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UPPER AND LOWER FUNCTIONS FOR MARTINGALES AND MIXING PROCESSES[1]

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An almost sure invariance principle due to Strassen for partial sums $\{S_n\}$ of martingale differences $\{X_n\}$ is sharpened. This result is then used to establish integral tests which characterize the asymptotic growth rates of S_n and $M_n = \max_{1 \leqq i \leqq n} |S_i|$. If, in addition, $\{X_n\}$ is a stationary ergodic sequence, then integral tests are established for nonrandom normalizers of $\{S_n\}$. Improving a decomposition due to Gordin for mixing sequences, integral tests are established for mixing sequences and Doeblin processes. In the independent case, the results obtained compare favorably with similar classical results due to Feller and strengthen a classical result due to Chung.

1. Introduction. Let $\{X_n\}$ be a sequence of random variables on some probability space $(\Omega, \mathscr{F}, P)$ and let $S_n = \sum_{i=1}^n X_i$. The asymptotic growth rates of S_n and $M_n = \max_{1 \leqq i \leqq n} |S_i|$ have been studied in two fundamental papers by Feller [8] and Chung [5], respectively, when the sequence $\{X_n\}$ consists of independent, but not necessarily identically distributed, random variables. Feller's paper is an excellent source for the historical development of the problem concerning the asymptotic growth rate of S_n; Chung was the first to consider M_n. The basic approach (by no means easy) in the above papers was to first obtain sufficiently good probability estimates for the appropriate tail, and then to apply Borel–Cantelli arguments to solve the problem.

Strassen [26] uses an entirely different approach to attack these problems. He lets $\{X_n\}$ be a generalized martingale difference sequence (that is, $E[X_n \mid X_1, \cdots, X_{n-1}]$ is well defined—see Section 2 for an explanation of "well-defined"—and $E[X_n \mid X_1, \cdots, X_{n-1}] = 0$ a.s.) with $E[X_n^2 \mid X_1, \cdots, X_{n-1}]$ well defined. By using the Skorokhod technique he embeds S_n into a Brownian motion process and under suitable conditions on $\{X_n\}$ proves an almost sure invariance principle. By this latter phrase one means that if $\{S(t), t \geqq 0\}$ is the process obtained by interpolating S_n (in some suitable manner) then $S(t)$ is sufficiently close to a Brownian motion ξ_t almost surely for all sufficiently large t, so that an asymptotic property known for Brownian motion also holds for $S(t)$. The main advantage of this approach is that it is easier to establish asymptotic properties of

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Brownian motion than of S_n mainly because the appropriate tail probabilities can be computed exactly for Brownian motion.

When Strassen's [26] results are specialized to the independent case, one does not get results quite as strong as Feller's [8]. Our main motivation was to see if this gap could be closed. We have succeeded in closing this gap considerably by refining Strassen's techniques.

Feller's problem is to give an integral test for non-decreasing functions φ so that the integral converges or diverges according as $P[S_n > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}]$ is 0 or 1, where $V_n = \sum_{i=1}^n E[X_i^2 \mid X_1, \cdots, X_{i-1}]$. If this probability is 0 we say φ is an "upper function" for S_n. The corresponding result for Brownian motion is due to Kolmogorov (see Itô and McKean [14] page 163). We note that for a martingale difference sequence Stout [22] has proved an exact analogue of Kolmogorov's law of the iterated logarithm. Stout's approach is classical and the reason he has been able to establish this *exact* analogue is that relatively crude tail probability estimates suffice; whereas, in the more delicate problem of establishing an integral test, such as Feller's, one needs much sharper estimates, and as far as we know, no such sharp estimates are available in the case of martingale difference sequences.

Chung's problem is to find an integral test for non-decreasing φ such that the integral converges or diverges according as $P[M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1} \text{ i.o.}] = 0$ or 1. If this probability is 0, we say φ is a "lower function" for M_n. Chung's [5] results can be used to derive the corresponding results for Brownian motion which can then be used in finding such tests for M_n via the almost sure invariance principle. A simpler direct proof for Brownian motion concerning the asymptotic behavior of $M(t) = \max_{0 \leq u \leq t} |\xi_u|$ has been recently given by Jain and Taylor [16]. Combining this proof with an almost sure invariance principle we considerably sharpen and at the same time get a simpler proof of Chung's result. Our results are, of course, proved for generalized martingale difference sequences. We also mention the recent work of Jain and Pruitt [15] and comments by Breiman [3] on the asymptotic behavior of M_n.

Gordin [11] has shown how a stationary mixing sequence $\{X_n\}$ can be written as the sum of a martingale difference sequence and a "negligible" sequence. By proving a suitable form of Gordin's representation, we have been able to establish integral tests for certain stationary mixing sequences $\{X_n\}$. This technique is most suited to functionals of a "Doeblin process" (studied in Doob [7] Chapter 5).

Before describing the organization of this paper one more remark is in order. The $V_n \equiv \sum_{i=1}^n E[X_i^2 \mid X_1, \cdots, X_{i-1}]$ are of course random quantities when $\{X_n\}$ is a generalized martingale difference sequence. In the independent case the V_n's are simply constants. If $\{X_n\}$ is a *stationary* martingale difference sequence with finite second moment, then one should expect to be able to replace V_n by $nE[X_1^2]$ in various results. One can, indeed, do this so far as the law of the iterated logarithm is concerned; see, for example, Stout [23] and Basu [1]. For this problem we formulate an almost sure invariance principle in Section 4, but we have to make an additional assumption (4.2). The justification of this assumption is indicated by a counter-example in Section 7, which may be of independent interest. It is known that if $\{X_n\}$ is a sequence of independent identically distributed random variables with mean 0, variance 1, then $S_n/n^\delta \to 0$ a.s. for each $\delta > \frac{1}{2}$, while our example shows that for a stationary and ergodic sequence $\{X_n\}$ no such improvement of the strong law of large numbers is possible even for *uniformly bounded* X_n.

We give preliminary lemmas in Section 2. An almost sure invariance principle for generalized martingale difference sequences, which represents a refinement of Strassen's, is given in Section 3. Its analogue for the stationary case, replacing random normalizers by constants, is given in Section 4. Analogues of Feller's results are discussed in Section 5, those of Chung's results in Section 6. The counter-example mentioned above and its relevance, so far as condition (4.2) is concerned, are explained in Section 7. Applications to mixing sequences are discussed in Section 8.

2. Preliminaries. Certain notation and conventions are adopted throughout the paper. The symbol $\sigma(X_1, \cdots, X_n)$ will always stand for the σ-field generated by random variables $X_1, \cdots, X_n$.

The conditional expectation $E[X \mid \mathscr{G}]$ will be said to be well defined if the measure $\mu(A) = \int_A |X|\, dP$, $A \in \mathscr{G}$, is σ-finite.

In some circumstances, in order to discuss properties of given random variables, we may need a probability space "richer" than the underlying one. In such cases a phrase such as "if necessary, redefining the X_i's on a new probability space" will imply that the joint distributions of the X_i's are kept the same.

In statements like $S_n = o(n)$ a.s., $n \to \infty$, it is to be understood that $n^{-1}S_n \to 0$ a.s.

We will only be concerned with large values of t whenever $\log t$ and $\log \log t$ (written henceforth as $\log_2 t$) are involved. Hence, to avoid cumbersome expressions, we adopt the convention that

$$\log t = 1 \quad \text{for} \quad 0 < t \leqq e \qquad \text{and}$$
$$\log_2 t = 1 \quad \text{for} \quad 0 < t \leqq e^e .$$

The use of abbreviations "a.s." for "almost surely" and "i.o." for "infinitely often" is standard and will be made.

The relation $a_n \sim b_n$ means $a_n b_n^{-1} \to 1$ as $n \to \infty$.

The following lemmas will be used in subsequent sections.

LEMMA 2.1 (Kronecker). *Let x_n be real for $n \geqq 1$ and $0 < b_n \nearrow \infty$. Then $\sum_{i=1}^\infty (x_i/b_i)$ converges implies that $(\sum_{i=1}^n x_i)/b_n \to 0$ as $n \to \infty$.*

DEFINITION 2.1. Let ξ be a Brownian motion on $[0, \infty)$ and $0 \leqq a < b < \infty$. Then we define $R_\xi(a, b) = \max_{a \leqq u, v \leqq b} |\xi(u) - \xi(v)|$.

LEMMA 2.2. *The distribution of $R_\xi(a, b)$ is the same as that of $R_\xi(0, 1)(b - a)^{\frac{1}{2}}$.*

Moreover, there exists $c > 0$ *such that for all* $\lambda > 0$,

$$P[R_\xi(0, 1) > \lambda] \leqq \frac{c}{\lambda}\exp(-\lambda^2/2) .$$

Lemma 2.2 is given in [10].

The following lemma is an obvious analogue of Lemma 2.14 in Jain and Taylor [16]. Since the proof is also analogous, it is omitted. First we need

DEFINITION 2.2. Let $A > 0$. Then Φ_A is the class of nonnegative, non-decreasing functions defined on $[A, \infty]$ which increase to ∞ with their arguments.

LEMMA 2.3. *Let* g *be an eventually non-increasing function from* $[0, \infty)$ *to* $[0, \infty)$ *and* h *be a measurable function from* $[A, \infty)$ *to* $[0, \infty)$, *for some fixed* $A > 0$. *For* $\varphi \in \Phi_A$, *define*

$$F(\varphi) = \int_A^\infty g(\varphi(t))h(t)\,dt ,$$

which may be either finite or infinite. Assume that

(a_1) *for every* $\varphi \in \Phi_A$ *and for every* B *such that* $B > A > 0$, $\int_A^B g(\varphi(t))h(t)\,dt < \infty$.

(a_2) *There exist* φ_1, φ_2, *two members of* Φ_A, *such that* $\varphi_1 \leqq \varphi_2$, $F(\varphi_2) < \infty$, *while* $F(\varphi_1) = \infty$ *and*

$$\lim_{B\to\infty} g(\varphi_1(B)) \int_A^B h(t)\,dt = \infty .$$

Define $\hat\varphi = \min[\max(\varphi, \varphi_1), \varphi_2]$. *Then for* $\varphi \in \Phi_A$,

(b_1) $F(\varphi) < \infty$ *implies that* $\hat\varphi \leqq \varphi$ *near* ∞ *and* $F(\hat\varphi) < \infty$.

Conversely,

(b_2) $F(\varphi) = \infty$ *implies that* $F(\hat\varphi) = \infty$.

REMARK. Note that $\hat\varphi$ is trapped between φ_1 and φ_2. In proving upper and lower class results this lemma allows us to consider only those $\varphi \in \Phi_A$ which satisfy $\varphi_1 \leqq \varphi \leqq \varphi_2$.

3. An almost sure invariance principle. Throughout this section $\{X_n\}$ will denote a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ such that $E[X_n^2 \mid \mathscr{L}_{n-1}]$ is well defined and $E[X_n \mid \mathscr{L}_{n-1}] = 0$ a.s. for $n \geqq 1$, where we write $\mathscr{L}_n$ for $\sigma(X_1, \cdots, X_n)$, $\mathscr{L}_0$ = trivial σ-field $\{\Omega, \phi\}$. Let

$$S_n = \sum_{i=1}^n X_i , \qquad S_0 = 0$$

$$V_n = \sum_{i=1}^n E[X_i^2 \mid \mathscr{L}_{i-1}] , \qquad V_0 = 0 .$$

To avoid trivialities we assume $V_1 = EX_1^2 > 0$.

The following theorem is analogous to an almost sure invariance principle due to Strassen [26]. It allows us to obtain sharper results than those of Strassen.

THEOREM 3.1. *For a fixed* $\alpha \geqq 0$ *let*

$$f_\alpha(t) = t(\log_2 t)^{-\alpha} , \qquad t > 0 .$$

Suppose the following conditions hold a.s.:

$$V_n \to \infty \qquad \text{as } n \to \infty , \tag{3.1}$$

and for all $\delta > 0$

$$\lim_{n\to\infty} (f_\alpha(V_n))^{-1} \sum_{k=1}^{n} E\{X_k^2 I[X_k^2 \geqq \delta f_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} = 0 \tag{3.2}$$

$$\sum_{k=1}^{\infty} (f_\alpha(V_k))^{-\frac{1}{2}} E\{|X_k| I[X_k^2 \geqq \delta f_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} < \infty \tag{3.3}$$

$$\sum_{k=1}^{\infty} (f_\alpha(V_k))^{-2} E\{X_k^4 I[X_k^2 \leqq \delta f_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} < \infty . \tag{3.4}$$

Let S be the random function defined on $[0, \infty)$, *obtained by setting* $S(t) = S_n$ *for* $t \in [V_n, V_{n+1})$. *Then, redefining* $\{S(t), t \geqq 0\}$, *if necessary, on a new probability space, there exists a Brownian motion* ξ *such that*

$$|S(t) - \xi(t)| = o(t^{\frac{1}{2}}(\log_2 t)^{(1-\alpha)/2}) \quad \text{a.s.,} \tag{3.5}$$

as $t \to \infty$.

Before proving this theorem, it will be convenient to formulate a lemma.

Let f be a positive function defined on $[0, \infty)$ such that $f(t) \nearrow \infty$ as $t \nearrow \infty$. Following Strassen [26] let

$$\begin{aligned} \tilde{X}_n &= X_n && \text{if} \quad X_n^2 \leqq f(V_n) \\ &= \operatorname{sgn} X_n \rho_{f(V_n)}(|X_n|) && \text{if} \quad X_n^2 > f(V_n) \end{aligned} \tag{3.6}$$

for $n \geqq 1$, where $\rho_r(x) = 2r^{\frac{1}{2}} - r/x$ for $r \geqq 0$ and $x > r^{\frac{1}{2}}$. Note that $\rho_r(x) \leqq x$. Let

$$Y_n = \tilde{X}_n - E[\tilde{X}_n \mid \mathscr{L}_{n-1}] . \tag{3.7}$$

The above truncation guarantees that $\sigma(Y_1, \cdots, Y_n) = \mathscr{L}_n$. Moreover,

$$|\tilde{X}_n| \leqq 2(f(V_n))^{\frac{1}{2}} \qquad \text{and} \qquad |Y_n| \leqq 4(f(V_n))^{\frac{1}{2}} , \tag{3.8}$$

so that $E[\tilde{X}_n^{2j} \mid \mathscr{L}_{n-1}]$ and $E[Y_n^{2j} \mid \mathscr{L}_{n-1}]$ are well defined for each $j \geqq 1$ and $n \geqq 1$ since V_n is $\mathscr{L}_{n-1}$ measurable for $n \geqq 1$. From (3.7), it is clear that

$$E[Y_n \mid \mathscr{L}_{n-1}] = 0 \quad \text{a.s.} \tag{3.9}$$

We apply Theorem 4.3 of Strassen [26] to the sequence $\{Y_i, i \geqq 1\}$, (redefined on a richer probability space if necessary) so that there is a Brownian motion $\{\xi(t), t \geqq 0\}$ together with a sequence of nonnegative random variables $\{T_i, i \geqq 1\}$ such that

$$\sum_{i=1}^{n} Y_i = \xi(\textstyle\sum_{i=1}^{n} T_i) \quad \text{a.s.} \qquad \text{for} \quad n \geqq 1 . \tag{3.10}$$

Let $\mathscr{U}_n$ be the σ-field generated by $\xi(t)$ for $0 \leqq t \leqq \sum_{i=1}^{n} T_i$ *and* $Y_1, \cdots, Y_n$. Let $\mathscr{U}_0$ be (Ω, ϕ), the trivial σ-field. Then for each $n \geqq 1$, T_n is $\mathscr{U}_n$ measurable, $E[T_n \mid \mathscr{U}_{n-1}]$ is well defined and for $n \geqq 1$,

$$\begin{aligned} E[T_n \mid \mathscr{U}_{n-1}] &= E[T_n \mid \mathscr{L}_{n-1}] \\ &= E[Y_n^2 \mid \mathscr{U}_{n-1}] = E[Y_n^2 \mid \mathscr{L}_{n-1}] \quad \text{a.s.} \end{aligned} \tag{3.11}$$

If $j > 1$, then

$$E[T_n^j \mid \mathscr{U}_{n-1}] \leqq c_j E[Y_n^{2j} \mid \mathscr{L}_{n-1}] \quad \text{a.s.,} \tag{3.12}$$

where the constants c_j depend only on j. Let

$$U_n = \sum_{i=1}^n T_i \quad \text{for } n \geqq 1, \tag{3.13}$$

and

$$\begin{aligned} W_n &= \sum_{i=1}^n E[Y_i^2 \mid \mathscr{L}_{i-1}] \\ &= \sum_{i=1}^n E[T_i \mid \mathscr{L}_{i-1}]. \end{aligned} \tag{3.14}$$

LEMMA 3.1. *For a fixed* $\alpha \geqq 0$, *let* $f_\alpha = f$ *be as in Theorem* 3.1. *If* (3.1)—(3.4) *hold, then we have*

$$|W_n - V_n| = o(f(V_n)) \tag{3.15}$$

$$|\sum_{i=1}^n Y_i - \sum_{i=1}^n X_i| = o((f(V_n))^{\frac{1}{2}}) \tag{3.16}$$

$$|U_n - V_n| = o(f(V_n)) \tag{3.17}$$

and

$$|V_n - V_{n+1}| = o(f(V_n)) \tag{3.18}$$

a.s. *as* $n \to \infty$.

PROOF. According to relation (147) of Strassen [26], for $i \geqq 2$,

$$|E[Y_i^2 \mid \mathscr{L}_{i-1}] - E[X_i^2 \mid \mathscr{L}_{i-1}]| \leqq 6E[X_i^2 I(X_i^2 > f(V_i)) \mid \mathscr{L}_{i-1}] \quad \text{a.s.} \tag{3.19}$$

This together with the assumption (3.2) gives (3.15). Again from relations (152) and (154) of Strassen [26] we get

$$P[\tilde{X}_n \neq X_n \quad \text{i.o.}] = 0 \tag{3.20}$$

and

$$|\sum_{i=1}^n Y_i - \sum_{i=1}^n \tilde{X}_i| \leqq 2 \sum_{i=1}^n E[|X_i| I(X_i^2 > f(V_i)) \mid \mathscr{L}_{i-1}] \quad \text{a.s.}$$

Condition (3.3) and Kronecker's lemma show that the right side of the preceding inequality is $o((f(V_n))^{\frac{1}{2}})$. Thus

$$|\sum_{i=1}^n Y_i - \sum_{i=1}^n \tilde{X}_i| = o((f(V_n))^{\frac{1}{2}}) \quad \text{a.s.} \tag{3.21}$$

Now (3.16) follows from (3.20), (3.21) and the fact that $f(V_n) \nearrow \infty$ a.s.

In order to prove (3.17) and (3.18), observe that the definition of Y_i and (3.12) yield

$$\begin{aligned} E[T_i^2 \mid \mathscr{U}_{i-1}] &\leqq c_2 E[Y_i^4 \mid \mathscr{L}_{i-1}] \leqq 16 c_2 E[\tilde{X}_i^4 \mid \mathscr{L}_{i-1}] \\ &= 16 c_2 \{E[\tilde{X}_i^4 I(X_i^2 \leqq f(V_i)) \mid \mathscr{L}_{i-1}] + E[\tilde{X}_i^4 I(X_i^2 > f(V_i)) \mid \mathscr{L}_{i-1}]\} \\ &\leqq 16 c_2 \{E[X_i^4 I(X_i^2 \leqq f(V_i)) \mid \mathscr{L}_{i-1}] \\ &\qquad + 8(f(V_i))^{\frac{3}{2}} E[|X_i| I(X_i^2 > f(V_i)) \mid \mathscr{L}_{i-1}]\}, \end{aligned}$$

by noting that V_i is $\mathscr{L}_{i-1}$ measurable, and $|\tilde{X}_i| \leqq \min[|X_i|, 2(f(V_i))^{\frac{1}{2}}]$. This together with conditions (3.3) and (3.4) gives

$$\sum_{i=1}^\infty f^{-2}(V_i) E[T_i^2 \mid \mathscr{U}_{i-1}] < \infty \quad \text{a.s.} \tag{3.22}$$

Now from a standard result in martingale theory ([7] page 320), it follows that

$$\sum_{i=1}^{n} \{T_i - E[T_i \mid \mathscr{U}_{i-1}]\}/f(V_i) \quad \text{converges a.s.} \tag{3.23}$$

as $n \to \infty$.

Combining (3.11) and (3.23) yields the a.s. convergence of $\sum_{i=1}^{\infty} \{T_i - E[T_i \mid \mathscr{L}_{i-1}]\}/f(V_i)$, which via the Kronecker lemma (Lemma 2.1) gives

$$|U_n - W_n| = o(f(V_n)) \quad \text{a.s.}$$

This together with (3.15) yields the desired relation (3.17). To prove (3.18) note that for $\delta > 0$

$$\begin{aligned} V_n - V_{n-1} = E[X_n^2 \mid \mathscr{L}_{n-1}] &= E[X_n^2 I(X_n^2 > \delta f(V_n)) \mid \mathscr{L}_{n-1}] \\ &\quad + E[X_n^2 I(X_n^2 \leqq \delta f(V_n) \mid \mathscr{L}_{n-1}] \\ &\leqq o(f(V_n)) + \delta f(V_n)\,, \end{aligned} \tag{3.24}$$

by (3.2). Since $\delta > 0$ is arbitrary, we have (3.18) by observing that this implies $V_{n+1}/V_n \to 1$ a.s., as $n \to \infty$, and the lemma is proved.

PROOF OF THEOREM 3.1. Since $\alpha \geqq 0$ is fixed throughout, we drop the subscripts from f_α. For $0 < \delta < 1$, define

$$p_j = e^{\delta j/(\log j)^\alpha} \tag{3.25}$$

and

$$n_j = \inf\{n\colon V_n \geqq p_j\}\,, \qquad j \geqq 2\,. \tag{3.26}$$

Given $\varepsilon > 0$, for all j sufficiently large (depending on $\omega \in \Omega$ and ε), we have

$$\begin{aligned} &\sup_{p_j \leqq t \leqq p_{j+1}} |S(t) - \xi(t)| \\ &\quad \leqq \sup_{V_{n_j-1} \leqq t \leqq V_{n_{j+1}}} |S(t) - \xi(t)| \\ &\quad \leqq \sup_{n_j-1 \leqq n \leqq n_{j+1}} |\textstyle\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i| \\ &\qquad + R_\xi(V_{n_j-1} - \varepsilon f(V_{n_{j+1}}), V_{n_{j+1}} + \varepsilon f(V_{n_{j+1}})) \quad \text{a.s.} \end{aligned} \tag{3.27}$$

where R is as in Definition 2.1. The last inequality follows from (3.17) and the facts that $S(t) = S_n$ on $[V_n, V_{n+1})$ for $t \geqq 0$, $\sum_{i=1}^{n} Y_i = \xi(U_n)$. Using (3.16), it follows that for all j sufficiently large (depending on ω),

$$\begin{aligned} &\sup_{p_j \leqq t \leqq p_{j+1}} |S(t) - \xi(t)| \\ &\quad \leqq o((f(V_{n_{j+1}}))^{\frac{1}{2}}) \\ &\qquad + R_\xi(V_{n_j-1} - \varepsilon f(V_{n_{j+1}}), V_{n_{j+1}} + \varepsilon f(V_{n_{j+1}})) \quad \text{a.s.} \end{aligned} \tag{3.28}$$

Elementary computations, using the definitions of f, p_j, n_j, and (3.18), show that the interval involved in R_ξ in (3.28) is contained in the interval $[p_{j-1}, p_{j+2}]$ a.s. for all sufficiently large j. Using this information in (3.28) we get

$$\sup_{p_j \leqq t \leqq p_{j+1}} |S(t) - \xi(t)| \leqq o((f(p_j))^{\frac{1}{2}}) + R_\xi(p_{j-1}, p_{j+2}) \quad \text{a.s.} \tag{3.29}$$

for all j sufficiently large.

From Lemma 2.2 it follows that for any $\gamma > 0$

$$P[R_\xi(p_{j-1}, p_{j+2}) > \gamma p_j^{\frac{1}{2}}(\log_2 p_j)^{(1-\alpha)/2}] = P\left[R_\xi(0, 1) > \gamma\left\{\frac{p_j(\log_2 p_j)^{1-\alpha}}{p_{j+2} - p_{j-1}}\right\}^{\frac{1}{2}}\right].$$

For all j sufficiently large and δ sufficiently small we have $(p_{j+2} - p_{j-1})p_j^{-1} \leqq 4\delta \log_2^{-\alpha} p_j$, hence for such δ and j we have

$$\begin{aligned} P[R_\xi(p_{j-1}, p_{j+2}) &> \gamma p_j^{\frac{1}{2}}(\log_2 p_j)^{(1-\alpha)/2} \\ &\leqq P\left[R_\xi(0, 1) > \frac{\gamma}{(4\delta)^{\frac{1}{2}}}\{\log_2 p_j\}^{\frac{1}{2}}\right] \\ &\leqq \frac{c}{2\{\log_2 p_j\}^{\frac{1}{2}}} \exp\{-2\log_2 p_j\} \leqq c\frac{(\log j)^{2\alpha}}{j^2}, \end{aligned}$$

by picking δ so that $\gamma^2/8\delta \geqq 2$. Thus for every $\gamma > 0$

$$\sum_{j=1}^\infty P[R_\xi(p_{j-1}, p_{j+2}) > \gamma p_j^{\frac{1}{2}}(\log_2 p_j)^{(1-\alpha)/2}] < \infty .$$

Hence, by the Borel–Cantelli lemma, almost surely,

$$R_\xi(p_{j-1}, p_{j+2}) \leqq \gamma p_j^{\frac{1}{2}}(\log_2 p_j)^{(1-\alpha)/2} \tag{3.30}$$

eventually. The assertion of the theorem now follows from (3.29) and (3.30) since $\gamma > 0$ can be chosen as small as we please.

It is worthwhile to note the following theorem which is a corollary of Theorem 3.1. It is simpler to state and almost as useful as Theorem 3.1. However, conditions (3.2)—(3.4) are generally weaker than the condition (3.31) below.

THEOREM 3.2. *For a fixed $\alpha \geqq 0$, let f_α be as in Theorem* 3.1. *Assume* (3.1) *and*

$$\sum_{k=1}^\infty \frac{(\log_2 V_k)^\alpha}{V_k} E\left\{X_k^2 I\left[X_k^2 > \frac{V_k}{\log V_k(\log_2 V_k)^{2(\alpha+1)}}\right]\middle|\mathscr{L}_{k-1}\right\} < \infty \quad \text{a.s.} \tag{3.31}$$

Let S be the random function defined on $[0, \infty)$, obtained by setting $S(t) = S_n$ for $t \in [V_n, V_{n+1})$. Then, redefining $\{S(t), t \geqq 0\}$, if necessary, on a new probability space, there exists a Brownian motion ξ such that (3.5) *holds.*

PROOF. Note that (3.31) implies

$$\sum_{k=1}^\infty (f_\alpha(V_k))^{-1} E\{X_k^2 I(X_k^2 > \delta f_\alpha(V_k)) \mid \mathscr{L}_{k-1}\} < \infty \tag{3.32}$$

a.s. for all $\delta > 0$. This implies (3.2) by Kronecker's lemma and it is obvious that it also implies (3.3). Now we will deduce (3.4) as a consequence of (3.31). For convenience we write

$$g_\alpha(V_k) = \frac{V_k}{\log V_k(\log_2 V_k)^{2(\alpha+1)}} .$$

Now

$$\begin{aligned} &E\{X_k^4 I[X_k^2 \leqq \delta f_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} \\ &\quad = E\{X_k^4 I[X_k^2 \leqq g_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} + E\{X_k^4 I[g_\alpha(V_k) < X_k^2 \leqq \delta f_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} \\ &\quad \leqq g_\alpha(V_k) E\{X_k^2 I[X_k^2 \leqq g_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} + \delta f_\alpha(V_k) E\{X_k^2 I[X_k^2 > g_\alpha(V_k)] \mid \mathscr{L}_{k-1}\} . \end{aligned}$$

Multiplying by $(f_\alpha(V_k))^{-2}$ and summing on k, we see that the expression in (3.4) is dominated by

$$\sum_{k=1}^{\infty} \frac{V_k - V_{k-1}}{V_k(\log V_k)(\log_2 V_k)^2} + \delta \sum_{k=1}^{\infty} (f_\alpha(V_k))^{-1} E\{X_k^2 I[X_k^2 > g_\alpha(V_k)] \mid \mathscr{L}_{k-1}\}.$$

The first term here is clearly a.s. convergent. Hence (3.31) implies (3.4).

4. An almost sure invariance principle when the summands are stationary. Throughout this section $\{X_n\}$ will denote a stationary ergodic sequence of martingale differences ($E[X_n \mid \mathscr{L}_{n-1}] = 0$ a.s. for each $n \geqq 2$). It will be assumed that $EX_1^2 = 1$. As before V_n will denote $\sum_{i=1}^n E[X_i^2 \mid \mathscr{L}_{i-1}]$ and $S_n = \sum_{i=1}^n X_i$. We write $\mathscr{L}_0 = \{\phi, \Omega\}$ and $\mathscr{L}_n = \sigma(X_1, \cdots, X_n)$, $n \geqq 1$.

THEOREM 4.1. *For a fixed $\alpha \geqq 0$, let*

$$f_\alpha(t) = t(\log_2 t)^{-\alpha} \qquad \textit{for } t > 0. \tag{4.1}$$

Assume that

$$|V_n - n| = o(f_\alpha(n)) \quad \text{a.s.}, \tag{4.2}$$

$$\lim_{n\to\infty} (f_\alpha(n))^{-1} \sum_{k=1}^n E\{X_k^2 I[X_k^2 \geqq \delta f_\alpha(k)] \mid \mathscr{L}_{k-1}\} = 0 \quad \text{a.s.} \tag{4.3}$$

for all $\delta > 0$, and

$$E\{X_1^2(\log_2 X_1^2)^\alpha\} < \infty. \tag{4.4}$$

Let S be the random function defined by $S_0 = 0$ and $S(t) = S_n$ for $t \in [n, n+1)$, $n \geqq 0$. Then upon redefining $\{S_t, t \geqq 0\}$ on a new probability space, if necessary, there exists a Brownian motion ξ such that

$$|S(t) - \xi(t)| = o(t^{\frac{1}{2}}(\log_2 t)^{(1-\alpha)/2}) \quad \text{a.s.}, \tag{4.5}$$

a.s. *as $t \to \infty$.*

REMARK. It is important to point out the difference between Theorems 3.1 and 4.1. In Theorem 3.1 the interpolation of S_t is done at random times V_n, whereas in Theorem 4.1 it is done at n. The additional condition (4.2) appears to be rather indispensable as we explain later in Section 7. Without such an additional restriction it seems that all one can get is random normalizers as in the nonstationary case, that is

$$|S_n - \xi(V_n)| = o(V_n^{\frac{1}{2}}(\log_2 V_n)^{(1-\alpha)/2}) \quad \text{a.s.}$$

as $n \to \infty$, instead of the preferable

$$|S_n - \xi(n)| = o(n^{\frac{1}{2}}(\log_2 n)^{(1-\alpha)/2}) \quad \text{a.s.}$$

For the special case of independent, identically distributed X_i's, we have $V_n = n$. Hence (4.2) is automatically satisfied. It is also easy to check that (4.4) implies (4.3). We thus get

THEOREM 4.2. *Let $\{X_i, i \geqq 1\}$ be a sequence of independent, identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$. Let $S_n = X_1 + \cdots + X_n$. Assume that for a fixed $\alpha \geqq 0$,*

$$E\{X_1^2(\log_2 X_1^2)^\alpha\} < \infty . \tag{4.6}$$

Let S be the random function defined by $S_0 = 0$ and $S(t) = S_n$ for $t \in [n, n+1)$, $n \geqq 0$. Then upon redefining $\{S(t), t \geqq 0\}$ on a new probability space, if necessary, there exists a Brownian motion ξ such that

$$|S(t) - \xi(t)| = o(t^{\frac{1}{2}}(\log_2 t)^{(1-\alpha)/2}) \tag{4.7}$$

a.s. *as $t \to \infty$.*

For $\alpha = 0$, this was proved by Strassen [25] and for $\alpha = 2$ by Breiman [3].

We also note that in general both conditions (4.3) and (4.4) are implied by the condition (4.3′) below. Hence we get

THEOREM 4.3. *Let $\{X_i, i \geqq 1\}$ be a stationary ergodic martingale difference sequence. For $\alpha \geqq 0$, let*

$$f_\alpha(t) = t(\log_2 t)^{-\alpha}, \qquad t > 0$$

and suppose the following conditions are satisfied:

$$|V_n - n| = o(f_\alpha(n)) \tag{4.2}$$

and

$$E\{X_1^2(\log X_1^2)(\log_2 X_1^2)^\alpha\} < \infty . \tag{4.3′}$$

Then the conclusion of Theorem 4.1 holds.

The proof of Theorem 4.1 follows essentially along the lines of the proof of Theorem 3.1, and we only give the necessary modifications.

Using $f(n)$ in place of $f(V_n)$ in the truncation (3.6), let $\tilde{X}_n$ be redefined. The following is an analogue of Lemma 3.1.

LEMMA 4.1. *Let $Y_n = \tilde{X}_n - E[\tilde{X}_n \mid \mathscr{L}_{n-1}]$, where $\mathscr{L}_{n-1}$ is $\sigma(X_1, \cdots, X_{n-1})$, T_i, $i \geqq 1$ be as in* (3.10) *and $U_n = \sum_{i=1}^n T_i$. Then* (4.2), (4.3) *and* (4.4) *together imply that* a.s.

$$|U_n - n| = o(f_\alpha(n)) \tag{4.8}$$

and

$$|\textstyle\sum_{i=1}^n X_i - \sum_{i=1}^n Y_i| = o((f_\alpha(n))^{\frac{1}{2}}) . \tag{4.9}$$

PROOF. Since α is fixed, we write $f = f_\alpha$. We will first show that (4.4) implies that a.s.

$$\textstyle\sum_{k=1}^\infty (f(k))^{-\frac{1}{2}} E\{|X_k| I[X_k^2 \geqq \delta f(k)] \mid \mathscr{L}_{k-1}\} < \infty \tag{4.10}$$

and

$$\textstyle\sum_{k=1}^\infty (f(k))^{-2} E\{X_k^4 I[X_k^2 \leqq \delta f(k)] \mid \mathscr{L}_{k-1}\} < \infty \tag{4.11}$$

for all $\delta > 0$. In fact, (4.4) implies more, i.e.

$$\sum_{k=1}^{\infty} (f(k))^{-\frac{1}{2}} E\{|X_k| I[X_k^2 \geqq \delta f(k)]\} < \infty \tag{4.12}$$

and

$$\sum_{k=1}^{\infty} (f(k))^{-2} E\{X_k^4 I[X_k^2 \leqq \delta f(k)]\} < \infty \tag{4.13}$$

for all $\delta > 0$. We will derive (4.12); (4.13) follows by a similar argument.

Writing $[x]$ for the greatest integer $\leqq x$, we see that the expression in (4.12) is dominated by

$$\sum_{k=1}^{\infty} (f(k))^{-\frac{1}{2}} \sum_{j=[(\delta f(k))^{\frac{1}{2}}]}^{\infty} \int_{\{j \leq |X_1| < j+1\}} |X_1|\, dP\,. \tag{4.14}$$

Changing the order of summation, we can dominate this expression by

$$\sum_{j=1}^{\infty} \int_{\{j \leq |X_1| < j+1\}} |X_1|\, dP \sum_{k=1}^{\psi(j)} (f(k))^{-\frac{1}{2}}\,,$$

where $\psi(j) = 2\delta^{-1}j^2(\log_2 j)^\alpha$, which is roughly twice the inverse function of $(\delta f(k))^{\frac{1}{2}}$ evaluated at j. Hence we get a bound

$$\text{const.} \sum_{j=1}^{\infty} j(\log_2 j)^\alpha \int_{\{j \leq |X_1| < j+1\}} |X_1|\, dP\,,$$

which, in turn, is dominated by

$$\text{const.} \sum_{j=1}^{\infty} \int_{\{j \leq |X_1| < j+1\}} X_1^2(\log_2 X_1^2)^\alpha\, dP < \infty\,.$$

Now, (4.3), (4.10) and (4.11) play the same role here as (3.2), (3.3) and (3.4) in the proof of Lemma 3.1. The fact that $V_n \to \infty$ a.s. follows from (4.2). Hence we conclude along the lines of the proof of Lemma 3.1 that

$$\left|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i\right| = o((f(n))^{\frac{1}{2}})$$

and

$$|U_n - V_n| = o(f(n))\,.$$

Combining the assumption (4.2) with this last conclusion, we get

$$|U_n - n| = o(f(n))$$

and the lemma is established.

Proof of Theorem 4.1. Note that

$$\begin{aligned} \sup_{p_j \leq t \leq p_{j+1}} |S(t) - \xi(t)| &\leqq \sup_{p_{j-1} \leq n \leq p_{j+1}} \left|S_n - \sum_{i=1}^{n} Y_i\right| \\ &\quad + \sup_{p_j \leq t \leq p_{j+1},\, p_{j-1} \leq n \leq p_{j+1}+1} |\xi(U_n) - \xi(t)|\,, \end{aligned} \tag{4.15}$$

where, as before, for $0 < \delta < 1$ we define

$$p_j = e^{\delta j/(\log j)^\alpha}\,.$$

Using (4.8) in the above, we get for $\varepsilon > 0$ and all j sufficiently large

$$\begin{aligned} &\sup_{p_j \leq t \leq p_{j+1}} |S(t) - \xi(t)| \\ &\quad \leqq \sup_{p_{j-1} \leq n \leq p_{j+1}+1} \left|S_n - \sum_{i=1}^{n} Y_i\right| + R_\xi(p_j - \varepsilon f(p_{j+1}), p_{j+1} + \varepsilon f(p_{j+1}))\,. \end{aligned}$$

The remainder of the proof is identical to the corresponding portion of the proof of Theorem 3.1 and hence is omitted.

5. Integral test for upper functions for martingales. In this section $\{X_n\}$ will denote a sequence of random variables with $E[X_n^2 \mid \mathscr{L}_{n-1}]$ well defined and $E[X_n \mid \mathscr{L}_{n-1}] = 0$ a.s. for $n \geqq 1$, where $\mathscr{L}_n = \sigma(X_1, \cdots, X_n)$, $\mathscr{L}_0 = \{\Omega, \phi\}$. Again, as in Section 3,

$$S_n = \sum_{i=1}^n X_i\,, \qquad S_0 = 0$$
$$V_n = \sum_{i=1}^n E[X_i^2 \mid \mathscr{L}_{i-1}]\,, \qquad V_0 = 0\,,$$

and we assume $V_1 = EX_1^2 > 0$.

The almost sure invariance principle of Section 3 will now be used to prove an integral test for upper functions of S_n.

THEOREM 5.1. *Suppose* $V_n \to \infty$ a.s. *and*

$$\sum_{n=1}^\infty V_n^{-1}(\log_2 V_n)^2 \times E[X_n^2 I(X_n^2 > V_n(\log V_n)^{-1}(\log_2 V_n)^{-6}) \mid \mathscr{L}_{n-1}] < \infty \quad \text{a.s.} \tag{5.1}$$

Let $\varphi > 0$ *be a non-decreasing function. Then*

$$P[S_n > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}] = 0 \quad \textit{or} \quad 1\,, \tag{5.2}$$

according as

$$I(\varphi) = \int_1^\infty \frac{\varphi(t)}{t} \exp(-\varphi^2(t)/2)\, dt < \infty \quad \textit{or} \quad = \infty\,. \tag{5.3}$$

REMARK. This should be compared with Corollary 4.5 of Strassen [26].

PROOF OF THEOREM 5.1. We first establish the result for functions φ such that

$$(\log_2 t)^{\frac{1}{2}} \leqq \varphi(t) \leqq 2(\log_2 t)^{\frac{1}{2}}\,, \tag{5.4}$$

for all t sufficiently large. We will then remove this restriction.

If $I(\varphi) < \infty$, then by Kolmogorov's test for Brownian motion ([14] page 163), we have for any $\beta > 0$,

$$P[\xi(t) > t^{\frac{1}{2}}(\varphi(t) - \beta/\varphi(t)) \text{ i.o. as } t \to \infty] = 0\,, \tag{5.5}$$

since $\varphi(t) - \beta/\varphi(t) \nearrow$ and $I(\varphi - \beta/\varphi) < \infty$. The conditions of Theorem 3.2 are satisfied by taking $\alpha = 2$. Using (3.5) in conjunction with (5.4) and (5.5) we obtain

$$P[S(t) > t^{\frac{1}{2}}\varphi(t) \text{ i.o. as } t \to \infty] = 0\,.$$

This clearly implies that

$$P[S_n > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}] = 0\,,$$

as desired. On the other hand, if $I(\varphi) = \infty$, then $I(\varphi + \beta/\varphi) = \infty$, for every $\beta > 0$. Further, $\varphi(t) + \beta/\varphi(t)$ is increasing for sufficiently large t, and similar analysis shows

$$P[S(t) > t^{\frac{1}{2}}\varphi(t) \text{ i.o. as } t \to \infty] = 1\,,$$

which in turn implies that

$$P[S_n > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}] = 1 .$$

It remains to show that (5.4) may be assumed without loss of generality. Let $\varphi_1(t) = (\log_2 t)^{\frac{1}{2}}$ and $\varphi_2(t) = 2\varphi_1(t)$, for $t > 0$. Let $\varphi \nearrow$ be an arbitrary nonnegative function and

$$\hat{\varphi}(t) = \min [\max (\varphi, \varphi_1), \varphi_2] .$$

The conditions of Lemma 2.3 are satisfied and by that lemma, $I(\varphi) < \infty$ implies $I(\hat{\varphi}) < \infty$ and $\hat{\varphi} \leqq \varphi$ near ∞. Since $\hat{\varphi}$ satisfies (5.4), we conclude that $P[S_n > V_n^{\frac{1}{2}}\hat{\varphi}(V_n) \text{ i.o.}] = 0$. Since $\hat{\varphi} \leqq \varphi$ near ∞, φ also satisfies this relation.

On the other hand suppose $I(\varphi) = \infty$. Then by Lemma 2.3, $I(\hat{\varphi}) = \infty$, and it follows that $P[S_n > V_n^{\frac{1}{2}}\hat{\varphi}(V_n) \text{ i.o.}] = 1$. Hence there exists a sequence $n_k \uparrow \infty$, depending on $\omega \in \Omega$, such that $S_{n_k} > V_{n_k}^{\frac{1}{2}}\hat{\varphi}(V_{n_k})$, for every positive integer k. Since $I(\varphi_2) < \infty$, we have $P[S_n > V_n^{\frac{1}{2}}\varphi_2(V_n) \text{ i.o.}] = 0$. Hence

$$\hat{\varphi}(V_{n_k}) < \varphi_2(V_{n_k}) \quad \text{a.s.} \tag{5.6}$$

for all sufficiently large k. Thus, from the definition of $\hat{\varphi}$, the inequality (5.6) implies that $\varphi(V_{n_k}) \leqq \hat{\varphi}(V_{n_k})$ a.s., for all sufficiently large k, and hence almost surely $S_{n_k} > V_{n_k}^{\frac{1}{2}}\varphi(V_{n_k})$ eventually also. This completes the proof.

When $\{X_n\}$ is a sequence of independent random variables, $V_n = \sum_{i=1}^n EX_i^2$, are constants. Condition (5.1), in this special case, becomes

$$\sum_{n=1}^\infty V_n^{-1}(\log_2 V_n)^2 E[X_n^2 I(X_n^2 > V_n(\log V_n)^{-1}(\log_2 V_n)^{-6})] < \infty .$$

It is interesting to compare this condition with that of Feller ([8] page 399):

$$\sum_{n=1}^\infty V_n^{-1}(\log_2 V_n)^3 E[X_n^2 I(X_n^2 > V_n(\log_2 V_n)^{-3})] < \infty .$$

COROLLARY 5.1. *Let $\{X_n\}$ be a sequence of independent random variables with $EX_n = 0$, and $V_n = \sum_{i=1}^n EX_i^2$ be such that $V_n \nearrow \infty$. Suppose there exist constants $\delta > 0$ and $\beta > 0$ such that for every $n \geqq 1$*

$$\int_1^\infty x^2(\log x)^{1+\delta}\, dP\,[|X_n| \leqq x] < \beta EX_n^2 .$$

Then the conclusion of Theorem 5.1 *holds.*

This result was obtained by Feller ([8] page 401) by applying his condition mentioned above.

Let $\{X_n\}$ be a martingale difference sequence which is stationary and ergodic with $EX_1^2 = 1$. We formulate a result similar to Theorem 5.1.

THEOREM 5.2. *Suppose the conditions* (4.2)—(4.4) *of Theorem* 4.1 *hold with $\alpha = 2$, i.e.*

$$\left|\sum_{i=1}^n E[X_i^2 \mid \mathscr{L}_{i-1}] - n\right| = o(n/(\log_2 n)^2) \quad \text{a.s.} \tag{5.7}$$

$$\lim_{n\to\infty} \frac{(\log_2 n)^2}{n} \sum_{k=1}^n E\{X_k^2 I[X_k^2 \geqq \delta k(\log_2 k)^{-2}] \mid \mathscr{L}_{k-1}\} = 0 \quad \text{a.s.} \tag{5.8}$$

for all $\delta > 0$, *and*

(5.9) $$E\{X_1^2(\log_2 X_1^2)^2\} < \infty .$$

Then for any $0 < \varphi \nearrow$

(5.10) $$P[S_n > n^{\frac{1}{2}}\varphi(n) \text{ i.o.}] = 0 \quad or \quad 1$$

according as

(5.11) $$I(\varphi) = \int_1^\infty \frac{\varphi(t)}{t} e^{-\varphi^2(t)/2}\, dt < \infty \quad or \quad = \infty .$$

The proof of Theorem 5.2 is identical to that of Theorem 5.1, except that (4.5) of Theorem 4.1 takes the place of (3.5) of Theorem 3.1. In the independent, identically distributed case, Theorem 4.2 leads to

COROLLARY 5.2. *Let* $\{X_n\}$ *be a sequence of independent identically distributed random variables with* $EX_1 = 0$, $EX_1^2 = 1$ *and*

(5.12) $$E\{X_1^2(\log_2 X_1^2)^2\} < \infty .$$

Then the conclusion of Theorem 5.2 *holds.*

REMARK 1. A weaker result than Corollary 5.2 was obtained by Feller in [8]. In [9], Feller uses a clever truncation argument to improve the above result by requiring only that $E[X_1^2 \log_2 X_1^2] < \infty$ instead of (5.12). M. Wichura points out, however, that Feller's proof is valid only for *symmetric* X_i (as was observed by H. Robbins and D. Siegmund).

REMARK 2. One can replace the conditions (5.8) and (5.9) in Theorem 5.2 by the single condition

(5.9′) $$E\{X_1^2(\log X_1^2)(\log_2 X_1^2)^2\} < \infty .$$

One can refer to Theorem 4.3 with $\alpha = 2$ for the proof.

6. Integral test for lower functions of absolute maxima of martingales. We continue to use the notation of the first paragraph of Section 5. We will be concerned here with the growth of small values of

$$M_n = \max_{1 \le i \le n} |S_i| .$$

Our aim here is to give an integral test for an increasing function φ under suitable conditions on $\{X_n\}$ such that

$$P[M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1} \text{ i.o.}] = 0 \quad \text{or} \quad 1 ,$$

according as the integral of a certain function of φ converges or diverges.

The case where $\{X_n\}$ is a sequence of independent random variables was considered by Chung [5]. His method was to obtain sufficiently good probability estimates for the appropriate tail of M_n and then use these to establish an integral test, a method reminiscent of Feller's [8]. A simpler proof for the case of Brownian motion was recently given by Jain and Taylor [16].

Our approach here is to use the almost sure invariance principle of Section 3 to obtain such results. Indeed, Strassen's almost sure invariance principle already gives results stronger than Chung's and any improvement of the invariance principle will naturally lead to stronger results.

THEOREM 6.1. *Assume that* $V_n \to \infty$ a.s. *and*

$$\sum_{n=1}^{\infty} \frac{(\log_2 V_n)^4}{V_n} E[X_n^2 I(X_n^2 > V_n(\log V_n)^{-1}(\log_2 V_n)^{-10}) \mid \mathscr{L}_{n-1}] < \infty \quad \text{a.s.} \tag{6.1}$$

Let $0 < \varphi \nearrow$. *Then*

$$P[M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1} \text{ i.o.}] = 1 \quad or \quad 0 \tag{6.2}$$

according as

$$I_1(\varphi) = \int_1^{\infty} \frac{\varphi^2(u)}{u} \exp[-8\varphi^2(u)/\pi^2]\, du \tag{6.3}$$

$$= \infty \quad or \quad < \infty\,.$$

Before proving this theorem, we state the corresponding theorem for Brownian motion ([16] page 547).

THEOREM 6.2. *Let* ξ *be a standard Brownian motion process,*

$$M(t) = \max_{0 \leqq u \leqq t} |\xi(u)| \qquad for \quad t \geqq 0\,,$$

and $0 < \varphi \nearrow$. *Then*

$$P[M(t) < t^{\frac{1}{2}}\{\varphi(t)\}^{-1} \text{ i.o. } as \quad t \to \infty] = 1 \quad or \quad 0$$

according as $I_1(\varphi) = \infty$ *or* $< \infty$.

PROOF OF THEOREM 6.1. As in the proof of Theorem 5.1, we first consider $\varphi \nearrow$ such that

$$\varphi_1(t) \equiv (\log_2 t)^{\frac{1}{2}} \leqq \varphi(t) \leqq \varphi_2(t) \equiv 2(\log_2 t)^{\frac{1}{2}} \tag{6.4}$$

for $t > 0$. Note that $I_1(\varphi_1) = \infty$ while $I_1(\varphi_2) < \infty$. Let S be the interpolated process as in Theorem 3.1 and let $M'(t) = \sup_{0 \leqq u \leqq t} |S(u)|$ for $t > 0$. In view of condition (6.1), Theorem 3.2 is applicable so that with $\alpha = 4$ we get

$$|\xi(t) - S(t)| = o(t^{\frac{1}{2}}(\log_2 t)^{-\frac{3}{2}}) \quad \text{a.s.} \tag{6.5}$$

as $t \to \infty$. But this implies

$$|M(t) - M'(t)| = o(t^{\frac{1}{2}}(\log_2 t)^{-\frac{3}{2}}) \quad \text{a.s.} \tag{6.6}$$

as $t \to \infty$.

Let $I_1(\varphi) < \infty$. Then it follows that $I_1(\varphi - \beta/\varphi) < \infty$ for each $\beta > 0$. By Theorem 6.2,

$$P[M(t) < t^{\frac{1}{2}}(\varphi(t) - \beta/\varphi(t))^{-1} \text{ i.o. as } \quad t \to \infty] = 0\,.$$

This then implies that

$$P[M(t) - \beta t^{\frac{1}{2}}\{\varphi(t)\}^{-3} < t^{\frac{1}{2}}\{\varphi(t)\}^{-1} \text{ i.o. as } \quad t \to \infty] = 0 \tag{6.7}$$

for every $\beta > 0$. Combining (6.6) and (6.7) we get for every $\beta > 0$

$$P[M'(t) < t^{\frac{1}{2}}(\varphi(t))^{-1} \text{ i.o. as } t \to \infty] = 0 .$$

Recalling that $\sup_{0 \leqq u \leqq V_n} |S(u)| = M_n$ we obtain from the above assertion that

$$P[M_n < V_n^{\frac{1}{2}}(\varphi(V_n))^{-1} \text{ i.o.}] = 0 .$$

The argument when $I_1(\varphi) = \infty$ is similar; one needs (3.18) at the last step.

It remains to show that (6.4) may be assumed without loss of generality. However, this argument is very similar to the one just given in the proof of Theorem 5.1 and is omitted.

The special case when $\{X_n\}$ is a sequence of independent random variables is again of special importance because it provides a useful comparison. We will show that Theorem 6.1 when specialized to this case implies the following improvement of the result of Chung [5].

COROLLARY 6.1. *Let $\{X_n\}$ be a sequence of independent random variables with $EX_n = 0$, $E|X_n|^3 < \infty$ for each $n \geqq 1$. Assume $V_n = \sum_{i=1}^n EX_i^2$ approaches ∞ with n. Let $0 < \varphi \nearrow$. Suppose*

$$E|X_n|^3/EX_n^2 = O(V_n^{\frac{1}{2}}/(\log V_n)^2) \tag{6.8}$$

as $n \to \infty$. Then

$$P[M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1} \text{ i.o.}] = 1 \quad \text{or} \quad 0 \tag{6.9}$$

according as $I_1(\varphi) = \infty$ or $< \infty$.

PROOF. It suffices to show that (6.8) implies the version of (6.1) for the case of independence. Clearly,

$$\begin{aligned} &E[X_n^2 I(X_n^2 > V_n(\log V_n)^{-1}(\log_2 V_n)^{-10})] \\ &\qquad \leqq V_n^{-\frac{1}{2}}(\log V_n)^{\frac{1}{2}}(\log_2 V_n)^5 E|X_n|^3 \\ &\qquad = O((\log V_n)^{-\frac{3}{2}}(\log_2 V_n)^5 EX_n^2) , \end{aligned} \tag{6.10}$$

where the last relation follows from (6.8). Using (6.10), it is seen that

$$\begin{aligned} &\sum_{n=1}^\infty \frac{(\log_2 V_n)^4}{V_n} E[X_n^2 I(X_n^2 > V_n(\log V_n)^{-1}(\log_2 V_n)^{-10}] \\ &\qquad \leqq \text{constant} \sum_{n=1}^\infty \frac{EX_n^2}{V_n(\log V_n)^{\frac{5}{4}}} \\ &\qquad = \text{constant} \sum_{n=1}^\infty \frac{V_n - V_{n-1}}{V_n(\log V_n)^{\frac{5}{4}}} < \infty . \end{aligned}$$

This completes the proof.

Pakshirajan [18] has improved Chung's result by replacing (6.8) with a condition on truncated moments. Although Theorem 6.1, when specialized to the case of independence, does not contain Pakshirajan's result, the latter can be derived by the method of this paper. We omit the details, except to observe that independence can be used to get better truncation.

The version of Theorem 6.1 for the stationary case can also be formulated in the same manner as Theorem 5.2.

THEOREM 6.3. *Let $\{X_n\}$ be a stationary ergodic martingale difference sequence with $EX_1^2 = 1$. Suppose the conditions of Theorem 4.1 or Theorem 4.3 are satisfied with $\alpha = 4$. Then for every $0 < \varphi \nearrow$*

$$P[M_n < n^{\frac{1}{2}}\{\varphi(n)\}^{-1} \text{ i.o.}] = 1 \quad or \quad 0 \tag{6.11}$$

according as $I_1(\varphi) = \infty$ or $< \infty$.

The proof of Theorem 6.3 is identical with that of Theorem 6.1 except that (4.5) replaces (3.5). We omit the proof.

REMARK. It should be pointed out that in the case of independent, identically distributed X_i's, the conditions reduce to $E\{X_1^2(\log_2 X_1^2)^4\} < \infty$.

REMARK. Breiman [3] proves the following result: Let $\{X_i, i \geqq 1\}$ be independent, identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and $E\{X_1^2(\log_2 X_1^2)^2\} < \infty$. Then

$$\liminf_{n\to\infty} \frac{M_n}{(n(\log_2 n)^{-1})^{\frac{1}{2}}} = \frac{\pi}{8^{\frac{1}{2}}} . \tag{6.12}$$

Since this is true for Brownian motion by Theorem 6.2, the result follows from Theorem 4.2 by taking $\alpha = 2$. Jain and Pruitt [15] have shown that $EX_1 = 0$, $EX_1^2 = 1$ alone imply that the lim inf in (6.12) is a constant. The question whether, in this latter case, the constant is $\pi 8^{\frac{1}{2}}$ is still open.*

7. A counterexample. The purpose of the counterexample given in this section is to explain the role played by the condition (4.2) in replacing the random normalizers by constant ones in the case of stationary martingale difference sequences.

We first remark that if $\{X_n\}$ is a stationary ergodic martingale difference sequence with $EX_1^2 < \infty$ then $\{T_n\}$ may be chosen to be a stationary ergodic sequence (see [1] for example). If in addition $\{X_n\}$ is an independent sequence, then $\{T_n\}$ may be chosen to be independent as well. In Theorem 4.1 the assumption (4.2) is made to conclude (4.8). In the independent case (4.8) is easily available if, for example, $EX_1^4 < \infty$, because one can use the classical Marcinkiewicz strong law of large numbers. Thus

$$|\textstyle\sum_{i=1}^n T_i - n| = o(n^{\frac{1}{2}+\alpha}) \quad \text{a.s.} \tag{7.0}$$

for each $\alpha > 0$. One may ask then whether finiteness of higher moments in the case of a stationary ergodic sequence yields a relation similar to (7.0) which would dislodge the unpleasant assumption (4.2).

The following example shows, however, that even if the random variables are assumed to be *uniformly bounded*, a relation such as (7.0) does not necessarily

* Added in proof: Jain and Pruitt have shown that the constant is indeed $\pi 8^{\frac{1}{2}}$.

hold for the partial sums of a stationary sequence. We should add, however, that the special stationary sequence $\{T_i, i \geqq 1\}$ of imbedding times may possess some special properties which could permit such a conclusion. Thus the counterexample only indicates that if the method is to succeed, just the stationarity of the sequence $\{T_i\}$ is not enough.

EXAMPLE. Let $A = \{0, \alpha a_1, \alpha a_2, \cdots\}$, where $\alpha > 0$, $0 < a_i \leqq a_{i+1} < 1$ for each $i \geqq 1$. Define

$$f_{00}(n) = \frac{c}{n^2(\log(n+2))^2}, \tag{7.1}$$

where c is chosen such that

$$\sum_{n=1}^{\infty} f_{00}(n) = 1 .$$

Then we can construct (see [6] page 60) a Markov chain $\{Z_n, n \geqq 0\}$ with state space A with stationary transition probabilities for which

$$P[Z_n = 0, Z_{n-1} \neq 0, \cdots, Z_1 \neq 0 \mid Z_0 = 0] = f_{00}(n), \tag{7.2}$$

for each $n \geqq 1$. The chain will be aperiodic and positive recurrent since $\sum_{n=1}^{\infty} nf_{00}(n) < \infty$. Hence there exists a unique stationary probability distribution π on A. Let Z_0 have distribution π. We thus have a stationary ergodic Markov chain. Clearly $0 < EZ_1 \leqq \alpha$. For notational convenience we choose α so that $EZ_1 = 1$. Since $\{Z_n, n \geqq 0\}$ is stationary ergodic, by the pointwise ergodic theorem

$$|\sum_{i=1}^{n} Z_i - n| = o(n) \quad \text{a.s.} \tag{7.3}$$

However, we will show that for every $\varepsilon > 0$

$$P[|\sum_{i=1}^{n} Z_i - n| = o(n^{1-\varepsilon})] = 0 . \tag{7.4}$$

(From the proof it will become clear that (7.4) holds even when $n/(\log_2 n)^2$ is substituted for $n^{1-\varepsilon}$.) Let

$$\tau_0 = 0 , \qquad \tau_n = \inf\{j > \tau_{n-1} : Z_j = 0\} . \tag{7.5}$$

Note that $\{\tau_j - \tau_{j-1}, j \geqq 2\}$ is a sequence of independent identically distributed random variables and that for $j \geqq 2$,

$$E(\tau_j - \tau_{j-1}) = E[\tau_1 \mid Z_0 = 0] = \sum_{n=1}^{\infty} nf_{00}(n) < \infty .$$

Hence, with probability 1,

$$\tau_n/n \to E(\tau_2 - \tau_1) < \infty . \tag{7.6}$$

Fix $\varepsilon > 0$ and let

$$U_n = \{\sum_{i=1}^{n} Z_i - n\}/n^{1-\varepsilon} . \tag{7.7}$$

Suppose that $U_n \to 0$ with positive probability. Since, by (7.6) $\tau_n \to \infty$ a.s.,

$$P[U_{\tau_n} \to 0] > 0 , \tag{7.8}$$

or equivalently,

$$P[\sum_{i=1}^{\tau_n} (Z_i - 1)/n^{1-\varepsilon} \to 0] > 0 . \tag{7.9}$$

Let

$$Y_i = \sum_{j=\tau_{i-1}+1}^{\tau_i} (Z_j - 1) \qquad . \qquad \text{for } i \geqq 1 .$$

Then $\{Y_i\}_{i=2}^{\infty}$ are independent identically distributed and (7.9) becomes

$$P[\sum_{i=1}^{n} Y_i/n^{1-\varepsilon} \to 0] > 0 .$$

By the Kolmogorov $0 - 1$ law,

$$P[\sum_{i=1}^{n} Y_i/n^{1-\varepsilon} \to 0] = 1 .$$

By the converse to the Marcinkiewicz strong law ([24])

$$E|Y_2|^{1/1-\varepsilon} < \infty . \tag{7.10}$$

We show that this leads to a contradiction. Let $\beta = P[Z_1 = 0] > 0$. Then $1 = EZ_1 \leqq (1 - \beta)\alpha$. Choose a_1 so that $1 - \beta < a_1 < 1$. Thus

$$1 = EZ_1 < a_1\alpha . \tag{7.11}$$

We have

$$\begin{aligned} E|Y_2|^{1/(1-\varepsilon)} &= E|\sum_{j=\tau_1+1}^{\tau_2} (Z_j - 1)|^{1/(1-\varepsilon)} \\ &= E|\sum_{j=\tau_1+1}^{\tau_2-1} (Z_j - 1) - 1|^{1/(1-\varepsilon)} , \end{aligned} \tag{7.12}$$

and

$$\sum_{j=\tau_1+1}^{\tau_2-1} (Z_j - 1) \geqq (\tau_2 - \tau_1 - 1)(a_1\alpha - 1) , \tag{7.13}$$

since, for $\tau_1 + 1 \leqq j \leqq \tau_2 - 1$, $Z_j \geqq a_1\alpha$. Now

$$E(\tau_2 - \tau_1)^{1/(1-\varepsilon)} = \sum_{n=1}^{\infty} n^{1/(1-\varepsilon)} f_{00}(n) = \infty$$

and combining (7.11), (7.12), (7.13) with this, it follows that

$$E|Y_2|^{1/(1-\varepsilon)} = \infty ,$$

contradicting (7.10). Hence (7.4) holds.

8. ψ-mixing stationary sequences. By using a representation due to Gordin [11] we extend our results to certain ψ-mixing sequences.

Let $\{\xi_n, -\infty < n < \infty\}$ be a stationary sequence, ξ_n taking values in a measurable space $(S, \mathscr{B})$, on some probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{M}_j^k = \sigma\{\xi_n, j \leqq n \leqq k\}$, for $-\infty < j \leqq k < \infty$, $\mathscr{M}_{-\infty}^k = \sigma\{\xi_n, n \leqq k\}$, $\mathscr{M}_k^\infty = \sigma\{\xi_n, n \geqq k\}$. Let $\psi_1 \geqq \psi_2 \geqq \cdots \downarrow 0$ be a sequence of nonnegative real numbers. The sequence $\{\xi_n, -\infty < n < \infty\}$ is said to be ψ-mixing if for each k, $-\infty < k < \infty$, and each $n \geqq 1$, $\Lambda_1 \in \mathscr{M}_{-\infty}^k$, $\Lambda_2 \in \mathscr{M}_{k+n}^\infty$ we have

$$|P(\Lambda_1 \cap \Lambda_2) - P(\Lambda_1)P(\Lambda_2)| \leqq \psi_n P(\Lambda_1) . \tag{8.1}$$

We assume that Ω consists of all doubly infinite sequences of real numbers, so that if $\omega \in \Omega$ then $\omega = (\cdots \omega_{-1}, \omega_0, \omega_1, \cdots)$, where ω_i are real numbers. The random variables ξ_n are defined by $\xi_n(\omega) = \omega_n$, for every integer n and $\omega \in \Omega$, letting $\mathscr{F} = \sigma\{\xi_n, -\infty < n < \infty\}$.

Let T denote the backward shift, that is,

$$(T\omega)_i = \omega_{i+1} , \qquad -\infty < i < \infty .$$

T is a measure preserving transformation on $(\Omega, \mathscr{F}, P)$. Note that, for every pair of integers j and k,

$$T^{-k}(\mathscr{M}_{-\infty}^{j}) = \mathscr{M}_{-\infty}^{j+k} .$$

For every random variable X on $(\Omega, \mathscr{F}, P)$, with values in *any* measurable space, we define as usual the transformation U given by:

$$UX(\omega) = X(T\omega) .$$

Clearly, X and UX have the same distribution because of stationarity. Also, for every real-valued random variable X,

$$U^n E[X \mid \mathscr{M}_{-\infty}^{j}] = E[U^n X \mid \mathscr{M}_{-\infty}^{j+n}] \quad \text{a.s.}$$

Using the ideas of Gordin [11] and Scott [20] we obtain the following theorem.

THEOREM 8.1. *Let $\{\xi_n, -\infty < n < \infty\}$ be a stationary $(S, \mathscr{B})$-valued sequence as above. Let g be a real-valued measurable function on $(S, \mathscr{B})$ and let $X_n = g(\xi_n)$. Assume that the stationary sequence $\{X_n, -\infty < n < \infty\}$ is ψ-mixing (and hence ergodic) with $EX_0 = 0$ and $E|X_0|^{2+\delta} < \infty$, for some $\delta \in [0, \infty)$. Suppose $\sum_{n=1}^{\infty} \psi_n^{\frac{1}{2}} < \infty$. Then*

(a) *there exist random variables Y_0 and Z_0 such that Y_0 is $\mathscr{M}_{-\infty}^{0}$ measurable, $EY_0 = 0 = EZ_0$, $E|Y_0|^{2+\delta} < \infty$, $E|Z_0|^{2+\delta} < \infty$ and*

$$X_0 = Y_0 - UZ_0 + Z_0 . \tag{8.2}$$

(b) *The sequence $(U^k Y_0, \mathscr{M}_{-\infty}^{k})$ is a stationary ergodic martingale difference sequence. Furthermore, for some $\alpha \geqq 0$*

$$E[(\textstyle\sum_{i=0}^{n} U^i X_0)^2] \sim \alpha n . \tag{8.3}$$

In proving this result we need the following lemma due to Ibragimov [13]. (See also [2] page 170.)

LEMMA 8.1. *Let $\{X_n, -\infty < n < \infty\}$ be a stationary ψ-mixing sequence. Let R_1 be measurable with respect to $\mathscr{M}_{-\infty}^{-k}$ and R_2 be measurable with respect to $\mathscr{M}_{-k+n}^{\infty}$, for some fixed k and $n \geqq 1$. Suppose $E[|R_1|^r]$ and $E[|R_2|^s]$ are finite for $r, s > 1$ and $1/r + 1/s = 1$. Then*

$$|E[R_1 R_2] - E[R_1]E[R_2]| \leqq 2\psi_n^{1/r} E^{1/r}\{|R_1|^r\} E^{1/s}\{|R_2|^s\} . \tag{8.4}$$

PROOF OF THEOREM 8.1. We will prove it for $0 \leqq \delta \leqq 2$; the proof clearly works also for any $\delta > 2$. It will be shown later that

$$\textstyle\sum_{k=1}^{\infty} ||U^k E[X_0 \mid \mathscr{M}_{-\infty}^{-k-1}]||_2 < \infty ,$$

where $||f||_p$ denotes $\{E[|f|^p]\}^{1/p}$. Assuming this now, we define

$$Y_0 = \textstyle\sum_{k=0}^{\infty} U^k\{E[X_0 \mid \mathscr{M}_{-\infty}^{-k}] - E[X_0 \mid \mathscr{M}_{-\infty}^{-k-1}]\} , \tag{8.5}$$

and

$$Z_0 = \textstyle\sum_{k=0}^{\infty} U^k E[X_0 \mid \mathscr{M}_{-\infty}^{-k-1}] . \tag{8.6}$$

Now Y_0 is $\mathscr{M}_{-\infty}^{0}$ measurable and (8.2) is satisfied. Note that $U^k Y_0 = Y_k$ is $\mathscr{M}_{-\infty}^{k}$ measurable for $k \geqq 1$.

Let V be the operator on $L^p(\Omega, \sigma(X_0))$, $2 \leqq p \leqq 4$, defined by (for f a Borel measurable function on the real line)

$$V(f(X_0)) = \sum_{k=0}^{\infty} U^k E[\{f(X_0) - Ef(X_0)\} \mid \mathscr{M}_{-\infty}^{-k-1}] \equiv \sum_{k=0}^{\infty} U^k E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}], \quad \text{say.} \tag{8.7}$$

We will first check that V is a bounded linear operator from $L^2(\Omega, \sigma(X_0))$ into $L^2(\Omega, \mathscr{M}_{-\infty}^{0})$ and from $L^4(\Omega, \sigma(X_0))$ into $L^4(\Omega, \mathscr{M}_{-\infty}^{0})$. It will then follow by the Marcinkiewicz interpolation theorem ([21] Appendix B, page 272) that V takes $L^{2+\delta}(\Omega, \sigma(X_0))$ into $L^{2+\delta}(\Omega, \mathscr{M}_{-\infty}^{0})$ for $0 \leqq \delta \leqq 2$.

For $\delta = 2$, using Lemma 8.1, we see that

$$\begin{aligned} E(\{U^k E(\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1})\}^4) &= E(E^4[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}]) \\ &= E(\bar{f}(X_0)\{E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}]\}^3) \\ &\leqq 2\phi_{k+1}^{\frac{3}{4}} E^{\frac{3}{4}}\{E^4[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}]\} E^{\frac{1}{4}}[\{\bar{f}(X_0)\}^4] . \end{aligned}$$

Therefore

$$E^{\frac{1}{4}}(E^4\{\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}\}) \leqq 2\phi_{k+1}^{\frac{3}{4}} \|\bar{f}(X_0)\|_4 \tag{8.8}$$

and hence

$$\|U^k E(\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1})\|_4 \leqq 2\phi_{k+1}^{\frac{3}{4}} \|\bar{f}(X_0)\|_4 ;$$

and

$$\sum_{k=0}^{\infty} \|U^k E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}]\|_4 \leqq 2(\textstyle\sum_{k=1}^{\infty} \phi_k^{\frac{3}{4}}) \|\bar{f}(X_0)\|_4 .$$

For $\delta = 0$ a similar argument with $r = s = 2$ in Lemma 8.1 shows that

$$\sum_{k=0}^{\infty} \|U^k E(\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k-1}\|_2 \leqq 2(\textstyle\sum_{k=1}^{\infty} \phi_k^{\frac{1}{2}}) \|\bar{f}(X_0)\|_2 .$$

Since $\sum \phi_k^{\frac{1}{2}} < \infty$ by assumption, we have shown that V is a bounded linear operator from $L^2(\Omega, \sigma(X_0))$ into $L^2(\Omega, \mathscr{M}_{-\infty}^{0})$ and from $L^4(\Omega, \sigma(X_0))$ into $L^4(\Omega, \mathscr{M}_{-\infty}^{0})$. Also, taking f to be the identity function and recalling that $EX_0 = 0$ and $EX_0^2 < \infty$, it follows that Y_0 and Z_0 are well defined as L_2 limits (indeed also as L_1 limits and almost sure limits). Moreover, since V takes $L^{2+\delta}(\Omega, \sigma(X_0))$ into $L^{2+\delta}(\Omega, \mathscr{M}_{-\infty}^{0})$, $E|X_0|^{2+\delta} < \infty$ implies that $\|V(X_0)\|_{2+\delta}^{2+\delta} = E|Z_0|^{2+\delta} < \infty$. By stationarity, $E|UZ_0|^{2+\delta} < \infty$ and hence $E|Y_0|^{2+\delta} < \infty$. Thus the proof of assertion (a) is complete.

Let $Y_k = U^k Y_0$. Now $(Y_k, \mathscr{M}_{-\infty}^{k}, k \geqq 0)$ is clearly a stationary ergodic sequence. An easy computation shows that $E[Y_k \mid \mathscr{M}_{-\infty}^{k-1}] = 0$ a.s. Assertion (b) of the theorem now follows from the convergence of $\sum_{n=1}^{\infty} \phi_n^{\frac{1}{2}}$ and is given in [13] (see also [2] page 172).

REMARK. The case when $\delta = 0$ was considered by Gordin [11] (see also Scott [20] for more details). One could also use a version of the Marcinkiewicz interpolation theorem to conclude that

$$E[X_0^2(\log X_0^2)(\log_2 X_0^2)^\alpha] < \infty \Rightarrow E[Z_0^2(\log_2 Z_0)^\alpha] < \infty ,$$

etc., and get a refinement of Theorem 8.1.

REMARK. Walter Philipp has pointed out that one could avoid the use of the interpolation theorem by observing that for $\delta \geqq 0$

$$E\{|E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k}]|^{2+\delta}\} = E\{\bar{f}(X_0)E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k}]|E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k}]|^{\delta}\}$$
$$\leqq 2\psi_k^{(1+\delta)/(2+\delta)} E^{(1+\delta)/(2+\delta)}\{|E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k}]|^{2+\delta}\}$$
$$\times E^{(2+\delta)^{-1}}[|\bar{f}(X_0)|^{2+\delta}]$$

by applying Lemma 8.1 with $r = (2+\delta)/(1+\delta)$, $s = 2+\delta$. Hence one gets

$$||E[\bar{f}(X_0) \mid \mathscr{M}_{-\infty}^{-k}]||_{2+\delta} \leqq 2\psi_k^{(1+\delta)/(2+\delta)} ||\bar{f}(X_0)||_{2+\delta} .$$

Thus $\sum_k \psi_k^{(1+\delta)/(2+\delta)} < \infty$ suffices in our Theorem 8.1 in place of $\sum_k \psi_k^{\frac{1}{2}} < \infty$. However, this approach does not seem to lead to the refinement indicated in the remark above without first obtaining a corresponding refinement of Lemma 8.1.

We now consider the derivation of Feller and Chung type results for the functionals of a Markov process satisfying a condition of Doeblin (condition D_0, Doob [7] page 192 and page 221).

We will briefly discuss this condition and give a lemma which will connect it to ψ-mixing sequences.

Let $(S, \mathscr{B})$ be an abstract state space, where $\mathscr{B}$ is a σ-field of subsets of S. Under the Doeblin condition, the Markov transition function has a unique stationary probability distribution π. Moreover, the n-step transition function $P^n(x, B)$, satisfies for some $\gamma > 0$ and $0 < \rho < 1$,

$$|P^n(x, B) - \pi(B)| \leqq \gamma\rho^n , \tag{8.9}$$

uniformly in $x \in S$ and $B \in \mathscr{B}$ ([7] Chapter 5, Section 5). With this in mind, we define a Doeblin process to be a Markov process $\{\xi_i, -\infty < i < \infty\}$ such that (8.9) holds, and each ξ_i is distributed according to π.

For our purposes, it is important to show that Doeblin processes are ψ-mixing. This will be done rather easily with the help of the following lemma ([7] Lemma 7.2, page 224).

LEMMA 8.2. *Let $\{\xi_n, -\infty < n < \infty\}$ be a Doeblin process. Then there exist $\gamma > 0$ and $0 < \rho < 1$ such that for $k \geqq 0$ and every $\mathscr{M}_{k+1}^{\infty}$ measurable random variable R satisfying $|R| \leqq M$* a.s., *where M is a positive constant, the following holds*:

$$|E[R \mid \xi_1] - ER| \leqq 2\gamma M\rho^k . \tag{8.10}$$

COROLLARY 8.1. *If $\{\xi_n, -\infty < n < \infty\}$ is a Doeblin process, then it is ψ-mixing with*

$$\psi_n = c\rho^n ,$$

for some $c > 0$, $0 < \rho < 1$.

PROOF. Note that $E[R \mid \xi_1] = E[R \mid \mathscr{M}_{-\infty}^{1}]$ a.s. by the Markov property. Specializing R to indicators and using stationarity of $\{\xi_n, -\infty < n < \infty\}$, the defining mixing relation (8.1) is seen to hold with ψ_n decreasing exponentially as

$$\psi_n = 2\gamma\rho^n .$$

Let f be a real valued measurable function defined on $(S, \mathscr{B})$. Let $\{\xi_j, -\infty < j < \infty\}$ be a Doeblin process. We take Ω as a product space as before. For the remainder of this section let $\mathscr{M}_{-\infty}^k = \sigma\{\xi_n, n \leqq k\}$, $\mathscr{M}_j^k = \sigma\{\xi_n, j \leqq n \leqq k\}$, and $\mathscr{M}_k^\infty = \sigma\{\xi_n, n \geqq k\}$. Also, we will write

$$X_j = f(\xi_j), \qquad -\infty < j < \infty .$$

Observe that $\{X_j, -\infty < j < \infty\}$ is a stationary ψ-mixing sequence with $\psi_n = c\rho^n$. Suppose $Ef(\xi_0) = 0$ and $E|f(\xi_0)|^{2+\delta} < \infty$ for some $\delta \geqq 0$. Then Theorem 8.1 applies, with the resulting decomposition

$$X_0 = Y_0 - UZ_0 + Z_0, \qquad X_i = Y_i - U^{i+1}Z_0 + U^iZ_0 \tag{8.11}$$

where $Y_i = U^iY_0$, for $i \geqq 0$. Consider the stationary ergodic martingale difference sequence $\{Y_i, i \geqq 0\}$.

LEMMA 8.3. *Suppose* $E|Y_0|^{2+2\delta} < \infty$ *for some* $\delta > 0$. *Then*

$$|\sum_{i=2}^n E[Y_i^2 \mid Y_1, \cdots, Y_{i-1}] - \alpha n| = o(n^{1-\varepsilon}), \tag{8.12}$$

for some $\varepsilon > 0$, *where* $\alpha = EY_0^2$.

PROOF. We may assume $0 < \delta < 1$. Let $R_i = Y_i^2 - E[Y_i^2 \mid Y_1, \cdots, Y_{i-1}]$ for $i \geqq 2$, and $R_1 = 0$ a.s. Then $\{R_i, i \geqq 1\}$ is a martingale difference sequence. Since $E(Y_0^2)^{1+\delta} < \infty$, by a result of Loève ([17] (iii), page 288)

$$\sum_{i=1}^n R_i = o(n^{1-\varepsilon}) \quad \text{a.s.} \tag{8.13}$$

for some $\varepsilon > 0$. From the defining (8.5) it follows that

$$Y_0 = \sum_{k=0}^\infty \{E[X_k \mid \mathscr{M}_{-\infty}^0] - E[X_k \mid \mathscr{M}_{-\infty}^{-1}]\},$$

and applying Markov property, we get

$$Y_0 = h(\xi_{-1}, \xi_0),$$

for some Borel measurable function h on R^2. This naturally leads to the representation

$$Y_k = U^kY_0 = h(\xi_{k-1}, \xi_k), \qquad k \geqq 1,$$

and from Corollary 8.1, it is clear that $\{Y_k^2, k \geqq 0\}$ is ψ-mixing with $\psi_k = c\rho^{k-1}$. Let

$$\eta_i = (Y_i^2 - \alpha)I(|Y_i^2 - \alpha| \leqq i^{1/1+\delta}),$$

for $i \geqq 1$. Then clearly $\{\eta_k\}$ is also ψ-mixing with $\psi_k = c\rho^{k-1}$. Since $E|Y_1|^{2+2\delta} < \infty$, it can be checked (exactly as in the proof of the Marcinkiewicz strong law of large numbers (see [24])) that

$$\sum_{i=1}^\infty P[\eta_i \neq Y_i^2 - \alpha] < \infty, \tag{8.14}$$

$$n^{-1/1+\delta} \sum_{i=1}^n E\eta_i \to 0, \tag{8.15}$$

and

$$\sum_{i=1}^\infty \mathrm{Var}\,(\eta_i/i^{1/1+\delta}) < \infty. \tag{8.16}$$

By an analogue of Kolmogorov's convergence theorem for independent random variables for a mixing sequence, due to Cohn ([4] Theorem 2.1), the above relation (8.16) implies that

$$\sum_{i=1}^{\infty} (\eta_i - E\eta_i)/i^{1/(1+\delta)}$$

converges a.s. By the Kronecker lemma

$$n^{-1/(1+\delta)} \sum_{i=1}^{n} (\eta_i - E\eta_i) \to 0 \quad \text{a.s.}$$

Thus by (8.14) and (8.15)

$$(8.17) \qquad (\sum_{i=1}^{n} Y_i^2 - n\alpha) = o(n^{1/1+\delta}) \quad \text{a.s.}$$

Combining (8.13) and (8.17) the required relation (8.12) follows and the proof of the lemma is completed.

Lemma 8.3 provides the important condition (4.2) so that the results concerning stationary sequences in Section 5 and 6 can now be applied to (recall that $X_i = f(\xi_i)$)

$$(8.18) \qquad S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i - U^{n+1}Z_0 + UZ_0 ,$$

where $n \geqq 1$.

LEMMA 8.4. *Suppose $E|X_0|^{2+\delta} < \infty$, for some $\delta \geqq 0$. Then*

$$\lim_{n\to\infty} (U^{n+1}Z_0 - UZ_0)/n^{1/(2+\delta)} = 0 , \quad \text{a.s.}$$

PROOF. Fix $\varepsilon > 0$.

$$(8.19) \qquad \begin{aligned} P[|U^n Z_0 - Z_0| > 2\varepsilon n^{1/(2+\delta)}] &\leqq P[|U^n Z_0| > \varepsilon n^{1/(2+\delta)}] + P[|Z_0| > \varepsilon n^{1/(2+\delta)}] \\ &= 2P[|Z_0| > \varepsilon n^{1/(2+\delta)}] , \end{aligned}$$

where the last equality follows from stationarity. By Theorem 8.1, $E|Z_0|^{2+\delta} < \infty$. Hence

$$\sum_{n=1}^{\infty} P[|Z_0| > \varepsilon n^{1/(2+\delta)}] < \infty$$

and by (8.19) and the Borel–Cantelli Lemma the assertion of the lemma follows.

REMARK. Note that $\delta = 0$ is admissible in the lemma.

THEOREM 8.2. *Let $\{\xi_n, -\infty < n < \infty\}$ be a Doeblin process, $X_n = f(\xi_n)$, $-\infty < n < \infty$, where f is a real, measurable function on $(S, \mathscr{B})$. Let $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, for some $\delta > 0$. Let $S_n = \sum_{i=0}^{n} X_i$ and $M_n = \max_{1\leqq i\leqq n} |S_i|$. Suppose α in Lemma 8.3 (recall that $\alpha = EY_0^2$) is positive.*

(i) *Then for every real function φ, $0 < \varphi \nearrow$,*

$$(8.20) \qquad P[S_n > (\alpha n)^{\frac{1}{2}}\varphi(\alpha n) \text{ i.o.}] = 1 \quad or \quad 0$$

according as $I(\varphi) = \infty$ or $< \infty$.

(ii) *Further, for every real function $0 < \varphi \nearrow$,*

$$(8.21) \qquad P[M_n < (\alpha n)^{\frac{1}{2}}\{\varphi(\alpha n)\}^{-1} \text{ i.o.}] = 1 \quad or \quad 0$$

according as $I_1(\varphi) = \infty$ or $< \infty$.

Here $I(\varphi)$ and $I_1(\varphi)$ are given by (5.3) and (6.3), respectively.

PROOF. (i) Consider the decomposition (8.18). By Theorem 8.1, $E|Y_0|^{2+\delta} < \infty$. We use Theorem 5.2 via Theorem 4.3. The condition (5.9′) is clearly satisfied for the stationary ergodic martingale difference sequence $\{Y_i, i \geqq 0\}$ since $E|Y_0|^{2+\delta} < \infty$ for some $\delta > 0$. By Lemma 8.3 the condition (5.7) is fulfilled. Using Lemma 8.4, we apply Theorem 5.2 to S_n in (8.18) and the first assertion follows.

(ii) This follows from Theorem 6.3 in the same manner as above.

REMARKS. It is known that stationary ergodic martingale difference sequences with finite second moments obey both the law of the iterated logarithm ([23]; see [1] for a proof using embedding in Brownian motion) and the functional central limit theorem ([2] pages 206–208). The results of Section 8 suggest that these results should hold for stationary ψ-mixing sequences with $\sum_{n=1}^{\infty} \psi_n^{\frac{1}{2}} < \infty$. This has been shown by Scott [20] and Heyde and Scott [12] using the Gordin decomposition.

Using the above remark we now state the law of the iterated logarithm for functionals of a Doeblin process. This was first proved by Pakshirajan and Sreehari [19] under $(2+\delta)$th moment condition for some $\delta > 0$.

THEOREM 8.3. *Let $\{\xi_n, -\infty < n < \infty\}$ be a Doeblin process, $X_n = f(\xi_n)$, $-\infty < n < \infty$, where f is a real, measurable function on $(S, \mathscr{B})$. Let $E[X_0] = 0$ and $E[X_0^2] = 1$. Let $S_n = \sum_{i=0}^{n} X_i$. If $\alpha > 0$ in (8.3) then*

$$\limsup_n \frac{S_n}{(2\alpha n \log_2 n)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

$$\liminf_n \frac{S_n}{(2\alpha n \log_2 n)^{\frac{1}{2}}} = -1 \quad \text{a.s.}$$

The functional central limit theorem also holds under the assumptions of Theorem 8.3.

Finally we would like to observe that the tool which Gordin [11] has presented has a serious drawback. In the Gordin representation it is not clear that the martingale difference $\{Y_n\}$ inherits the mixing property of the original sequence $\{X_n\}$. In the case of a Doeblin process the Markov property helped in preserving mixing for $\{Y_n\}$. Our results of Sections 5 and 6 for the stationary case would apply via Theorem 4.3 whenever mixing for $\{Y_n\}$ is preserved in the Gordin representation of $\{X_n\}$.

If we only require random normalizers then we do, as a consequence of Theorem 8.1 and Lemma 8.4, obtain integral tests for stationary ψ-mixing sequences without having to assume the Markov structure.

THEOREM 8.4. *Let $\{X_n, -\infty < n < \infty\}$ be a stationary ψ-mixing sequence with $EX_0 = 0$, $EX_0^2 = 1$ and $E|X_0|^{2+\delta} < \infty$ for some $\delta > 0$. Referring to (8.2), let $V_n = \sum_{k=0}^{n} E[Y_k^2 \mid \mathscr{M}_{-\infty}^{k-1}]$, where $Y_k = U^k Y_0$. Suppose $\sum_{k=1}^{\infty} \psi_k^{\frac{1}{2}} < \infty$, and $\alpha = EY_0^2 > 0$. Let $S_n = \sum_{i=0}^{n} X_i$ and $M_n = \max_{1 \leq i \leq n} |S_i|$.*

(a) *Then for every real function* φ, $0 < \varphi \nearrow$,

$$P[S_n > V_n^{\frac{1}{2}}\varphi(V_n) \text{ i.o.}] = 0 \quad or \quad 1 \tag{8.22}$$

according as $I(\varphi) < \infty$ *or* $= \infty$.

(b) *Further, for every real function* φ, $0 < \varphi \nearrow$,

$$P[M_n < V_n^{\frac{1}{2}}\{\varphi(V_n)\}^{-1} \text{ i.o.}] = 0 \quad or \quad 1 \tag{8.23}$$

according as $I_1(\varphi) < \infty$ *or* $= \infty$.

Here $I(\varphi)$ and $I_1(\varphi)$ are as given in (5.3) and (6.3) respectively.

PROOF. (a) $V_n/n \to \alpha$ a.s. follows from the pointwise ergodic theorem. Thus by Lemma 8.4

$$\lim_{n\to\infty} (U^{n+1}Z_0 - UZ_0)/V_n^{1/(2+\delta)} = 0 \quad \text{a.s.}$$

It is easy to show that (5.1) holds for the stationary ergodic martingale difference sequence $\{Y_i, i \geqq 0\}$. Thus (8.22) follows by Theorem 5.1.

(b) This follows from Theorem 6.1 in the same manner as above.

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