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MARTINGALE CONVERGENCE TO MIXTURES OF INFINITELY DIVISIBLE LAWS

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Sufficient conditions are found for the row sums of a double array of martingale differences to converge in law to a mixture of infinitely divisible distributions.

1. Introduction. Sufficient conditions for the row sums of a triangular array of martingale differences to converge in law to an infinitely divisible distribution were found by Brown and Eagleson (1971). Dvoretsky (1972) gave an example to show that in general one could not expect convergence to mixtures of normals without some sort of measurability assumptions. In this paper, the necessary measurability assumptions are written out and sufficient conditions for the row sums of a triangular array of martingale differences to converge in law to a mixture of infinitely divisible distributions are obtained.

Two theorems are proved. In the first, the conditions require the almost sure (a.s.) convergence of certain quantities and the conclusion is strong in that certain conditional characteristic functions are shown to converge a.s. In the second theorem, the conditions are weakened to convergence in probability and the conclusion is correspondingly weakened to obtain only the convergence of certain characteristic functions.

The results of this paper have been used to discuss central limit theorems for stationary processes, see Eagleson (1974).

Throughout the paper only convergence to infinitely divisible laws with finite variance and their mixtures is discussed. While the most general infinitely divisible laws could be considered, it would only complicate the analysis without any essential change in the ideas and the conditions obtained would be difficult to verify in a particular situation.

2. Convergence to mixtures of infinitely divisible laws. Consider a double array of random variables (rv's) whose rows are martingale difference sequences, i.e. for each $n = 1, 2, \cdots$ we have rv's $X_{n1}, \cdots, X_{nk_n}$ on a probability space $(\Omega, \mathscr{A}, P)$ with sub-σ-fields $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \cdots \subset \mathscr{F}_{nk_n}$ of $\mathscr{A}$ such that X_{nj} is $\mathscr{F}_{nj}$-measurable and $E(X_{nj} \mid \mathscr{F}_{n,j-1}) = 0$ a.s. for $j = 1, \cdots, k_n$ where $k_n \to \infty$ as $n \to \infty$. Let

$$S_n = X_{n1} + \cdots + X_{nk_n}, \qquad \sigma_{nj}^2 = E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}),$$
$$V_{nk}^2 = \sum_{j=1}^k \sigma_{nj}^2, \qquad b_n = \max_{j \leq k_n} \sigma_{nj}^2 .$$

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Brown and Eagleson (1971) gave sufficient conditions for S_n to converge in law to an infinitely divisible distribution. Their result is (for the sake of reference) restated here as a Theorem.

THEOREM 1. *If*

$$b_n \to_p 0 \qquad \textit{as } n \to \infty\,, \tag{1}$$

there exists a finite constant C for which

$$\lim_{n\to\infty} P(V_{nk_n}^2 > C) = 0\,, \tag{2}$$

and there exists a bounded non-decreasing function G for which

$$\sum_{j=1}^{k_n} E(X_{nj}^2 I(a < X_{nj} \leqq b) \mid \mathscr{F}_{n,j-1}) \to_p G(b) - G(a) \tag{3}$$

as $n \to \infty$ for all continuity points a, b of G, then S_n converges in law to an infinitely divisible limit law whose characteristic function ϕ is given by

$$\log \phi(t) = \int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2}\, dG(x)\,. \tag{4}$$

There seems to be no reason why the function G, appearing in condition (3), should be nonrandom. Let $\mathscr{F} = \bigcap_{n=1}^{\infty} \mathscr{F}_{n0}$ and suppose G is an $\mathscr{F}$-measurable function, in the sense that G is a function, $G(t, \omega)$, from $R^1 \times \Omega$ into R^1 which for fixed t is an $\mathscr{F}$-measurable rv. Suppose further that, for almost all fixed ω, G is a bounded non-decreasing function of t. One would hope that by conditioning on $\mathscr{F}$, it should still be possible to discuss the asymptotic behavior of S_n and then, by averaging over $\mathscr{F}$, to obtain limit laws which are mixtures. The aim of this Section is to formalize this idea and to show how in this case S_n will converge in distribution to a mixture of infinitely divisible laws.

As the double array $\{X_{nj}, j \leqq k_n, n \geqq 1\}$ contains only countably many rv's, we may order them and take the sample space to be R^∞. Denote the minimum σ-field generated by all the X_{nj} by $\mathscr{H}$. We will restrict ourselves to considering the probability space $(R^\infty, \mathscr{H}, P)$ so that all rv's considered from now on will be Borel-measurable functions of the $\{X_{nj}\}$. Given any σ-field $\mathscr{F} \subset \mathscr{H}$, there exists (see Breiman (1968) pages 77 ff) a regular conditional probability on $\mathscr{H}$ given $\mathscr{F}$. That is, for each fixed $\omega' \in R^\infty$, there exists a function $Q_{\omega'}(B, \mathscr{F}) = Q_{\omega'}(B)$ such that

(a) for fixed $B \in \mathscr{H}$, $Q_{\omega'}(B)$ is a version of $P(B \mid \mathscr{F})$, and
(b) for fixed $\omega' \in R^\infty$, $Q_{\omega'}(\cdot)$ is a probability measure on $\mathscr{H}$.

For each fixed $\omega' \in R^\infty$, consider the measure space $(R^\infty, \mathscr{H}, Q_{\omega'})$. Denote the expectation with respect to $Q_{\omega'}$ by $E_{\omega'}$ and note that any set $B \in \mathscr{H}$ which has P-measure zero must have $Q_{\omega'}$-measure zero, at least for P-almost all ω'. Suppose that $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ is a double array of martingale differences on $(R^\infty, \mathscr{H}, P)$. Then $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ is still an array of martingale differences on $(R^\infty, \mathscr{H}, Q_{\omega'})$—at least for P-almost all ω'. To see this we prove:

LEMMA. *Let $\mathscr{G}$ be a sub-σ-field of $\mathscr{H}$ such that $\mathscr{F} \subset \mathscr{G}$. For P-almost all $\omega' \in R^\infty$, if Y is a* rv *such that $E|Y| < \infty$, then $E_{\omega'}(Y \mid \mathscr{G})(\omega) = E(Y \mid \mathscr{G})(\omega)\, Q_{\omega'}$*-a.s.

PROOF. First assume that $EY^2 < \infty$. For fixed ω', set $E_{\omega'}(Y \mid \mathscr{G})(\omega) = Z(\omega)$. Thus Z is defined to be any one of the $Q_{\omega'}$-equivalence class of $\mathscr{G}$-measurable rv's such that for all $A \in \mathscr{G}$,

$$\int_A Z(\omega) Q_{\omega'}(d\omega) = \int_A Y(\omega) Q_{\omega'}(d\omega) .$$

By the definition of $Q_{\omega'}$, it follows that $E(ZI(A) \mid \mathscr{F})(\omega') = E(YI(A) \mid \mathscr{F})(\omega')$, P-a.s.

Hence, by a standard argument, for any fixed rv $U \in \mathscr{G}$, such that $EU^2 < \infty$, $E(ZU \mid \mathscr{F})(\omega') = E(YU \mid \mathscr{F})(\omega')$, P-a.s.

As both Z and $E(Y \mid \mathscr{G})$ are $\mathscr{G}$-measurable, $E((Z - E(Y \mid \mathscr{G}))^2 \mid \mathscr{F})(\omega') = 0$, P-a.s. and hence for P-almost all ω'

$$Z(\omega) = E_{\omega'}(Y \mid \mathscr{G})(\omega) = E(Y \mid \mathscr{G})(\omega) , \quad Q_{\omega'}\text{-a.s.}$$

Now approximate the original rv Y by a monotonically increasing sequence of rv's Y_n for which $EY_n^2 < \infty$ to obtain the result.

COROLLARY. *If $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ is an array of martingale differences on $(R^\infty, \mathscr{H}, P)$, then for P-almost all ω', $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ is an array of martingale differences on $(R^\infty, \mathscr{H}, Q_{\omega'})$.*

PROOF. Setting $Y = X_{nj}$ and $\mathscr{G} = \mathscr{F}_{n,j-1}$ in the lemma, shows that for P-almost all ω'

$$E_{\omega'}(X_{nj} \mid \mathscr{F}_{n,j-1})(\omega) = 0 , \quad Q_{\omega'}\text{-a.s.} \qquad \text{for all } j \leqq k_n,\ n \geqq 1 .$$

If conditions on the $\{X_{nj}\}$ could be found so that for 'most' ω' the double array of martingale differences on $(R^\infty, \mathscr{H}, Q_{\omega'})$ satisfied the conditions of Theorem 1, one could first condition on $\mathscr{F}$, apply Theorem 1 and then integrate over $\mathscr{F}$.

Let $C(\omega)$ denote an (a.s. finite) positive, $\mathscr{F}$-measurable rv, and $G(t, \omega)$ a random function which for fixed t is an $\mathscr{F}$-measurable rv and for almost all ω is bounded and nondecreasing in t. As an example of what can be proved, we have:

THEOREM 2. *Suppose that $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ is a double array of martingale differences on $(R^\infty, \mathscr{H}, P)$. If*

$$(1') \qquad b_n \to 0 \quad \text{a.s.} \qquad \textit{as } n \to \infty ,$$

there exists a $C(\omega)$ for which

$$(2') \qquad P\{V_{nk_n}^2(\omega) > C(\omega) \text{ for infinitely many } n\} = 0 ,$$

and there exists a $G(t, \omega)$ for which

$$(3') \qquad \sum_{j=1}^{k_n} E(X_{nj}^2 I(a < X_{nj} \leqq b) \mid \mathscr{F}_{n,j-1}) \to G(b) - G(a) \quad \text{a.s.}$$

as $n \to \infty$ for all a, b belonging to some countable dense set D, then $S_n = \sum_{j=1}^{k_n} X_{nj}$, conditional on $\mathscr{F}$, converges in law as $n \to \infty$ to an infinitely divisible distribution. In fact

$$(5) \qquad \lim_{n\to\infty} E(e^{itS_n} \mid \mathscr{F}) = \exp\left(\int_{-\infty}^{\infty} (e^{itx} - 1 - itx) x^{-2}\, dG(x)\right) \quad \text{a.s.}$$

PROOF. Set $Y = X_{nj}^2$ and $\mathscr{G} = \mathscr{F}_{n,j-1}$ in the lemma. Then for P-almost all $\omega' \in R^\infty$,

$$E_{\omega'}(X_{nj}^2 \mid \mathscr{F}_{n,j-1})(\omega) = E(X_{nj}^2 \mid \mathscr{F}_{n,j-1})(\omega) \quad Q_{\omega'}\text{-a.s.}$$

Thus (1′) implies that there exists a set $N_1 \in \mathscr{H}$ such that $P(N_1) = 0$ and such that for fixed $\omega' \notin N_1$

$$(6) \qquad Q_{\omega'}(\omega : \sup_{j \leq k_n} E_{\omega'}(X_{nj}^2 \mid \mathscr{F}_{n,j-1})(\omega) \to 0 \text{ as } n \to \infty) = 1 .$$

As both $C(\omega)$ and $G(t, \omega)$ are $\mathscr{F}$-measurable, for fixed ω' and t

$$C(\omega) = C(\omega') \qquad \text{and}$$
$$G(t, \omega) = G(t, \omega') \qquad Q_{\omega'}\text{-a.s.}$$

i.e. except for a set of ω of $Q_{\omega'}$-measure zero, $C(\omega)$ and $G(t, \omega)$ are the same as the constants $C(\omega')$ and $G(t, \omega')$.

So from (2′) and (3′), using the lemma again, we see that for all fixed $\omega' \notin N_2$, where $P(N_2) = 0$,

$$(7) \qquad \lim_{n\to\infty} Q_{\omega'}(\omega : \textstyle\sum_{j=1}^{k_n} E_{\omega'}(X_{nj}^2 \mid \mathscr{F}_{n,j-1})(\omega) > C(\omega')) = 0$$

and

$$(8) \qquad Q_{\omega'}(\omega : \textstyle\sum_{j=1}^{k_n} E_{\omega'}(X_{nj}^2 I(a < X_{nj} \leqq b) \mid \mathscr{F}_{n,j-1})(\omega)$$
$$\to G(b, \omega') - G(a, \omega') \text{ as } n \to \infty \text{ for all } a, b \in D) = 1 .$$

But (6), (7) and (8) imply that the conditions (1), (2) and (3) of Theorem 1 are satisfied for the array of martingale differences $\{X_{nj}, \mathscr{F}_{nj}, j \leqq k_n, n \geqq 1\}$ on $(R^\infty, \mathscr{H}, Q_{\omega'})$ at least for all ω' outside a set of P-measure zero. Thus for P-almost all ω'

$$\lim_{n\to\infty} E_{\omega'}(e^{itS_n}) = E(e^{itS_n} \mid \mathscr{F})(\omega')$$
$$= \exp(\textstyle\int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2}\, dG(x, \omega')) .$$

COROLLARY 1. *Under the conditions of Theorem* 2, S_n *converges in law as* $n \to \infty$ *to a limit law whose characteristic function* ϕ *is given by*

$$\phi(t) = E(\exp(\textstyle\int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2}\, dG(x, \omega))) .$$

PROOF. The corollary follows from (5), using dominated convergence.

COROLLARY 2. *If there exists an* $\mathscr{F}$*-measurable*, a.s. *finite positive* rv η *such that*

$$(9) \qquad V_{nk_n}^2 \to \eta \quad \text{a.s.} \qquad \textit{as } n \to \infty ,$$

and if for any $\varepsilon > 0$

$$(10) \qquad \textstyle\sum_{j=1}^{k_n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1}) \to 0 \quad \text{a.s.} \qquad \textit{as } n \to \infty ,$$

then $S_n = \sum_{j=1}^{k_n} X_{nj}$, *conditional on* $\mathscr{F}$, *converges in law as* $n \to \infty$ *to a normal distribution. In fact*,

$$\lim_{n\to\infty} E(e^{itS_n} \mid \mathscr{F}) = e^{-\frac{1}{2}t^2\eta} \quad \text{a.s.} ,$$

and hence S_n *converges in law as* $n \to \infty$ *to a limit law whose characteristic function*

ϕ is given by

$$\phi(t) = E(\exp(-\tfrac{1}{2}t^2\eta)) .$$

PROOF. As (9) clearly implies (2′), we need only prove that (10) implies (1′). But

$$\sup_{j \le k_n} E(X_{nj}^2 \mid \mathscr{F}_{n,j-1}) \leqq \varepsilon^2 + \textstyle\sum_{j=1}^{k_n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1})$$
$$\to \varepsilon^2 \quad \text{a.s.} \qquad \text{as } n \to \infty .$$

As $\varepsilon > 0$ is arbitrary, the result follows.

If only an asymptotic distributional result (rather than the a.s. convergence of the conditional characteristic function) is required, then the conditions of Theorem 2 can be substantially weakened.

THEOREM 3. *Suppose that $\{X_{nj}, \mathscr{F}_{nj}, j < k_n, n \geqq 1\}$ is a double array of martingale differences on $(R^\infty, \mathscr{H}, P)$. If*

$$(1'') \qquad b_n \to_p 0 \qquad \textit{as } n \to \infty ,$$

there exists a $C(\omega)$ for which

$$(2'') \qquad \lim_{n\to\infty} P(V_{nk_n}^2(\omega) > C(\omega)) = 0 ,$$

and there exists a $G(t, \omega)$ for which

$$(3'') \qquad \textstyle\sum_{j=1}^{k_n} E(X_{nj}^2 I(a < X_{nj} \leqq b) \mid \mathscr{F}_{n,j-1}) \to_p G(b) - G(a)$$

as $n \to \infty$ for all a, b belonging to some countable dense set D, then $S_n = \sum_{j=1}^{k_n} X_{nj}$ converges in law as $n \to \infty$ to a limit law whose characteristic function ϕ is given by

$$(11) \qquad \phi(t) = E(\exp(\textstyle\int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2}\, dG(x, \omega))) .$$

PROOF. Choose any subsequence of $\{S_n\}$, say $\{S_{n_k}\}$, which converges in distribution; such a subsequence must exist. Choose a further subsequence of $\{n_k\}$, $\{m_j\}$, such that (1″) and (3″) hold a.s. and such that

$$\textstyle\sum_{j=1}^{\infty} P(V_{m_j k_{m_j}}^2(\omega) > C(\omega)) < \infty .$$

As there are only countably many conditions, such an $\{m_j\}$ can be found by the usual diagonalization procedure. It follows from Theorem 2 that S_{m_j} (and hence S_{n_j}) converges in distribution to the law with characteristic function (11). As this is true for all convergent subsequences of $\{S_n\}$, S_n itself must converge in law to the same distribution.

COROLLARY. *If there exists an $\mathscr{F}$-measurable,* a.s. *finite, positive* rv η *such that*

$$(12) \qquad V_{nk_n}^2 \to_p \eta \qquad \textit{as } n \to \infty$$

and if for any $\varepsilon > 0$

$$(13) \qquad \textstyle\sum_{j=1}^{k_n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1}) \to_p 0 \qquad \textit{as } n \to \infty ,$$

then $S_n = \sum_{j=1}^{k_n} X_{nj}$ converges in law as $n \to \infty$ to a mixture of normal distributions whose characteristic function ϕ is given by

$$\phi(t) = E(\exp(-\tfrac{1}{2}t^2\eta)) .$$

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