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CRITICAL AGE-DEPENDENT BRANCHING PROCESSES

BY WARREN W. ESTY

The Cleveland State University

The asymptotic behavior of the joint generating function of a critical age-dependent branching process is derived and used to obtain conditioned limit laws and a limiting diffusion.

1. Introduction. In this paper we derive a new technique for obtaining limit theorems in age-dependent branching processes.

Since the useful functional iteration formula of the Galton–Watson and continuous time Markov branching processes, namely,

$$F(s_1, s_2; t_1, t_2) = F(s_1 F(s_2, t_2), t_1)\,, \tag{1.1}$$

does not hold for the age-dependent process, we derive a lemma giving natural conditions under which (1.1) is asymptotically true. There are numerous applications of the lemma. In the last section we obtain a limit theorem and a limiting diffusion as examples. Others will be found in succeeding papers.

Let $Z(t)$ be the number of particles at time t in an age-dependent branching process (as defined in Athreya-Ney) with particle production generating function $f(s)$ and lifetime distribution $G(t)$. The conditions on $f(s)$ and $G(t)$ are the same for each result in this paper. Therefore, for conciseness, we state them once only.

Assume throughout that $f'(1) = 1$ (the process is "critical"), $f''(1-) = \sigma^2 < \infty$, $\int_0^\infty t\, dG(t) = \mu < \infty$, and $\lim_{t\to\infty} t^2(1 - G(t)) = 0$. Define $K = \sigma^2/2\mu$.

We proceed with the statement of the main lemma after introducing the following notation for the continued fraction associated with a sequence $\{x_i;\, i = 1, 2, \cdots\}$:

(1.2) $y_1(x_1) = 1/x_1$, and having specified $y_n(x_1, x_2, \cdots, x_n)$, let $y_{n+1}(x_1, \cdots, x_{n+1}) = y_n(x_1, \cdots, x_{n-1}, x_n + x_{n+1}^{-1})$.

MAIN LEMMA. *Let* $0 < d_i < \infty$, $t_i = d_i t$, *and* $0 \leqq s_i(t) \leqq 1$ *such that* $\lim Kt(1 - s_i(t)) = L_i$, $0 \leqq L_i \leqq \infty$ *for* $i = 1, 2, \cdots, n$. *Then, writing* s_i *for* $s_i(t)$,

$$\lim t[1 - F(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n)] \tag{1.3}$$

$$= \lim t[1 - F(s_1 F(s_2 F(\cdots s_{n-1} F(s_n, t_n), t_{n-1}), \cdots, t_1))] \tag{1.4}$$

$$= \frac{1}{K} y_{2n}(d_1, L_1, d_2, L_2, \cdots, d_n, L_n)$$

with the interpretation that $1/\infty = 0$, *i.e., if there exists a least index* j *such that*

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$L_j = \infty$, *then the limit* (1.4)

$$(1.5) \qquad = \frac{1}{K} y_{2j-1}(d_1, L_1, d_2, L_2, \cdots, L_{j-1}, d_j) \qquad \text{if} \quad L_j = \infty .$$

Note that this is > 0 if $L_i > 0$ for at least one i.

Before turning to the proof (Section 4), we insert a section of routine preliminary lemmas without proof, and a section on the joint generating function integral equation.

2. Preliminary lemmas. The following result of Goldstein plays a central role.

(2.1) LEMMA. $$\lim_{t\to\infty}\left[\frac{1-F(s,t)}{1-s}\right][K(1-s)t+1] = 1 ,$$

uniformly in $0 \leqq s < 1$.

The proof may be found in Athreya-Ney as well as in Goldstein.

We state a number of consequences of this lemma without proof.

(2.2) LEMMA. *If* $0 \leqq x(t) \leqq 1$ *and* $\lim t(1 - x(t)) = L \leqq \infty$, *then*

$$\lim t[1 - F(x(t), t)] = \frac{L}{KL + 1} \qquad (\text{interpreted as } 1/K \text{ if } L = \infty) .$$

(2.3) LEMMA. *Let* $0 \leqq x(t), y(t), s(t) \leqq 1$. *If* $\lim t(1 - x(t)) = \lim t(1 - y(t)) = L \leqq \infty$ *and* $\lim t(1 - s(t)) = M \leqq \infty$, *then*

$$\lim t(1 - F(x(t)s(t), t)) = \lim t(1 - F(y(t)s(t), t)) .$$

(2.4) LEMMA. *If* $0 \leqq z(t) \leqq 1$ *and* $n = [t/\mu]$ (*where* $[x]$ *is the greatest integer in* x), *then*

$$\lim t|F(z(t), t) - f_n(z(t))| = 0 .$$

(2.5) LEMMA. *For any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and a* t_0 *such that for* $0 \leqq z(t) \leqq 1$ *and* $t > t_0$

$$t|f_{[t/\mu]}(z(t)) - f_{[(t/\mu)(1\pm\delta)]}(z(t))| < \varepsilon .$$

(2.6) LEMMA. *If* $f'(1) = 1$ *and* $0 \leqq s, r \leqq 1$, *then*

$$f(sr) \geqq sf(r) .$$

PROOF. Comparing slopes of lines through the origin and $(r, f(r))$ and $(sr, f(sr))$ yields $f(sr)/(sr) \geqq f(r)/r$ and (2.6).

3. The joint generating function integral equation. In this section we give an integral equation for the joint generating function associated with $(Z(t_1), Z(t_1 + t_2), \cdots, Z(t_1 + t_2 + \cdots + t_n))$. We will then indicate the proof that this equation does have a unique bounded solution, which is, in fact, a joint generating function.

The integral equation is known in the one dimensional case. The nth dimensional equation is in terms of the $n - 1$st and lower dimensional joint generating functions, which will satisfy the existence and uniqueness theorem by the induction hypothesis.

Let $T_i = \sum_{j=1}^{i} t_j$, $i = 1, 2, \cdots, n$ and

$$
\begin{aligned}
&F(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) \\
(3.1) \qquad &= \textstyle\sum_{(j_1, j_2, \cdots, j_n) \geqq 0} s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \\
&\times P(Z(T_1) = j_1, Z(T_2) = j_2, \cdots, Z(T_n) = j_n) .
\end{aligned}
$$

THEOREM. *Let f be a (not necessarily critical) probability generating function and $G(\cdot)$ a distribution on $[0, \infty)$ with $G(0+) = 0$. Then the equation*

$$
\begin{aligned}
(3.2) \quad & F(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) \\
(a) \quad & = \textstyle\int_0^{t_1} f(F(s_1, s_2, \cdots, s_n; t_1 - y, t_2, \cdots, t_n))\, dG(y) \\
& + s_1 \textstyle\int_{t_1}^{T_2} f(F(s_2, s_3, \cdots, s_n; T_2 - y, t_3, \cdots, t_n))\, dG(y) \\
& \vdots \\
(k) \quad & + s_1 s_2 \cdots s_{k-1} \textstyle\int_{T_{k-1}}^{T_k} f(F(s_k, \cdots, s_n; T_k - y, t_{k+1}, \cdots, t_n))\, dG(y) \\
& \vdots \\
& + s_1 s_2 \cdots s_{n-1} \textstyle\int_{T_{n-1}}^{T_n} f(F(s_n; T_n - y))\, dG(y) \\
(n) \quad & + s_1 s_2 \cdots s_n (1 - G(T_n))
\end{aligned}
$$

has a solution which is an n-dimensional joint generating function for each $(t_1, t_2, \cdots, t_n) \geqq 0$, and which is the unique bounded solution.

(3.2) may also be written

$$
\begin{aligned}
&F(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) \\
(3.3) \qquad &= \textstyle\int_0^{t_1} f(F(s_1, s_2, \cdots, s_n; t_1 - y, t_2, \cdots, t_n))\, dG(y) \\
&- s_1 \textstyle\int_0^{t_1} f(F(s_2, s_3, \cdots, s_n; T_2 - y, t_3, \cdots, t_n))\, dG(y) \\
&+ s_1 F(s_2, s_3, \cdots, s_n; T_2, t_3, \cdots, t_n) .
\end{aligned}
$$

Furthermore, if $f'(1) = 1$,

$$(3.4) \qquad s_1 s_2 \cdots s_j F(s_{j+1}, \cdots, s_n; t_{j+1}, \cdots, t_n) \leqq F(s_1, \cdots, s_n; t_1, \cdots, t_n) , \quad j \leqq n .$$

(3.5) $F(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n)$ *is non-decreasing in t_i, $i = 1, 2, \cdots, n$.*

The derivation of (3.2) is similar to that of the one-dimensional equation and is omitted. The reader is referred to Athreya-Ney for that derivation and other "known" results.

PROOF. The theorem is well known for $n = 1$ and reads then

$$F(s, t) = s(1 - G(t)) + \textstyle\int_0^t f(F(s, t - y))\, dG(y) .$$

(3.6) Assume (3.2) for $n = k$ and define

$$(3.7) \qquad F_0(s_1, \cdots, s_k, s_{k+1}; t_1, \cdots, t_k, t_{k+1}) = s_1 F(s_2, s_3, \cdots, s_{k+1}; t_2, t_3, \cdots, t_{k+1})$$

and

$$\tag{3.8} \begin{aligned} &F_{j+1}(s_1, s_2, \cdots, s_{k+1}; t_1, t_2, \cdots, t_{k+1}) \\ &\quad = \int_0^{t_1} f(F_j(s_1, s_2, \cdots, s_{k+1}; t_1 - y, t_2, \cdots, t_{k+1}))\, dG(y) \\ &\qquad \vdots \\ &\qquad + s \cdots s_{i-1} \int_{T_{i-1}}^{T_i} f(F(s_i, \cdots, s_{k+1}; T_i - y, t_{i+1}, \cdots, t_{k+1}))\, dG(y) \\ &\qquad \vdots \\ &\qquad + s_1 \cdots s_{k+1}(1 - G(T_{k+1})) \end{aligned}$$

where the last $(k + 1)$ terms are as in (3.2). Write F_{j+1} for (3.8). Thus F_j is defined recursively, with terms of the kth and lower dimensional joint generating functions, for which the theorem is assumed true by the induction hypothesis (3.6). That these F_j have a limit F, which is the unique bounded solution of (3.2), and which is a joint generating function, follows as in the one-dimensional proof.

That (3.2) may be written as (3.3) follows by adding and subtracting

$$s_1 \int_0^{t_1} f(F(s_2, s_3, \cdots, s_{k+1}; T_2 - y, t_3, \cdots, t_{k+1}))\, dG(y) .$$

Factoring out s_1, we recognize

$$\int_0^{T_2} f(F(s_2, \cdots, s_{k+1}; T_2 - y, \cdots, t_{k+1}))\, dG(y)$$

plus the last k terms of (3.8) to be just $F(s_2, \cdots, s_{k+1}; T_2, t_3, \cdots, t_{k+1})$.

(3.4) and (3.5) are properties of the critical joint generating function necessary to the proof of the main lemma, but, because they extend one-dimensional results, the reader may wish, at first reading, to skip now to Section 4, even though the proofs do contain some fine points.

We now proceed with the proofs of (3.4) and (3.5). They are known for $n = 1$.

(3.9) Assume (3.4) and (3.5) for $n = k$.

To prove (3.4) we show F_j is non-decreasing to F. Therefore $F_0 \leqq F$. Repeated application of this fact in lower dimensions yields (3.4).

(3.10) If $f'(1) = 1$, $F_{i+1} - F_i \geqq 0$.

The proof is by induction. For $i = 0$,

$$\begin{aligned} F_1 - F_0 = &\int_0^{t_1} f(F_0(s_1, \cdots, s_{k+1}; t_1 - y, \cdots, t_{k+1}))\, dG(y) \\ &- s_1 \int_0^{t_1} f(F(s_2, \cdots, s_{k+1}; T_2 - y, t_3, \cdots, t_{k+1}))\, dG(y) \\ &+ s_1 F(s_2, \cdots, s_{k+1}; T_2, t_3, \cdots, t_{k+1}) \\ &- s_1 F(s_2, \cdots, s_{k+1}; t_2, t_3, \cdots, t_{k+1}) , \end{aligned}$$

writing F_1 as in (3.3). Recalling the definition of F_0, (3.7), and applying Lemma

(2.6) and the induction hypothesis (3.9), yields

$$
\begin{aligned}
F_1 - F_0 \geqq\ & G(t_1)s_1 f(F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1})) \\
& - s_1 G(t_1) f(F(s_2, \cdots, s_{k+1}; T_2, t_3, \cdots, t_{k+1})) \\
& + s_1 F(s_1, \cdots, s_{k+1}; T_2, t_3, \cdots, t_{k+1}) \\
& - s_1 F(s_1, \cdots, s_{k+1}; t_2, t_3, \cdots, t_{k+1}) \\
\geqq\ & s_1([F(s_2, \cdots, s_{k+1}; T_2, \cdots, t_{k+1}) - F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1})] \\
& - G(t_1)[f(F(\cdots; T_2, \cdots, t_{k+1})) - f(F(\cdots; t_2, \cdots, t_{k+1}))]) \\
\geqq\ & 0 ,
\end{aligned}
$$

since $A - B \geqq f(A) - f(B)$ for $A \geqq B$, and using the induction hypothesis (3.9) and (3.4). Thus $F_1 \geqq F_0$.
Assume (3.10) for $i = j$. Then

$$
\begin{aligned}
F_{j+2} - F_{j+1} = & \int_0^{t_1} (f(F_{j+1}(s_1, \cdots, s_{k+1}; t_1 - y, \cdots, t_{k+1})) \\
& - f(F_j(s_1, \cdots, s_{k+1}; t_1 - y, \cdots, t_{k+1}))) \, dG(y) \\
\geqq\ & 0 ,
\end{aligned}
$$

by the induction hypothesis, and the fact that f is increasing. Thus

(3.11) if $f'(1) = 1$, F_i is non-decreasing to F.

Repeated application of (3.11) to F_0 in lower dimensions yields (3.4).

Since the F_j converge to F, to show F non-decreasing in t_i we show that the F_j are non-decreasing in t_i. We do so by induction on j. F_0 is non-decreasing in t_i, $i = 1, 2, \cdots, k + 1$ by the induction hypothesis (3.9). Assume

(3.12) F_j is non-decreasing in t_i, $i = 1, 2, \cdots, k + 1$. Let $0 < t_i' - t_i < \min_i t_i$.

We will show $F_{j+1}(\cdots t_i' \cdots) - F_{j+1}(\cdots t_i \cdots) \geqq 0$ by subdividing the ranges of integration in (3.8) into intervals in which the integrand is $\geqq 0$ by either the induction hypothesis (3.9) on k, or (3.12) on j.

Let $T_k' = T_k$ if $k < i$ and $T_k' = T_k + (t_i' - t_i)$ if $k \geqq i$.

$$
\begin{aligned}
& F_{j+1}(\cdots t_i' \cdots) - F_{j+1}(\cdots t_i \cdots) \\
& \quad = \int_0^{t_1} (f(F_j(s_1, \cdots, s_{k+1}; t_1 - y, \cdots, t_i', \cdots, t_{k+1})) \\
& \qquad\qquad - f(F_j(s_1, \cdots, s_{k+1}; t_1 - y, \cdots, t_i, \cdots, t_{k+1}))) \, dG(y) \\
& \qquad \vdots \\
& \qquad + s_1 s_2 \cdots s_{i-1} \int_{T_{i-1}}^{T_i} (f(F(s_i, \cdots, s_{k+1}; T_i' - y, \cdots, t_{k+1})) \\
& \qquad\qquad - f(F(s_i, \cdots, s_{k+1}; T_i - y, \cdots, t_{k+1}))) \, dG(y) \\
& \qquad + s_1 \cdots s_{i-1} \int_{T_i}^{T_i'} (f(F(s_i, \cdots, s_{k+1}; T_i' - y, \cdots, t_{k+1})) \\
& \qquad\qquad - s_i f(F(s_{i+1}, \cdots, s_{k+1}; T_{i+1} - y, \cdots, t_{k+1}))) \, dG(y) \\
& \qquad \vdots \\
& \qquad + s_1 \cdots s_k \int_{T_{k+1}}^{T_{k+1}'} f(F(s_{k+1}, T_{k+1}' - y)) - s_{k+1} \, dG(y) \\
& \qquad + s \cdots s_{k+1}(1 - G(T_{k+1}')) .
\end{aligned}
$$

The first integral is $\geqq 0$ by the induction hypothesis on j. From the ith integral on we break up the range of integration $(T_i, T_{i+1}] = (T_i, T_i'] \cup (T_i', T_{i+1}]$. In the second of these ranges the integrand is $\geqq 0$ by the induction hypothesis on k. In the first we note that $T_{i+1} - y \leqq t_{i+1}$ and the result follows from (3.4) and Lemma (2.6).

4. The main lemma. The main lemma is the primary tool used to obtain conditioned limit laws and limiting diffusions in critical age-dependent branching processes. It relates joint generating functions to compositions of simple generating functions and enables the explicit calculation of certain types of asymptotic expressions which regularly occur in the derivation of limit theorems. Two examples may be found in Section 5.

To prove the main lemma we show the difference between the expressions in (1.3) is 0. The right side can then be evaluated explicitly using the one-dimensional lemma, (2.2).

The proof of (1.3) is complex. By induction we reduce (1.3) to the simpler (4.4), which is then rewritten by adding and subtracting appropriate terms. The resulting expression, (4.6), has three pairs of terms, the first two of which are easily shown to be small. Then, by showing the iterates F_i to be close to f_i as well as to F ((4.11) and (4.12)) we obtain Lemma (4.10) which proves the third pair also small.

To show the first equality, (1.3), we proceed by induction.

(1.3) is trivial for $n = 1$.

(4.1) Assume (1.3) for $n = k$.

$$\text{(4.2)} \quad \lim t([1 - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})] - [1 - F(s_1 F(s_2 \cdots F(s_{k+1}, t_{k+1}) \cdots, t_1))])$$

$$\begin{aligned} \text{(4.3)} \quad = \lim t\,([1 &- F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})] \\ &- [1 - F(s_1 F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1}), t_1)] \\ &+ [1 - F(s_1 F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1}), t_1)] \\ &- [1 - F(s_1 F(s_2 \cdots F(s_{k+1}, t_{k+1}) \cdots, t_1))]) \end{aligned}$$

adding and subtracting the same term.

By the induction hypothesis (4.1),

$$\lim t(1 - F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1})) = \lim t(1 - F(s_2 F(s_3 \cdots F(s_{k+1}, t_{k+1}), \cdots, t_2))) .$$

Applying Lemma (2.3) with $1 - F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1}) = x(t)$ to the second pair in (4.3) we get (4.2)

$$\text{(4.4)} \quad = \lim t(F(s_1 F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1}), t_1) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})) .$$

Thus to prove (1.3) we need only show (4.4) $= 0$. We will show that for any

$\varepsilon > 0$ there exists a t_0 such that for $t > t_0$

$$(4.5) \qquad -\varepsilon < t[F(s_1 F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1}), t_1) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})] < \varepsilon .$$

Again let $z = F_0 = s_1 F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1})$. On the one hand the inside of (4.5)

$$(4.6) \qquad = t(F(z, t_1) - f_n(z) + f_n(z) - f_{n(1+\delta)}(z) + f_{n(1+\delta)}(z) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})) .$$

For appropriate n and δ this will be shown to be $> -\varepsilon$.

On the other hand, replacing $(1 + \delta)$ with $(1 - \delta)$, (4.6) will be shown to be $< \varepsilon$. The conclusion (4.5) and therefore (1.3) follows when these steps are completed.

Choose $\varepsilon > 0$. By Lemma (2.4), with $n = [t_1/\mu]$,

$$(4.7) \qquad t|F(z, t_1) - f_n(z)| < \frac{\varepsilon}{3} \qquad \text{for } t \text{ large enough.}$$

By Lemma (2.5)

$$(4.8) \qquad t|f_n(z) - f_{n(1+\delta)}(z)| < \frac{\varepsilon}{3} \qquad \text{and}$$

$$(4.9) \qquad t|f_n(z) - f_{n(1-\delta)}(z)| < \frac{\varepsilon}{3} \qquad \text{for } \delta \text{ small enough and } t \text{ large enough.}$$

Thus the first two pairs in (4.6) can be made small.

To prove the third pair in (4.6) small we need the following:

(4.10) LEMMA. *Let* $z = F_0(s_1, \cdots, t_{k+1})$. *Then*

$$-G^{*i}(t_1) \leqq f_i(z) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1}) \leqq 1 - G^{*i}(t_1) , \qquad i = 1, 2, \cdots .$$

PROOF. Define, as in the existence and uniqueness proof for the joint generating function integral equation (3.7) and (3.8), the iterates F_i.

(4.10) will follow immediately from the following two inequalities:

$$(4.11) \qquad 0 \leqq F - F_i \leqq G^{*i}(t_1) \qquad \text{and}$$

$$(4.12) \qquad 0 \leqq f_i(z) - F_i \leqq 1 - G^{*i}(t_1) \qquad \text{for } i = 0, 1, 2, \cdots .$$

We prove each by induction on i.

PROOF OF (4.11). That $0 \leqq F - F_i$ was shown in (3.11). For $i = 0$, $F - F_0 \leqq G^{*0}(t_1) = 1$ is trivial.

(4.13) Assume (4.11) for $i = j$. Then

$$F - F_{j+1} = \int_0^{t_1} (f(F(\cdots, t_1 - y, \cdots)) - f(F_j(\cdots, t_1 - y, \cdots)))\, dG(y) .$$
$$(4.14) \qquad \leqq \int_0^{t_1} F(\cdots, t_1 - y, \cdots) - F_j(\cdots, t_1 - y, \cdots)\, dG(y)$$
$$\leqq \int_0^{t_1} G^{*j}(t_1 - y)\, dG(y) = G^{*(j+1)}(t_1) ,$$

and (4.11) is proved.

PROOF OF (4.12). $f_0(z) - F_0 = z - z = 0$ and (4.12) is true for $i = 0$.

(4.15) Assume (4.12) for $i = j$. Then

$$
\begin{aligned}
f_{j+1}(z) - F_{j+1} = &\int_0^{t_1} f(f_j(z)) - f(F_j(\cdots, t_1 - y, \cdots))\, dG(y) \\
&+ (1 - G(t_1)) f_{j+1}(s_1 F(s_2, \cdots, t_{k+1})) \\
&- s_1 \int_{T_1}^{T_2} f(F(s_2, \cdots, s_{k+1}; T_2 - y, \cdots, t_{k+1}))\, dG(y) \\
&\vdots \\
&- s_1 s_2 \cdots s_{k+1}(1 - G(T_{k+1}))\,.
\end{aligned} \tag{4.16}
$$

Noting that $f_{j+1}(x) \geqq f(x)$, (4.16)

$$
\begin{aligned}
\geqq &\int_0^{t_1} f(f_j(z)) - f(F_j(\cdots, t_1 - y, \cdots))\, dG(y) \\
&+ (1 - G(T_{k+1}) + G(T_{k+1}) - G(T_k) + \cdots - G(T_1)) f(s_1 F(s_2, \cdots, t_{k+1})) \\
&- s_1(G(T_2) - G(T_1)) f(F(s_2, \cdots, s_{k+1}; t_2, \cdots, t_{k+1})) \\
&- s_1 s_2(G(T_3) - G(T_2)) f(F(s_3, \cdots, s_{k+1}; t_3, \cdots, t_{k+1})) \\
&\vdots \\
&- s_1 s_2 \cdots s_{k+1}(1 - G(T_{k+1}))\,.
\end{aligned}
$$

Grouping the $(G(T_{r+1}) - G(T_r))$ terms, each is $\geqq 0$ by (3.4) and Lemma (2.6). The first term is greater than or equal to 0 by the induction hypothesis (4.15). We have shown the left-hand inequality in (4.12). The right-hand side follows more easily. Dropping negative terms from (4.16) we get

$$
\begin{aligned}
f_{j+1}(z) - F_{j+1} &\leqq \int_0^{t_1} f(f_j(z)) - f(F_j(\cdots, t_1 - y, \cdots))\, dG(y) + (1 - G(t_1)) \\
&\leqq \int_0^{t_1} f_j(z) - F_j(\cdots, t_1 - y, \cdots)\, dG(y) + (1 - G(t_1)) \\
&\leqq \int_0^{t_1} (1 - G^{*j}(t_1 - y))\, dG(y) + (1 - G(t_1))\,, \qquad \text{by (4.15)} \\
&= 1 - G^{*(j+1)}(t_1)\,.
\end{aligned}
$$

Thus (4.12) is proved. (4.11) and (4.12) yield (4.10).

We use the following consequence of Baum and Katz to prove the third pair in (4.6) small.

(4.17) LEMMA. *If* $\mu = \int_0^\infty t\, dG(t) < \infty$ *and* $\lim t^2(1 - G(t)) = 0$, *and if* $\delta > 0$, *then*

$$
\text{if} \quad i = \left[\frac{t}{\mu}(1 + \delta)\right] \quad \text{then} \quad tG^{*i}(t) \to 0
$$

$$
\text{if} \quad i = \left[\frac{t}{\mu}(1 - \delta)\right] \quad \text{then} \quad t(1 - G^{*i}(t)) \to 0\,. \tag{4.18}
$$

Applying (4.17) with $t = t_1$ to the left-hand side of Lemma (4.10) we get

$$
t[f_{n(1+\delta)}(z) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})] \geqq -\frac{\varepsilon}{3} \tag{4.19}
$$

for t large enough, recalling $n = [t_1/\mu]$. Similarly, applying (4.18) to the right-hand side of Lemma (4.10), we get

$$t[f_{n(1-\delta)}(z) - F(s_1, \cdots, s_{k+1}; t_1, \cdots, t_{k+1})] \leqq \frac{\varepsilon}{3} \tag{4.20}$$

for t large enough.

Combining (4.19), (4.8) and (4.7) as in (4.6) we have the left-hand side of (4.5). Combining (4.20), (4.18) and (4.7) as in (4.6) but with $(1 - \delta)$ replacing $(1 + \delta)$, we have the right-hand side of (4.5). Thus (4.5) is true and the first equality (1.3) is proved.

The second equality of the main lemma, (1.4), follows by induction using Lemma (2.2) in a lengthy but straightforward calculation which we omit.

5. Applications. The main lemma, (1.3), may be used to prove convergence in finite-dimensional distributions of several conditioned and scaled critical age-dependent branching processes to diffusions. We give one example here, (5.7), and mention three others, the proofs of which are left to a succeeding paper. With two variables, the main lemma is the tool we use to determine the limit of $\{Z(ct)/Kt \mid Z(t) > 0, 0 \leqq c \leqq 1\}$, (5.1), as $t \to \infty$, as well as other limit laws of a similar nature which will be found in a later paper.

If we condition a critical age-dependent branching process on non-extinction at time t, as in the expression (5.1), we get a limiting diffusion in c, $0 \leqq c \leqq 1$, as $t \to \infty$. If we condition on extinction in the interval $(t, t(1 + \varepsilon)]$, we get a limiting diffusion, as $t \to \infty$ and $\varepsilon \downarrow 0$. Another approach is to define the "age-dependent Q-process" by defining its transition probabilities to be the limits of the transition probabilities of a process conditioned on non-extinction at time T, and letting $T \to \infty$. Again, a limiting diffusion may be obtained using the main lemma.

(5.1) THEOREM. $\{Z(ct)/Kt \mid Z(t) > 0, 0 \leqq c \leqq 1\}$ *converges in distribution, as* $t \to \infty$, *to the sum of two independent exponential random variables with means* c *and* $c(1 - c)$.

PROOF. The Laplace transform of $\{Z(ct)/Kt \mid Z(t) > 0\}$ is

$$\begin{aligned} L(u, c, t) &= E\left(\exp\left(-\frac{uZ(ct)}{Kt}\right)\middle| Z(t) > 0\right) \\ &= \sum e^{-uj/Kt} P(Z(ct) = j \mid Z(t) > 0) \\ &= \sum e^{-uj/Kt} \frac{P(Z(ct) = j) - P(Z(ct) = j, Z(t) = 0)}{P(Z(t) > 0)} \\ &= \frac{F(e^{-u/Kt}, ct) - F(e^{-u/Kt}, 0; ct, t - ct)}{1 - F(0, t)}, \end{aligned}$$

recalling the definition of the joint generating function in (3.1). Thus

$$L(u, c, t) = \frac{t(1 - F(e^{-u/Kt}, 0; ct, t - ct))}{t(1 - F(0, t))} - \frac{t(1 - F(e^{-u/Kt}, ct))}{t(1 - F(0, t))}. \tag{5.2}$$

By (2.1),

$$\lim t(1 - F(0, t)) = \frac{1}{K}. \tag{5.3}$$

By Lemma (2.2),

$$\lim t(1 - F(e^{-u/Kt}, ct)) = \lim \frac{t(1 - e^{-u/Kt})}{K(1 - e^{-u/Kt})ct + 1} = \frac{u}{K(cu + 1)}. \tag{5.4}$$

Using the main lemma with $n = 2$, $s_1 = e^{-u/Kt}$, $s_2 = 0$, $d_1 = c$, and $d_2 = 1 - c$, $L_1 = \lim Kt(1 - e^{-u/Kt}) = u$, and $L_2 = \infty$. Thus

$$\lim t(1 - F(e^{-u/Kt}, 0; ct, t - ct)) = \frac{1}{K} y_3(c, u, 1 - c). \tag{5.5}$$

Using (5.2)—(5.5) we get

$$\begin{aligned} \lim_t L(u, c, t) &= y_3(c, u, 1 - c) - \frac{u}{cu + 1} \\ &= \frac{(1 - c)u + 1}{(1 - c)cu + 1} - \frac{u}{cu + 1} \\ &= \left(\frac{1}{uc(1 - c) + 1}\right)\left(\frac{1}{cu + 1}\right) \end{aligned} \tag{5.6}$$

the product of the Laplace transforms of the stated distributions. Theorem (5.1) is proved.

We may consider $\{Z_t(tT)/Kt \mid Z_t(0) = x_0 Kt + o(t)\}$ to be a sequence of processes in T, indexed by t. These converge in distribution to a limiting diffusion.

(5.7) THEOREM. *The finite dimensional distributions of*

$$\left\{X_t(T) = \frac{Z_t(tT)}{Kt};\ 0 \leqq T \,\middle|\, Z_t(0) = x_0 Kt + o(t),\ x_0 > 0\right\}$$

converge, as $t \to \infty$, to those of a diffusion $\{X(T)\}$ with initial state x_0, infinitesimal mean 0, *and infinitesimal variance* $2x$.

The transition function has associated density

$$f(y, T + t \mid x, T) = e^{-y/t} \frac{1}{t}\left(\frac{x}{y}\right)^{\frac{1}{2}} I_1\left(\frac{2(xy)^{\frac{1}{2}}}{t}\right),$$

$$P(X(T + t) = 0 \mid X(T) = x) = e^{-x/t}.$$

This process is also the limit of the corresponding sequence of Galton–Watson processes. Feller (1951) derived a related limiting diffusion assuming $f'(1) \neq 1$.

We use the following relation of the diffusion and its finite dimensional Laplace transform.

(5.8) LEMMA. *Let* $0 = T_0 < T_1 < T_2 < \cdots < T_n$ *and* $d_i = T_i - T_{i-1}$. *If the joint Laplace transform of a process* $\{X(T);\ T > 0\}$, *is*

$$E[\exp(-\textstyle\sum u_i X(T_i))] = \exp(-x_0 y_{2n}(d_1, u_1, d_2, u_2, \cdots, d_n, u_n)), \qquad \text{if } x_0 > 0,$$

and where $y_{2n}(\cdot)$ is the continued fraction notation of (1.2), *then* $\{X(T)\}$ *is a diffusion with infinitesimal mean* 0 *and infinitesimal variance* $2x$ *and initial state* x_0, *with the above transition function.*

The transition function density is obtained by inverting the Laplace transform.

PROOF OF (5.7): Let $b_0 = 0 < b_1 < b_2 < \cdots < b_k$ and define

$$W_{tk} = \frac{1}{Kt} \sum_{i=1}^{k} u_i Z(b_i t) . \tag{5.9}$$

The u_i will be the variables of the joint transform.
Let $d_i = b_i - b_{i-1}$, $t_i = d_i t$, and $s_i = s_i(t) = \exp(-(u_i/Kt))$, $i = 1, 2, \cdots, k$. If $x_0 > 0$,

$$\begin{aligned} \text{(5.10)} \qquad & \lim E(e^{-W_{tk}} \mid Z_t(0) = Kx_0 t + o(t)) \\ & = \lim [1 - (1 - F(s_1, \cdots, s_k; t_1, \cdots, t_k))]^{Z_t(0)} . \end{aligned}$$

By the main lemma (1.3)

$$\lim Z_t(0)(1 - F(s_1, \cdots, s_k; t_1, \cdots, t_k)) = x_0 y_{2k}(d_1, u_1, \cdots, d_k, u_k) . \tag{5.11}$$

Thus (5.10) =

$$\exp(-x_0 y_{2k}(d_1, u_1, \cdots, d_k, u_k)) , \tag{5.12}$$

which is the joint transform of $X(T)$ by (5.8). (5.7) is proved.

The one-dimensional limit law when $T = 1$ is of special interest.

(5.13) THEOREM. $\{Z_t(t)/Kt \mid Z_t(0) = x_0 Kt + o(t), x_0 > 0\}$ *converges in distribution, as* $t \to \infty$, *to a Poisson* (*with mean* x_0) *sum of independent identically distributed exponential random variables with mean* 1.

PROOF. Let

$$\begin{aligned} L(u, t) &= E\left(\exp\left[\left(-u\left(\frac{Z_t(t)}{Kt}\right)\right) \middle| Z_t(0) = x_0 Kt + o(t)\right]\right) \\ &= [F(e^{-u/Kt}, t)]^{x_0 Kt + o(t)} . \end{aligned}$$

Since $(x_0 Kt + o(t))(1 - F(e^{-u/Kt}, t)) \to x_0 u/(u+1) = x_0(1 - 1/(u+1))$, by Lemma (2.2), $\lim L(u, t) = \exp[-x_0(1 - 1/(u+1))]$ which is the composition of the Poisson and exponential transforms.

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DEPARTMENT OF MATHEMATICS
THE CLEVELAND STATE UNIVERSITY
CLEVELAND, OHIO 44115