

CONDITIONS FOR FINITE MOMENTS OF THE NUMBER OF ZERO CROSSINGS FOR GAUSSIAN PROCESSES¹

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Let $M_k(0, T)$ denote the k th (factorial) moment of the number of zero crossings in time T by a stationary Gaussian process. We present a necessary and sufficient condition for $M_k(0, T)$ to be finite. This condition is then applied to processes whose covariance functions $\rho(t)$ satisfy the local condition.

$$\rho(t) = 1 - \frac{t^2}{2} + \frac{C|t|^3}{6} + o|t|^3$$

for t near zero ($C > 0$). In this case we show all the crossing moments $M_k(0, T)$ are finite. In the course of the proof of this result, we point out an error which vitiates the related work of Piterbarg (1968) and Mirošin (1971, 1973, 1974a, 1974b). We also find a counterexample to Piterbarg's results.

1. Introduction. Let $X(t)$ denote a real separable continuous parameter stationary Gaussian process. Assume $EX(0) = 0$ and let $X(t)$ have covariance function $\rho(t) = EX(0)X(t)$. Then there is a (spectral) distribution function F so that $\rho(t) = \int_0^\infty \cos wt dF(w)$ and the quantities $\lambda_n = \int_0^\infty w^n dF(w)$ denote the spectral moments. We shall always assume that $\lambda_2 < \infty$ and take $\lambda_0 = \lambda_2 = 1$ for convenience. The covariance function now takes the form

$$\rho(t) = 1 - \frac{t^2}{2} + \phi(t)$$

where $\phi(t)$ is $o(t^2)$ for t near zero. The function $\phi(t) = \phi''(t)$ will be of major importance in our work. One can easily verify that $\phi(t) = \frac{1}{2}E(X'(t) - X'(0))^2$ is half the increments variance for the derivative of the process.

We further assume that $X(t)$ is nonsingular, i.e. that when $X(t)$ has n quadratic mean derivatives, for any distinct t_1, \dots, t_k , the random variables $\{X^{(j)}(t_i)\}_{i=1, \dots, k; j=0, \dots, n}$ are linearly independent. Cramér and Leadbetter (1967, page 203-204) have shown that a sufficient condition for nonsingularity is that the spectral distribution function F possess a continuous component.

In this paper, we are interested in the number of zero crossings by the process $X(t)$. As distributional information about zeros is difficult to obtain, we study the moments of the number of axis crossings in some time interval, and we focus especially on conditions that guarantee the finiteness of these moments. Itô (1964) and Ylvisaker (1965) have shown that the mean number of crossings is finite exactly when $\lambda_2 < \infty$. More recently Geman (1972) has shown that

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$\int_0^\infty \phi(t)/t < \infty$ is a necessary and sufficient condition for the variance of the number of zeros to be finite. We shall study only higher moments. Some sufficient conditions for these higher moments to be finite are given in Belyaev (1966). Some interesting uses for moments are given in Longuet-Higgins (1962), Leadbetter (1969), and Cramér, Leadbetter and Serfling (1971).

In Section 2 we define some useful concepts and develop some elementary results about crossing moments. Section 3 contains a necessary and sufficient condition for the moments to be finite.

One would still like simpler conditions for the moments to be finite than those given in Theorem 3.1. In Section 4 we obtain complete results about the finiteness of the moments for an important special class of processes. We also point out an error which nullifies the results in the works of Piterbarg (1968) and Mirošin (1971) there.

Our results can be extended in a straightforward manner to nonstationary Gaussian processes. Simple sufficient conditions for finite moments in the case of general (stationary) Gaussian processes require some tedious analysis. They will be presented at a later time.

2. Preliminaries. Some care must be taken in defining crossings, since $X'(t)$ may not be sample continuous (although $X(t)$ is sample continuous). We refer the reader to Cramér and Leadbetter (1967) for the definition of a crossing and the relationships between crossings, upcrossings, downcrossings, and tangencies. It can be shown that the number of zero crossings in time interval $[S, T]$ is a random variable. We denote this random variable by $N(0, [S, T])$ or when $S = 0$ simply $N(0, T)$. We now attempt to determine when its moments are finite.

We note that the magnitude of T is of little importance in determining the finiteness of the moments, for when $E(N(0, T))^k$ is finite for some T , it is finite for all T . To see this, first observe that crossings at any fixed time occur with probability zero, so that

$$(2.1) \quad N(0, [0, 2T]) = N(0, [0, T]) + N(0, [T, 2T]) \quad \text{w.p. 1.}$$

Now for any random variables X and Y with the same distribution, we use the Hölder inequality to obtain

$$\begin{aligned} E(X + Y)^k &= \sum_{j=0}^k \binom{k}{j} E(X^j Y^{k-j}) \\ &\leq \sum_{j=0}^k \binom{k}{j} E(X^k)^{j/k} E(Y^k)^{(k-j)/k} = 2^k E(X^k). \end{aligned}$$

Applying this to (2.1) we see that

$$E(N(0, 2T))^k \leq 2^k E(N(0, T))^k.$$

Since $N(0, T)$ is an increasing function of T , the result follows. Thus it is clear that the conditions governing finite crossing moments must be local ones.

Factorial moments arise naturally in studying crossing moments. The k th factorial moment of a random variable N is defined as

$$M_k = E(N(N-1) \cdots (N-k+1)).$$

We denote the k th factorial moment of $N(0, T)$ by $M_k(0, T)$. Clearly the k th factorial moment is finite exactly when the k th moment is.

In their book, Cramér and Leadbetter (1967) have given a formula for the factorial moments of the number of zero crossings, viz.

$$(2.2) \quad M_k(0, T) = \int_0^T dt_1 \cdots \int_0^T dt_k E(\prod_{i=1}^k |X'(t_i)| | X(t_j) = 0, j = 1, \dots, k) \times p(0, \dots, 0)$$

where $p(x_1, \dots, x_k)$ is the joint density of the random variables $X(t_1), \dots, X(t_k)$. Here and throughout, when no confusion arises we will suppress the explicit dependence of various quantities on the times (t_1, \dots, t_k) .

This formula, whether it gives finite or infinite values, holds for all processes $X(t)$ described above. It has served as a foundation for all further work on higher moments.

3. A necessary and sufficient condition. Using this formula (2.2), Belyaev (1966) obtained a sufficient condition for the k th moment to be finite. He showed that $M_k(0, T) < \infty$ if

$$(3.1) \quad \int_0^T dt_1 \cdots \int_0^T dt_k \left[\frac{\prod_{i=1}^k \sigma_i^2}{\det R_k} \right]^{\frac{1}{2}} < \infty .$$

Here $\det R_k = \det \text{Cov}(X(t_1), \dots, X(t_k))$ is the determinant of the covariance matrix of the random vector $(X(t_1), \dots, X(t_k))$, and $\sigma_i^2 = \text{Var}(X'(t_i) | X(t_j), j = 1, \dots, k)$ is the variance of $X'(t_i)$ conditioned by $X(t_j), j = 1, \dots, k$. Belyaev went on to show that $M_k(0, T)$ is finite when $X(t)$ has k mean continuous derivatives.

We can simplify (3.1) by first noting that the integrand is a symmetric function of its arguments. Thus we need only integrate over the region $\{0 \leq t_1 < \dots < t_k \leq T\}$. The change of variables

$$\Delta_i = t_{i+1} - t_i \quad i = 1, \dots, k - 1$$

$$z = t_1$$

has a bounded nonzero Jacobian. The terms in (3.1) are functions of the Δ_i only, and since the finiteness of (3.1) is independent of T , a sufficient condition for $M_k(0, T) < \infty$ is that the integral

$$(3.2) \quad \int_0^\varepsilon d\Delta_1 \cdots \int_0^\varepsilon d\Delta_{k-1} \left[\frac{\prod_{i=1}^k \sigma_i^2}{\det R_k} \right]^{\frac{1}{2}}$$

be finite for some $\varepsilon > 0$. Notice that the multiple integral (3.2) is of one less dimension than (3.1).

With the help of the following lemma, we will show that this condition is also necessary for $M_k(0, T) < \infty$.

LEMMA 3.1. *Let $\{X_i\}_{i=1, \dots, n}$ have a multivariate normal distribution and let $\sigma_i^2 = \text{Var}(X_i)$. Then*

$$E|\prod_{i=1}^n X_i| > \alpha_n \prod_{i=1}^n \sigma_i$$

where $\alpha_n > (n + 1)^{-(n+1)}$.

PROOF. Without loss of generality assume $\sigma_i = 1$ for all i . Define

$$\begin{aligned}
 P^*(u) &\equiv P(\min_{1 \leq i \leq n} |X_i| \leq u) \\
 (3.3) \quad &= P(\bigcup_{i=1}^n \{|X_i| \leq u\}) \leq \sum_{i=1}^n P(|X_i| \leq u) \\
 &\leq n \left(\frac{2u}{(2\pi)^{\frac{1}{2}}} \right) < nu.
 \end{aligned}$$

Then for any $v \geq 0$, we have

$$\begin{aligned}
 E|\prod_{i=1}^n X_i| &\geq v^n P(\min_{i \leq n} |X_i| > v) = v^n(1 - P^*(v)) \\
 &\geq v^n(1 - nv) \quad \text{by (3.3)}.
 \end{aligned}$$

Letting $v = (n + 1)^{-1}$, the result follows.

We can now prove

THEOREM 3.1. *A necessary and sufficient condition for $M_k(0, T) < \infty$ is that for some $\epsilon > 0$*

$$(3.4) \quad \int_0^\epsilon d\Delta_1 \cdots \int_0^\epsilon d\Delta_{k-1} \left[\frac{\prod_{i=1}^k \sigma_i^2}{(\det R_k)} \right]^{\frac{1}{2}} < \infty.$$

PROOF. Belyaev (1966) has shown that (3.4) is a sufficient condition. We now prove necessity. Recall that $M_k(0, T)$ is given by (2.2). For zero crossings,

$$p(0, \dots, 0) = ((2\pi)^k \det R_k)^{-\frac{1}{2}}.$$

From Lemma 3.1 we have that

$$E(\prod_{i=1}^k |X'(t_i)| | X(t_j) = 0, j = 1, \dots, k) > \alpha_k \prod_{i=1}^k \sigma_i.$$

These two estimates show that (2.2) is larger than a constant times

$$(3.5) \quad \int_0^T dt_1 \cdots \int_0^T dt_k \left[\frac{\prod_{i=1}^k \sigma_i^2}{\det R_k} \right]^{\frac{1}{2}}.$$

The proof is completed by applying the steps which lead to (3.2).

4. A special class of processes. In this section we study the class of processes $X(t)$ with covariance functions of the form

$$(4.1) \quad \rho(t) = 1 - \frac{t^2}{2} + \frac{C|t|^3}{6} + o(t^3) \quad \text{for } t \text{ near zero } (C > 0).$$

These processes have received considerable attention, especially in applications. (See Slepian (1962), Longuet-Higgins (1962a).) The special case in which $(X(t), X'(t))$ is a vector Markov process is included among them. (See Wong (1966)). We will prove the following theorem which is implicitly stated in Longuet-Higgins (1962b).

THEOREM 4.1. *If $X(t)$ has a covariance function of the form (4.1), then $M_k(0, T) < \infty$ for all k .*

REMARK 4.1. If any of the derivatives $X^{(n)}(t)$ have a covariance function of

the form (4.1), then again for $X(t)$ we have $M_k(0, T) < \infty$ for all k . To see this, let $N_Y(0, T)$ be the number of zero crossings by some process $Y(t)$ in time T . Then, when $X'(t)$ is sample continuous we have

$$N_X(0, T) \leq N_{X'}(0, T) + 1.$$

Clearly there is at least one zero of X' between every adjacent pair of zeros of X . Iterating we find when $X^{(n)}(t)$ is sample continuous that

$$N_X(0, T) \leq N_{X^{(n)}}(0, T) + n.$$

so that the k th moment of $N_X(0, T)$ is finite when the same is true for $N_{X^{(n)}}(0, T)$.

REMARK 4.2. In the course of the proof of this result, we will point out the error made by both Piterbarg (1968) and Mirošin (1971). We also note that Theorem 4.1 contradicts the result (Theorem 2) of Piterbarg. More will be said at the completion of our proof.

Before proceeding, it is useful to introduce the notion of divided differences.

DEFINITION 4.1. The n th *divided difference* of a function $X(t)$ at the distinct points $t_1 < \dots < t_{n+1}$, denoted $X[t_{n+1}, \dots, t_1]$, is defined iteratively by

$$X[t_{n+1}, \dots, t_1] = \frac{X[t_{n+1}, \dots, t_2] - X[t_n, \dots, t_1]}{t_{n+1} - t_1}$$

with

$$X[t_2, t_1] = \frac{X(t_2) - X(t_1)}{t_2 - t_1}.$$

When $X'(t)$ exists we define $X[t_1, t_1] = X'(t_1)$.

We further define the n th *extended divided difference* by

$$X_\phi[t_{n+1}, \dots, t_1] = \frac{X[t_{n+1}, \dots, t_2] - X[t_n, \dots, t_1]}{\phi(t_{n+1} - t_1)}$$

for functions $\phi(t)$ which are strictly increasing on $[0, t_{n+1} - t_1]$ with $\phi(0) = 0$.

Note that

$$X_\phi[t_{n+1}, \dots, t_1] = \frac{t_{n+1} - t_1}{\phi(t_{n+1} - t_1)} X[t_{n+1}, \dots, t_1].$$

Consider the determinant of the covariance matrix of the random vector $(X(t_1), \dots, X(t_n))$, denoted $\det \text{Cov}(X(t_1), \dots, X(t_n))$. We shall make considerable use of equalities of the form

$$(4.2) \quad \det \text{Cov}(X(t_1), \dots, X(t_n)) = \Delta_1^2 \det \text{Cov}(X[t_1, t_2], X(t_2), \dots, X(t_n))$$

and

$$(4.3) \quad \det \text{Cov}(X(t_1), \dots, X(t_n)) = \phi(\Delta_1) \det \text{Cov}(X_{\phi^{\frac{1}{2}}}[t_1, t_2], X(t_2), \dots, X(t_n))$$

where $\Delta_i = t_{i+1} - t_i$.

These equations are obtained by subtracting the second column of the covariance matrix from the first and then dividing the first column by Δ_1 (or $\phi^{\frac{1}{2}}(\Delta_1)$);

and then doing the same to the first row. These manipulations can be iterated to generate higher divided differences of the X 's. When $X(t)$ is nonsingular, this procedure will be used to generate random variables which remain linearly independent as $t_n \rightarrow t_1$.

The following lemma is crucial to the proof of Theorem 4.1, and is of interest in its own right.

LEMMA 4.1. *Let $X(t)$ have covariance function*

$$(4.4) \quad \rho(t) = 1 - \frac{t^2}{2} + \frac{C|t|^3}{6} \quad \text{for } |t| < \varepsilon, \text{ for some } \varepsilon > 0$$

so that $\phi(t) = C|t|$, $|t| < \varepsilon$. Then, when $0 \leq t_1 < \dots < t_n \leq \varepsilon$ and for $3 \leq i, j \leq n$, we have

$$(4.5) \quad \begin{aligned} E(X_{\phi^{\frac{1}{2}}}[t_i, t_{i-1}, t_{i-2}] X_{\phi^{\frac{1}{2}}}[t_j, t_{j-1}, t_{j-2}]) & \\ &= \frac{2}{3} && i = j \\ &= \frac{C\Delta_{i-1}}{3\phi^{\frac{1}{2}}(\Delta_{i-1} + \Delta_{i-2})\phi^{\frac{1}{2}}(\Delta_i + \Delta_{i-1})} && i = j - 1 \\ &= \frac{C\Delta_{j-1}}{3\phi^{\frac{1}{2}}(\Delta_{j-1} + \Delta_{j+2})\phi^{\frac{1}{2}}(\Delta_j + \Delta_{j-1})} && i = j + 1 \\ &= 0 && |i - j| \geq 2. \end{aligned}$$

PROOF. The proof follows by direct calculation.

REMARK 4.3. This lemma shows that the random variables $Y_i = X[t_i, t_{i-1}] - X[t_{i-1}, t_{i-2}]$ are independent when their defining intervals $[t_{i-2}, t_i]$ do not overlap. Dividing Y_i by $(t_i - t_{i-2})^{\frac{1}{2}}$ normalizes its variance, and the degree of correlation between Y_i 's depends linearly on the percentage of overlap of the defining intervals.

REMARK 4.4. The existence of a process satisfying (4.4) is demonstrated as follows. Let $X(t)$ have the triangular covariance function

$$\begin{aligned} \rho_X(t) &= 1 - |t| \quad |t| < 1 \\ &= 0 \quad |t| \geq 1. \end{aligned}$$

Then $Y(t) = \int_t^{t+\frac{1}{2}} X(s) ds$ has covariance function

$$\rho_Y(t) = \frac{7}{48} - \frac{t^2}{2} + \frac{|t|^3}{3} \quad \text{for } |t| < \frac{1}{2}.$$

Finally $Z(t) = (\frac{48}{7})^{\frac{1}{2}} Y((\frac{7}{48})^{\frac{1}{2}} t)$ has covariance function

$$\rho_Z(t) = 1 - \frac{t^2}{2} + \left(\frac{7}{12}\right)^{\frac{1}{2}} \frac{|t|^3}{6} \quad \text{for } |t| \leq \frac{1}{4} \left(\frac{7}{12}\right)^{\frac{1}{2}}.$$

Other values of C in (4.4) can be obtained by a variation of the parameters defining $X(t)$ and $Y(t)$.

The following lemma for the case $\phi(t) = t^\beta$ was given by Mirošin (1971, Lemmas 6–7).

LEMMA 4.2. *Suppose $\phi(t)/t^\beta$ is monotonically increasing on $(0, 2\varepsilon]$ for some $\varepsilon > 0$ and $\beta > 0$ ($\phi(0) = 0$). Then all the integrals*

$$(4.6) \quad I_k(\phi) = \int_0^\varepsilon dx_1 \cdots \int_0^\varepsilon dx_k \frac{\prod_{i=1}^k \phi(x_i)}{(\prod_{i=1}^k x_i)(\prod_{i=1}^{k-1} \phi(x_i + x_{i+1}))}$$

are finite.

PROOF. Assume $\phi(0) = \phi'(0) = 0$ and that both $\phi(t)$ and $\phi(t)/\phi'(t)$ are monotonically increasing on $(0, \varepsilon]$. Then

$$\frac{\phi(t)}{\phi'(t)} \leq \frac{\phi(s+t)}{\phi'(s+t)} \quad \text{for } 0 \leq s, \quad t \leq \varepsilon.$$

Equivalently

$$\frac{\phi(t)}{\phi(s+t)} \leq \frac{\phi'(t)}{\phi'(s+t)},$$

so that if $I_k(\phi)$ is finite, we have that $I_k(\phi')$ is also finite. Thus we need only show that $I_k = I_k(\phi)$ is finite when $\phi(t) = t^\beta$, $\beta > 0$. The proof in this case is sketched in Mirošin (1971) and expanded in Cuzick (1974).

PROOF OF THEOREM 4.1. We know that $\phi(t) = C|t| + o(t)$. Since $\phi(t)$ is used only as a divisor in the proof below, we may assume for convenience that $\phi(t) = C|t|$. We begin by estimating the terms in the sufficient condition (3.4) for $M_k(0, T) < \infty$.

We shall use the divided difference manipulations (4.2) and (4.3) to simplify the determinant

$$\det R_k = \det \text{Cov} (X(t_1), \dots, X(t_k)).$$

Applying (4.2) to the last two entries yields

$$\det R_k = \Delta_{k-1}^2 \det \text{Cov} (X(t_1), \dots, X(t_{k-1}), X[t_{k-1}, t_k]).$$

Repeating this all other adjacent pairs shows that

$$\det R_k = \prod_{i=1}^{k-1} (\Delta_i)^2 \det \text{Cov} (X(t_1), X[t_1, t_2], \dots, X[t_{k-1}, t_k]).$$

Now repeat this entire operation on all but the first two entries except that we use the extended divided difference manipulation (4.3) to obtain

$$(4.7) \quad \det R_k = \prod_{i=1}^{k-1} (\Delta_i)^2 \prod_{i=1}^{k-2} \phi(\Delta_i + \Delta_{i+1}) \det \text{Cov} (\bar{Z})$$

where

$$(4.8) \quad \bar{Z} = (X(t_1), X[t_1, t_2], X_{\phi^{\frac{1}{2}}}[t_1, t_2, t_3], \dots, X_{\phi^{\frac{1}{2}}}[t_{k-2}, t_{k-1}, t_k]).$$

When $(t_k - t_1)$ is small, we see from Lemma 4.1 that all the diagonal terms in the lower $(k - 2) \times (k - 2)$ submatrix of $\text{Cov} (\bar{Z})$ approach $\frac{2}{3}$. Terms one away from the diagonal are less than $\frac{1}{3}$, and all other terms go to zero. In row and

column one and two of $\text{Cov}(\bar{Z})$ the diagonal elements approach unity, while the nondiagonal elements approach zero. Hence, for $(t_k - t_1)$ small enough, the matrix $\text{Cov}(\bar{Z})$ is diagonally dominant and using Ostrowski's (1952) estimate, the determinant is greater than 3^{2-k} . Since $\det \text{Cov} \bar{Z}$ also remains bounded as t_k approaches t_1 , we see that (4.7) gives us an asymptotic estimate for $\det R_k$.

We now estimate the terms in the numerator of (3.4). From the underlying multivariate normal distribution, one can verify that the conditional variances in (3.4) are given by (see Belyaev (1966))

$$\sigma_i^2 = \frac{\det \text{Cov}(X'(t_i), X(t_1), \dots, X(t_k))}{\det \text{Cov}(X(t_1), \dots, X(t_k))}.$$

Let $\bar{X} = (X(t_1), \dots, X(t_k))$. In the numerator, form the vector $E(X[t_i, t_{i+1}]|\bar{X})$ from columns $(i + 1)$ and $(i + 2)$ and subtract it from column 1. Divide column 1 by $\phi^{\frac{1}{2}}(\Delta_i)$. Do the same to the first row. We obtain for $i \leq k - 1$

$$\sigma_i^2 = \phi(\Delta_i) \frac{\det \text{Cov}(X_{\phi^{\frac{1}{2}}}[t_i, t_i, t_{i-1}], \bar{X})}{\det \text{Cov}(\bar{X})}.$$

Now operate on the vector \bar{X} in both the numerator and denominator as at (4.7) to see that

$$\sigma_i^2 = \phi(\Delta_i) \left[\frac{\det \text{Cov}(X_{\phi^{\frac{1}{2}}}[t_i, t_i, t_{i-1}], \bar{Z})}{\det \text{Cov} \bar{Z}} \right]$$

where \bar{Z} is the vector (4.8). As before the denominator of the last term is bounded and stays away from zero. The numerator is also bounded and thus we obtain for some constant K that for $t_k - t_1$ small enough

$$(4.9) \quad \begin{aligned} \sigma_i^2 &< K\phi(\Delta_i) \quad i \leq k - 1 \\ \sigma_k^2 &< 1. \end{aligned}$$

The last estimate is weaker than the others, but suffices for our purposes.

Now apply the estimates (4.8) and (4.9) to (3.4). We see that $M_k(0, T) < \infty$ when

$$\int_0^\epsilon d\Delta_1 \dots \int_0^\epsilon d\Delta_{k-1} \left[\frac{\prod_{i=1}^{k-1} \phi(\Delta_i)}{\prod_{i=1}^{k-1} \Delta_i^2 \prod_{i=1}^{k-2} \phi(\Delta_i + \Delta_{i+1})} \right]^{\frac{1}{2}} < \infty.$$

The truth of this statement for all k follows from Lemma 4.2 with $\beta < \frac{1}{2}$.

REMARK 4.5. Both Piterbarg and Mirošin assume that *all* off diagonal terms in $\text{Cov} \bar{Z}$ asymptotically approach zero. Lemma 4.1 exhibits a case in which this is not so. With other more general covariance functions the situation is worse, since for covariance functions of the form (4.1) entries two or more from the diagonal go to zero, which is not generally true. This is crucial in the estimate of the determinant in (4.7).

More recent work by Mirošin (1973, 1974a, 1974b) has presented further results on moments. Here he claims that all off-diagonal of $\text{Cov}(\bar{Z})$ are negative. This is again untrue as demonstrated by Lemma 4.1. In fact for *any* covariance

function for which $\phi(t)$ is continuous at zero, we can pick increments Δ_{i-1} , Δ_i , Δ_{i+1} so that some element one from the diagonal remains nonnegative as $t_n - t_1$ goes to zero. The problem has root in the fallacious attempt to treat the divided differences $X[t_1, t_2], \dots, X[t_{k-1}, t_k]$ as if they are the derivatives $X'(t_1), \dots, X'(t_{k-1})$.

REMARK 4.6. Theorem 4.1 still holds when studying the crossings of curves. This is treated in Cuzick (1974) where it is required that the curve $\alpha \in C_{k-1}[0, T]$ to assure that $M_k(\alpha, T) < \infty$.

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REFERENCES

- BELYAEV, YU. K. (1966). On the number of intersections of a level by a Gaussian stochastic process I. *Teor. Veroyatnost. i Primenen* **11** 120-128.
- CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.
- CRAMÉR, H., LEADBETTER, M. R., and SERFLING, R. J. (1971). On distribution function-moment relationships in a stationary point process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **18** 1-8.
- CUZICK, J. M. (1974). On the moments of the number of curve crossings by a stationary Gaussian process. Ph. D. Dissertation, Claremont Graduate School.
- GEMAN, D. (1972). On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **43** 977-982.
- ITÔ, K. (1964). The expected number of zeros of continuous stationary Gaussian processes. *J. Math. Kyoto Univ.* **3** 207-216.
- LEADBETTER, M. R. (1969). On the distributions of the times between events in a stationary stream of events. *J. Roy. Statist. Soc. Ser. B* **31** 295-302.
- LONGUET-HIGGINS, M. S. (1962a). Bounding approximations to the distribution of intervals between zeros of a stationary Gaussian process. *Time Series Analysis* (Rosenblatt, ed.). Wiley, New York, 63-88.
- LONGUET-HIGGINS, M. S. (1962b). The distribution of intervals between zeros of a stationary random function. *Proc. Roy. Soc. Ser. A* **254** 557-599.
- MIROŠIN, R. N. (1971). On the finiteness of the number of zeros of a differentiable Gaussian stationary process. *Dokl. Akad. Nauk SSSR*. **200** 1321-1324.
- MIROŠIN, R. N. (1973). A sufficient condition for finiteness of the number of zeros of a differentiable Gaussian process. *Teor. Veroyatnost. i Primenen* **18** 454-463.
- MIROŠIN, R. N. (1974a). Sufficient conditions for finite moments of zeros of a stationary Gaussian process. *Vestnik Lenigrad. Univ.* **1** 48-65 (in Russian).
- MIROŠIN, R. N. (1974b). A necessary condition for the moments of the number of zeros of a differentiable Gaussian process to be finite. *Teor. Veroyatnost. i Primenen* **19** 596-603 (in Russian).
- OSTROWSKI, A. M. (1952). Note on bounds for determinants with dominant principal diagonal. *Proc. Amer. Math. Soc.* **3** 26-30.
- PITERBARG, V. I. (1968). The existence of moments for the number of level crossings of a stationary Gaussian process. *Dokl. Akad. Nauk SSSR*. **182** 46-48.
- SLEPIAN, D. (1962). On the zeros of Gaussian noise. *Time Series Analysis* (Rosenblatt, ed.). Wiley, New York, 104-115.
- WONG, E. (1966). Some results concerning the zero crossings of Gaussian noise. *SIAM J. Appl. Math.* **14** 1246-1254.

YLVISAKER, N. D. (1965). The expected number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **36** 1043-1046.

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