac{\int_A^\infty P(zL_{d/w} > x)\,dx}{E(zL)}\right| = 0\,.$$

PROOF. Fix $d > 0$; then choose $u$ so large that $d/w < 1$. It follows that $L_{d/w} \leqq L$, and so

$$P(zL > x) \leqq P(zL_{d/w} > x)\,;$$

therefore,

$$0 \leqq F_u(A) - \frac{\int_A^\infty P(zL_{d/w} > x)\,dx}{E(zL)} \leqq F_u(0) - \frac{\int_0^\infty P(zL_{d/w} > x)\,dx}{E(zL)}$$
$$= \frac{E(L - L_{d/w})}{EL}\,.$$

As in the proof of Lemma 11.1 we express $E(L - L_{d/w})$ as a double integral of the form (11.3) but with the domain of the inner integral from $d/w$ to 1. Then, by the same estimates as in that proof, we obtain

$$\frac{E(L - L_{d/w})}{EL} \sim \int_{-\infty}^\infty G(y)e^{-y}\,dy \int_d^\infty e^{-s^\beta}\,ds\,.$$

Let $d \to \infty$ to complete the proof. □

The following is an analogue of Theorem 6.1:

THEOREM 11.1. *If*

$$\mathscr{B}(A; t, y) = \lim_{u\to\infty} P\{zL_{t/w} > A \mid u(X(t/w) - u - f(t/w)) = y\} \tag{11.9}$$

*exists for almost all $A$, $t$ and $y$, then so does*

$$F(A) = \lim_{u\to\infty} F_u(A)\,, \qquad A > 0\,. \tag{11.10}$$

*$F$ is then given by*

$$F(A) = \frac{\int_0^\infty \{\int_{-\infty}^\infty \mathscr{B}(A; t, y)G(y)e^{-y}\,dy\}e^{-t^\beta}\,dt}{\Gamma(1 + 1/\beta)\int_{-\infty}^\infty G(y)e^{-y}\,dy}\,. \tag{11.11}$$

*$F$ is absolutely continuous, and*

$$\lim_{u\to\infty} \frac{P(zL > x)}{E(zL)} = -F'(x) \tag{11.12}$$

*at all positive continuity points of $F'$.*

PROOF. According to Lemma 11.2 we can find the limit of $F_u$ by replacing $L$ in the numerator of (11.8) by $L_{d/w}$ and then letting $u \to \infty$ and $d \to \infty$. Apply Lemma 6.1 with the process $X(t) - f(t)$ in the place of $X(t)$ and the time interval $[0, d/w]$ in place of $[0, 1]$:

$$\begin{aligned}(E(zL))^{-1} &\int_A^\infty P(zL_{d/w} > x)\,dx \\ &= (EL)^{-1}\int_{A/z}^\infty P(L_{d/w} > x)\,dx \\ &= (EL)^{-1}\int_0^{d/w} E[I\{zL_t > A\}G(u(X(t) - f(t) - u))]\,dt\,.\end{aligned}$$

As in the calculations between (6.8) and (6.9) above we find the last expression to be equal to

$$(EL)^{-1}\int_0^{d/w}\int_{-\infty}^\infty P\{zL_t > A \mid u(X(t) - f(t) - u) = y\}G(y)\phi(u + y/u + f(t))\,dy\,dt\,.$$

Change the variable of integration from $t$ to $tw$, and estimate the $\phi$-kernel as in the proof of Lemma 11.1. Then the expression above is asymptotic to

$$\frac{\int_0^d \int_{-\infty}^\infty P\{zL_{t/w} > A \mid u(X(t/w) - f(t/w) - u) = y\}G(y)e^{-y-t^\beta}\,dy\,dt}{\Gamma(1 + 1/\beta)\int_{-\infty}^\infty G(y)e^{-y}\,dy}\,.$$

The proof of (11.10) and (11.11) is completed by letting $u \to \infty$ and then $d \to \infty$; (11.12) follows as (6.13). □

**12. A limit theorem for the sojourn time above a moving barrier.** In order to obtain the limit of $F_u$ for the moving barrier we assume, as in [5], page 366, conditions on the relation between the growth of the function $K$ in (8.2) and the growth of $f$ in (11.1). We suppose the existence of the following limit, finite or infinite:

$$p = \lim_{t\to 0} f(t)/K^{\frac{1}{2}}(t)\,. \tag{12.1}$$

It follows, as in [5], Lemma 3.3, that

$$\lim_{u\to\infty} w/v = p^{1/\alpha}\,, \tag{12.2}$$

where $p = 0$ if $\alpha/2 < \beta$ and $p = \infty$ if $\alpha/2 > \beta$. Recall that $\alpha$ and $\beta$ are the indices of variation of $K$ and $f$, respectively.

There are three cases to consider in the limit theorem for the moving barrier, corresponding to $p = 0$, $0 < p < \infty$, and $p = \infty$. If $p = 0$, then $f$ grows at a rate slower that $K^{\frac{1}{2}}$, and the limiting distribution of the sojourn time is very similar to that for the level barrier. As in that case, the normalizing function is taken to be $v$. If $0 < p < \infty$, then $f$ is of the same order as $K^{\frac{1}{2}}$, and the moving part of the barrier plays a role in the limiting distribution. The functions $v$ and $w$ are of the same order, and either may be used as the normalizing function. The case $p = \infty$ is very different from the first two: The barrier rises so rapidly that the sojourn time is dominated by the value of $X(0)$. The normalizing function is $w$. The first two cases, but not the third, were considered in [5] for stationary $X$ and $G$ the indicator of the positive axis (*ibid.*, Section 1).

THEOREM 12. 1. *Under the assumptions on $X$ and $G$ stated in Theorem* 9.1, *and conditions* (11.5) *and* (12.1) *on $f$, the limit* (11.9) *exists and*

$$\begin{aligned}
\mathscr{B}(A; t, y) &= P\{\textstyle\int_0^\infty G(W_0(s) + y)\, ds > A\}\,, && \textit{for } p = 0,\ z = v\,;\\
&= P\{\textstyle\int_0^t G(W_0(sp^{-1/\alpha}) - (t-s)^\beta + t^\beta + y\}\, ds > A\}\,, && \textit{for } 0 < p < \infty,\ z = w\,;\\
&= I\{\textstyle\int_0^t G(y + t^\beta - s^\beta)\, ds > A\}\,, && \textit{for } p = \infty,\ z = w\,,
\end{aligned} \tag{12.3}$$

*where $I\{\cdots\}$ is the indicator of $\{\cdots\}$.*

PROOF. The proof is based on the methods used for the level barrier in Theorem 9.1, and for the moving barrier for the stationary case in [5], Theorem 6.1. Since many of the details have already been given there, these will not be repeated, and the present proof will only be sketched.

We analyze the conditional probability on the right-hand side of (11.9). By a change of variable of integration we write $zL_{t/w}$ as

$$(z/v) \textstyle\int_0^{tv/w} G(u(X(t/w - s/v) - u) - uf(t/w - s/v))\, ds\,.$$

This can be expressed as a sojourn time of the $X^*$-process in (5.4):

$$(z/v) \textstyle\int_0^{tv/w} G(X^*(-s) - uf(t/w - s/v))\, ds\,; \tag{12.4}$$

and the conditioning in (11.9) may be written as

$$X^*(0) = y + uf(t/w)\,. \tag{12.5}$$

Here the unwritten parameter $t$ in $X^*$ has been replaced by $t/w$. Since the latter tends to 0 for $u \to \infty$, it can be shown, as in the proof of Theorem 8.2, that the process $\{X^*(s)\}$, conditioned by $X^*(0) = y$, converges weakly to the process $\{W_0(s) + y\}$. Now the condition (12.5) is asymptotically the same as

$$X^*(0) = y + t^\beta\,, \tag{12.6}$$

by virtue of (11.7); therefore, it is to be expected that $\{X^*(s)\}$, under the conditioning (12.5), converge weakly to $\{W_0(s) + y + t^\beta\}$. Since $\{W_0(-s)\}$ is equivalent to $\{W_0(s)\}$, it follows also that

$$(12.7) \qquad \{X^*(-s)\}\,, \quad \text{conditioned by} \quad X^*(0) = y + uf(t/w)\,, \quad \text{converges weakly to} \quad \{W_0(s) + y + t^\beta\}\,, \quad \text{over} \quad C(I)\,,$$

for any compact interval $I$. We apply this and (11.7) to determine the conditional limiting distribution of (12.4).

*Case* $p = 0$. Here $z = v$ and $w/v \to 0$, so that $uf(t/w - s/v) \sim uf(t/w) \to t^\beta$; therefore (12.4) converges in conditional distribution to $\int_0^\infty G(W_0(s) + y)\, ds$.

*Case* $0 < p < \infty$. Here $\alpha = \beta/2$, and, by (12.2), we may take $z = w = vp^{1/\alpha}$; therefore (12.4) converges in distribution to

$$p^{1/\alpha} \int_0^{tp^{-1/\alpha}} G(W_0(s) + y + t^\beta - (t - sp^{1/\alpha})^\beta)\, ds\,.$$

This is equivalent to the random variable in (12.3)

*Case* $p = \infty$. Here $z = w$, and $v/w \to 0$. Change the variable of integration in (12.4):

$$\int_0^t G(X^*(-sv/w) - uf((t - s)/w))\, ds\,.$$

For each $s$, $X^*(-sv/w)$ converges in conditional probability to $W_0(0) = 0$; thus, it can be shown that the conditional distribution of the above integral converges to the (degenerate) distribution of the constant $\int_0^t G(y + t^\beta - s^\beta)\, ds$. This completes the proof. □

In the case $p = 0$, the $\mathscr{B}$-function in (12.3) does not depend on $t$; here the function $F$ in (11.11) takes the form

$$F(A) = \frac{\int_{-\infty}^{\infty} P(\int_0^\infty G(W_0(s) + y)\, ds > A) G(y) e^{-y}\, dy}{\int_{-\infty}^{\infty} G(y) e^{-y}\, dy}\,.$$

This does not depend on $\beta$, the index of variation of $f$. This suggests that this case of the theorem is true for a larger class of functions $f$ satisfying $f/K^{\frac{1}{2}} \to 0$.

When $p = \infty$, $F$ can be put in the form

$$F(A) = \frac{\int_{-\infty}^{\infty} [\int_0^\infty G(y - t^\beta)\, dt - A]^+ e^{-y}\, dy}{\Gamma(1 + 1/\beta) \int_{-\infty}^{\infty} G(y) e^{-y}\, dy}\,.$$

This does not depend on $\alpha$, the index of variation of $K$. This suggests that this case of the theorem is true for a larger class of processes such that $f/K^{\frac{1}{2}} \to \infty$. When $G$ is the indicator of the positive axis, we obtain the simple form $F$ such that

$$-F'(x) = \frac{e^{-x^\beta}}{\Gamma(1 + 1/\beta)}\,.$$

Finally we remark that it can also be shown, as in Theorem 10.1, that $-F'(0)$ exists, is positive and finite, and

$$\lim_{u\to\infty} \frac{P(\max_{[0,1]} (X(t) - f(t)) > u)}{(v/w)\phi(u)/u} = -F'(0) \cdot \Gamma(1 + 1/\beta) \int_{-\infty}^{\infty} G(y) e^{-y}\, dy\,.$$

We point out the relation between the functions $F$ defined in this section and the one defined in [5], formulas (6.11) and (6.12). Let $F^*$ represent the latter. We consider only the case of finite $p$ and $G$ the indicator of the positive axis, which is the case treated in [5]. $F^*(x)$ is the limit of $\int_0^x P(vL > y)\,dy/[(v/w)\phi(u)/u]$. $F(x)$ is defined here as the limit of $\int_x^\infty P(vL > y)\,dy/E(vL)$. From Lemma 11.1 and the particular form of $G$ we get

$$E(vL) \sim \Gamma(1 + 1/\beta)(v/w)\phi(u)/u\,.$$

It follows that

$$F^*(x) = \Gamma(1 + 1/\beta)(F(0) - F(x))\,.$$

**13. Remarks on further extensions.** The hypothesis (8.3) was used throughout the calculations in Section 8. This prompts the question whether (8.3) can be replaced by other assumptions; for example, $K(s) \sim |\log s|^{-\alpha}$ for small $s$, for $\alpha > 0$. Assumptions 1–4 are valid for such $K$ under certain restrictions on $\alpha$; however, this is not sufficient for the proof of the limit theorems of the kind proved in Sections 8–12. For this choice of $K$, we find

$$q(t) \sim \text{constant} \cdot |\log t|^{-\alpha/2}\,, \qquad Q(t) \sim \text{constant} \cdot |\log t|^{(1-\alpha)/2}\,.$$

It is evident that Assumption 1 holds if and only if $\alpha > 1$. Put

$$v = \exp(u^m)\,. \tag{13.1}$$

It follows from the form of $Q$ and the definition (5.3) of $Q^*$ that Assumption 2 holds if and only if $m > 2/(\alpha - 1)$. From the calculation in the proof of Theorem 8.1 we see that $E(X^*(s)\,|\,X^*(0) = y) - y$ is uniformly asymptotic to a multiple of $-\frac{1}{2}(u^2 + y)|\log s - u^m|^{-\alpha}$; hence Assumption 3 holds if and only if $m > 2/\alpha$. Assumption 4 holds with any $\alpha > 0$ and $v$ of the above form but with $m \leqq 2/\alpha$. Assumption 5 does not involve the form of $K(s)$ for small $s$, and can be separately assumed.

In Section 8 $v$ is given in (8.8), and it is shown that Assumptions 2, 3 and 4 hold for *this* choice of $v$. When $K(s) \sim |\log s|^{-\alpha}$, the solution of (8.8) is (13.1) with $m = 2/\alpha$. Therefore Assumptions 2 and 3 are not satisfied for this $v$. Note also that the conditions on $m$ for Assumptions 2 and 3 contradict the condition for Assumption 4.

One of the main points in our calculation is that if $v$ is a solution of (8.8), then $L(t) = \lim_{u\to\infty} u^2K(t/v)$ exists and is continuous at $t = 0$. However, if $K$ is slowly varying, then $L(t) = 1$ for $t > 0$, and $L(0) = 0$. This suggests that the methods and results for such $K$ are very different from those in the case (8.3).

## REFERENCES

[1] BERMAN, S. M. (1971a). Excursions above high levels for stationary Gaussian processes. *Pacific J. Math.* **36** 63–80.

[2] BERMAN, S. M. (1971b). Maxima and high level excursions of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **160** 65–85.

[3] BERMAN, S. M. (1972a). A class of limiting distributions of high level excursions of Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 121–134.
[4] BERMAN, S. M. (1972b). Maximum and high level excursion of a Gaussian process with stationary increments. *Ann. Math. Statist.* **43** 1247–1266.
[5] BERMAN, S. M. (1973). Excursions of stationary Gaussian processes above high moving barriers. *Ann. Probability* **1** 365–387.
[6] BILLINGSLEY, P. (1968). Convergence of Probability Measures, Wiley, New York.
[7] DUDLEY, R. M. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Functional Analysis* **1** 290–330.
[8] FERNIQUE, X. (1964). Continuité des processes Gaussiens, *C. R. Acad. Sci. Paris* **258** 6058–6060.
[9] GARSIA, A. M., RODEMICH, E. and RUMSEY, H. (1970). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana J. Math.* **20** 565–578.
[10] KARAMATA, J. (1933). Sur un mode de croissance régulière. Théorèmes fondamentaux. *Bull. Soc. Math. France* **61** 55–62.
[11] MARCUS, M. B. (1970). A bound for the distribution of the maximum of continuous Gaussian processes. *Ann. Math. Statist.* **41** 305–309.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012