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ANOTHER UPPER BOUND FOR THE RENEWAL FUNCTION

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The general renewal equation and real variable methods are used to show that for a renewal process with generic lifetime random variable $X \geqq 0$ having distribution F and finite first and second moments $EX = \lambda^{-1}$ and EX^2, the renewal function $U(x) = \sum_0^\infty F^{n*}(x)$ satisfies $U(x) \leqq \lambda x_+ + C\lambda^2 EX^2$ for a certain constant C independent of F. Stone (1972) showed that $1 \leqq C \leqq 2.847 \cdots$; it is proved here that $C \leqq 1.3186 \cdots$ and conjectured that $C = 1$.

1. Introduction. It has been shown by Stone (1972) that the renewal function

$$U(x) = \sum_{n=0}^\infty F^{n*}(x)$$

of a random walk whose generic step length X has right-continuous distribution function (df) F with finite first and second moments $EX = \lambda^{-1} > 0$ and EX^2 satisfies

$$U(x) \leqq \lambda x_+ + C\lambda^2 EX^2 \tag{1.1}$$

for some finite constant C independent of F. He showed by example that $C \geqq 1$, and established by Fourier analytic methods that $C \leqq \eta = 2.846753 \cdots$ where η is the positive root of

$$2 \int_0^\eta (\eta - u)\{\sin \tfrac{1}{2}u / \tfrac{1}{2}u\}^2 \, du = 1 + 2\pi \,. \tag{1.2}$$

Below, we use real variable methods to show in the less general case that $F(0-) = 0$, so that the random walk is a renewal process, firstly that

$$C \leqq 1.5 \,, \tag{1.3}$$

and then, by refining the argument, that

$$C \leqq 1.3185649 \cdots \,. \tag{1.4}$$

The method used may be capable of further refinement and extension: to date we have not been successful in applying it to the general random walk where by refining Stone's Fourier transform argument we have shown (Daley, 1976) that $C < 2.081$. Certainly though, the evidence lends credence to the conjecture that $C = 1$.

We recall for later use some of the motivation for (1.1). It is known (e.g. Theorem XI. 3.1 of Feller (1971)) that for a renewal process, either F is non-arithmetic and

$$U(x) - \lambda x_+ \to \beta/2 \qquad x \to \infty \tag{1.5}$$

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where $\beta = \lambda^2 EX^2 = EX^2/(EX)^2$, or else F is a lattice distribution, with lattice span h, say, and for $0 \leqq t < h$,

$$U(nh + t) - \lambda(nh + t) \to \beta/2 + \lambda h(\tfrac{1}{2} - t/h) \qquad n \to \infty . \tag{1.6}$$

In either case then,

$$\sup_x \{U(x) - \lambda x_+\} = \gamma\beta \tag{1.7}$$

for some finite

$$\gamma = \gamma(F) \geqq (1 + \lambda h\beta^{-1})/2 , \tag{1.8}$$

putting $h = 0$ in the nonlattice case. If equality holds here, since $\lambda h \leqq \beta$, we must have $\gamma(F) \leqq 1$. Otherwise (and henceforth we shall assume it to be the case), the right-continuity of U then ensures that for some not necessarily unique finite ζ,

$$U(\zeta) - \lambda(\zeta) = \gamma\beta . \tag{1.9}$$

2. Proof of (1.3). For the moment, let F be the df of any (not necessarily nonnegative) rv X with mean $\lambda^{-1} > 0$ and finite standardized second moment $\beta = EX^2/(EX)^2$. Let $I(x) = 0$ or 1 as $x <$ or $\geqq 0$, and set

$$G(x) = \lambda \int_{-\infty}^{x} (I(u) - F(u))\, du = \lambda \int_0^\infty (I(x - u) - F(x - u))\, du . \tag{2.1}$$

Then $I(x) - G(x) \geqq 0$ (all x), $I(x) - G(x)$ is convex on $(-\infty, 0)$ and $(0, \infty)$, and

$$\beta = \lambda^2 EX^2 = 2\lambda \int_{-\infty}^{\infty} (I(u) - G(u))\, du . \tag{2.2}$$

Recall (Chapter XI of Feller (1971)) that $U = \sum_0^\infty F^{n*}$ is that solution Z of the general renewal equation $Z = z + Z * F$, i.e.,

$$Z(x) = z(x) + \int_{-\infty}^{\infty} Z(x - y)\, dF(y) \tag{2.3}$$

for which $z(x) = I(x)$. The function λx_+ is the solution of (2.3) for which $z(x) = G(x)$, and since in general (2.3) has the solution $Z = z * U$, we can write

$$V(x) \equiv U(x) - \lambda x_+ = \int_{-\infty}^{\infty} (I(x - u) - G(x - u))\, dU(u) . \tag{2.4}$$

Observe that the nonnegativity of $I - G$ and the nondecreasing nature of U ensure that $U(x) - \lambda x_+ \geqq 0$ (all x).

The strategy used below to bound V is to bound $I - G$ and appeal to (2.4); this principle is essentially used also in Feller (1948). So far we have been successful in bounding $I - G$ in a useful manner only when $F(0\,-) = 0$. Then $U(x) = 0$ $(x < 0)$, and also

$$1 - G(x) \leqq 2G(x)(1 - G(x)) \leqq 2(G(x) - G^{2*}(x)) \tag{2.5}$$

provided $G(x) \geqq .5$, which is certainly the case for all $x \geqq \beta/2\lambda$ because $1 - G$ is convex and has integral $\beta/2\lambda$ (see (2.2)).

Define $z_0(x) = 1$ or $2(G(x) - G^{2*}(x))$ as $\lambda x <$ or $\geqq \beta/2$. Then writing $\xi = \beta/2\lambda$,

$$(2.6)\qquad V(x+\xi) \leqq (z_0 * U)(x+\xi) \leqq U(x+\xi) - U(x) + 2((I-G)*G*U)(x+\xi)\,.$$

Now

$$(2.7)\qquad (I-G)*G*U = (I-G)*(\lambda x_+)*(I-F)*U = (I-G)*(\lambda x_+) \leqq \beta/2\,,$$

so putting $x = \zeta$, rearranging, and recalling (1.9),

$$(2.8)\qquad U(\zeta) - \lambda(\zeta+\xi) = \gamma\beta - \lambda\xi \leqq \beta\,,$$

whence $C = \sup \gamma(F) \leqq 1.5$ and (1.3).

3. Refinement and proof of (1.4). Our more detailed analysis depends in the first place on examining

$$(3.1)\qquad C_\beta \equiv \sup_{F \in \mathscr{G}'_\beta} \gamma(F)$$

where $\mathscr{G}_\beta'$ denotes the class of df's G defined as at (2.1) in terms of F with the properties stated there.

LEMMA 1. *C_β is a nondecreasing function of β.*

PROOF. Given a rv X with the df F, define a new rv X_q $(0 < q < 1)$ by $X_q = X$ with probability q, $= 0$ otherwise. Let F_q be the df of X_q, and let

$$(3.2)\qquad U_q = \textstyle\sum_0^\infty F_q^{n*} = \sum_0^\infty ((1-q)I + qF)^{n*} = \sum_0^\infty F^{n*}/q$$

be its renewal function. Then since

$$\lambda_q \equiv 1/EX_q = 1/qEX = \lambda/q\,, \qquad \beta_q \equiv \lambda_q^2 EX_q^2 = q\lambda_q^2 EX^2 = \beta/q\,,$$
$$\gamma(F_q)\beta_q = \sup_x (U_q(x) - \lambda_q x_+) = \sup_x (U(x) - \lambda x_+)/q = \gamma(F)\beta/q\,,$$

and so $\gamma(F_q) = \gamma(F)$. Thus, the family of sets $\{\tau : \gamma(F) = \tau \text{ for some } F \in \mathscr{G}_\beta'\}$ is monotone nondecreasing in β, and hence the lemma.

It will be convenient from this point on to take $\lambda = 1$, and to define $\mathscr{G}_\beta$ as $\mathscr{G}_\beta'$ so restricted (there is no loss of generality in this procedure) and also restricted to the df's of nonnegative rv's X (the lemma above does not need this restriction). Then any G in $\mathscr{G}_\beta$ has

$$(3.3)\qquad G(0) = 0 \leqq G'(0+) = 1 - F(0+) \leqq 1\,,$$

and since $G(x)$ is concave on $(0, \infty)$ and satisfies (2.2) with $\lambda = 1$,

$$(3.4)\qquad x \geqq G(x) \geqq x/\beta \qquad \text{for} \quad 0 \leqq x \leqq \beta/2\,.$$

Indeed, it is somewhat tedious but not difficult to use (2.2) and the first part of (3.4) to show that the class $\mathscr{G}_\beta$ generates the set

$$(3.5)\qquad \mathscr{D}_\beta = \{(x, G(x)) : G \in \mathscr{G}_\beta,\, x \geqq 0\} \equiv \{(x,y) : 0 \leqq y < 1,\, y \leqq x \leqq \xi(\beta, y)\} \cup \{(x, 1) : x \geqq \beta\}$$

where

(3.6) $$\xi(\beta, y) = \beta y \qquad 0 \leqq y \leqq .5$$
$$= (\beta - (2y-1)^2)/4(1-y) \qquad .5 \leqq y < 1 .$$

Refer back to (2.5)—(2.8), interpret ξ more generally, and define

(3.7) $$z_1(x) = 1 \qquad 0 \leqq x \leqq \xi$$
$$= (G(x) - G^{2*}(x))/y_1 \qquad x > \xi$$

for some $y_1 > 0$. It is clear that

(3.8) $$z_1(x) \geqq 1 - G(x)$$

for $x \leqq \xi$, and also for x and G such that $G(x) = y \geqq y_1$ (cf. (2.5)). If (3.8) holds for all x, then the argument from (2.6) to (2.8) shows that

(3.9) $$C \leqq \xi/\beta + 1/2y_1 .$$

It remains to determine ξ and y_1 jointly, in an optimal fashion. We give an outline of the argument in which a key step is Lemma 2.

LEMMA 2. *Given $y = G(x)$ and $x = O(\beta)$ for large β,*

(3.10) $$G(x) - G^{2*}(x) \geqq y(1-y) + (y-\eta)^2/2 + o(1)$$

(3.11) $$= y(1-y) - \frac{(1-y)^2}{2} + \frac{(1-y)^2}{1 + \left[1 - \left(\frac{\beta}{2x(1-y)} - 1\right)^{-2}\right]^{\frac{1}{2}}} + o(1) .$$

Fix y_0 in $.5 \leqq y_0 < 1$ so that by (3.6),

(3.12) $$x \leqq \xi_0 \equiv \beta/4(1-y_0) + O(1) .$$

We shall use (3.11) to show that if $x \geqq \xi_0$ and

(3.13) $$y = G(x) \leqq y_1 = 1 - b + by_0$$

for a certain constant b, then

(3.14) $$G(x) - G^{2*}(x) \geqq y_1(1-y) ,$$

and hence (3.9) does indeed hold. As an immediate consequence we have that

(3.15) $$C \leqq \inf_{.5 \leqq y_0 < 1} \{1/4(1-y_0) + 1/2(1 - b(1-y_0))\}$$
$$= (2 + (2b)^{\frac{1}{2}})^2/8 .$$

PROOF OF LEMMA 2. Let $y = G(x)$. Then

(3.16) $$G^{2*}(x) = \int_0^x G(x-u)\, dG(u) \leqq \int_0^x G_\eta(x-u)\, dG(u)$$
$$= \int_0^x G(x-u)\, dG_\eta(x-u) \leqq G_\eta^{2*}(x)$$

where $\eta \equiv \eta(x, y)$ is the index of the extremal $G_\eta \in \mathscr{G}_\beta$ for which $G_\eta(x) = y$ and

(3.17) $$G_\eta(z) = z \qquad 0 \leqq z \leqq \eta$$
$$= 1 - (1-\eta)(\beta - \eta - (1-\eta)z)_+/(\beta - 1 + (1-\eta)^2) \qquad z > \eta$$

observing that η (in case of possible ambiguity) is the larger positive root of

$$(\beta - 1 + (1-\eta)^2)(1-y) = (1-\eta)(\beta - \eta - (1-\eta)x) . \tag{3.18}$$

For $x = O(\beta)$ and large β, (3.18) can be written as

$$(y-\eta)^2 - 2(y-\eta)[\beta/2x - (1-y)] + (1-y)^2 + o(1) = 0 . \tag{3.19}$$

Also, rewriting the part of (3.17) relating to $z = O(\beta) \gg \eta$ in the form

$$G_\eta(z) = \eta + (1-\eta)^2 z/\beta + o(1) ,$$

it follows that

$$G_\eta^{2*}(x) = y^2 - (y-\eta)^2/2 + o(1) . \tag{3.20}$$

Combining (3.16) and (3.20) yields (3.10), and substituting from the solution of (3.19) into (3.10) yields (3.11).

Take $1 > y_1 > y_0$, and suppose that the right-hand side of (3.11) exceeds $y_1(1-y)$ for given y in $y_0 < y < y_1$ for all $\xi_0 < x < \beta/4(1-y)$. Then since the infimum of (3.11) with respect to x occurs at $x = \xi_0$, we can define

$$y_1 = \inf_{y_0<y<y_1}\left\{y - (1-y)/2 + \frac{1-y}{1+\left[1-\left(\frac{2(1-y_0)}{1-y}-1\right)^{-2}\right]^{\frac{1}{2}}}\right\} . \tag{3.21}$$

Put $Y = (1-y_0)/(1-y)$, and $Y_1 = (1-y_0)/(1-y_1)$. Then

$$y_1 = 1 - (1-y_0)\sup_{1\leq Y\leq Y_1}\frac{1}{Y}\left\{\frac{3}{2} - \frac{1}{1+[1-(2Y-1)^{-2}]^{\frac{1}{2}}}\right\} . \tag{3.22}$$

Assuming the supremum occurs at some point interior to the interval $(1, Y_1)$, differentiation shows that it occurs where

$$1.5 - 1/(1+W) = 2Y/\{(1+W)^2 W(2Y-1)^3\} \tag{3.23}$$

where $W^2 = 1-(2Y-1)^{-2}$. Simplifying, we get first

$$W(3W+1) = 2(1-W)(1+(1-W^2)^{\frac{1}{2}}) ,$$

and then (rejecting the root $W = 0$)

$$13W^3 + 10W^2 - 3W - 4 = 0 .$$

The only root in (0, 1) is at $W = .5725 \cdots$, whence $Y = 1.1098 \cdots$ and the supremum equals $.778562774 = b$ as at (3.13). Substitution in (3.15) yields (1.4). Note that $(1-y_0)^{-1} = b + (2b)^{\frac{1}{2}} = 2.026 \cdots$ so $y_0 > .5$ and the infimum at (3.15) does occur in $.5 < y_0 < 1$. Also, $Y_1 = (1-y_0)/(1-y_1) = b^{-1} > 1.1098$, so the supremum at (3.22) is interior to $(1, Y_1)$.

4. Concluding remarks. If X is a bounded rv, $X \leqq \sigma\beta$ a.s. say, for some finite $\sigma > 1$, then we can put ξ at (2.6) equal to $\sigma\beta$, dispense with the other term bounding $1 - G(x)$ for $x \geqq \sigma\beta$, and conclude that $U(x) \leqq \lambda x_+ + \sigma\beta$ (all x). This refinement is useful only if $\sigma \leqq 1.3186 \cdots$.

It should be noted that $\lambda^{-1}U(x) - x = V(x)EX$ is the expected length of 'overshoot' of a renewal process beyond x: that is, if $S_0 = 0, S_1, \cdots, S_n = S_{n-1} + X_n, \cdots$ are the successive epochs of a renewal process with $\{X_n\}$ i.i.d. like X, and $N(x) = \inf\{n: S_n \geqq x\}$, then $V(x)EX = ES_{N(x)} - x$. This interpretation does not appear to be of any use for studying $\gamma(F)$.

For any particular df F for which $EX = 1$, and with $G(G^{-1}(y)) = y$ for $0 < y < 1$, the argument behind (3.9) yields

$$\gamma(F) \leqq \inf_{0<y<1}\{G^{-1}(y)/\beta + 1/2y\}. \tag{4.1}$$

For example, if F is such that $G(\beta/4) = \frac{2}{3}$, then immediately $\gamma(F) \leqq 1$. However, since $G^{-1}(y) \geqq y$ (all $0 < y < 1$) (cf. (3.4)), the infimum at (4.1) can be $\leqq 1$ only if $\beta \geqq 2$.

In cases where F has a density function f and the hazard rate $f(x)/R(x)$ is a monotone function, rather better bounds than (1.1) and (2.6) may be available. If the mean residual life $E(X - x \mid X > x)$ is a bounded function of x (which requires that $R(x) = o(x^{-\alpha})$ for every positive α as $x \to \infty$), then Marshall (1973) has given bounds that may be still tighter. A review of related results is contained in Butterworth and Marshall (1974).

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