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ADMISSIBLE TRANSLATES FOR PROBABILITY DISTRIBUTIONS

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A real number t is an admissible translate of a probability φ if $\varphi(A) = 0$ implies that $\varphi_t(A) \equiv \varphi(A - t) = 0$. Conditions are given on its set of admissible translates which ensure that φ has a density. The theorems also describe the set where the density is positive and contain as a corollary the result that if φ is not absolutely continuous, then the set of admissible translates has an empty interior.

1. Introduction. Let φ be a probability distribution on $(R^1, \mathscr{B})$ where $\mathscr{B}$ is the Borel σ-field. A real number t is an *admissible translate* of φ if whenever a Borel set A has φ-measure zero, then $\varphi_t(A) \equiv \varphi(A - t) = 0$. Let $\mathscr{A}(\varphi)$ and $S(\varphi)$ denote respectively the set of admissible translates and the support of φ. Also let $\mathscr{L}$ be Lebesgue measure. In this note are two improvements to the following theorem of Skorokhod (see [3], pages 562–563).

THEOREM (Skorokhod). *Let φ be a probability distribution on $(R^1, \mathscr{B})$ and suppose that $(0, \infty) \subset \mathscr{A}(\varphi)$. Then*

(1) $\varphi \ll \mathscr{L}$,
(2) $S(\varphi) = [a, \infty), \ -\infty \leqq a < \infty$,
and
(3) $d\varphi/d\mathscr{L} > 0$ $\mathscr{L}$-a.e. *on* $S(\varphi)$.

The first theorem to be proved here has weaker hypotheses but yields the same conclusion.

THEOREM 1. *If φ is a probability distribution on $(R^1, \mathscr{B})$, if there is a Borel set $E \subset \mathscr{A}(\varphi) \cap [0, \infty)$ of positive Lebesgue measure, and if there exist admissible translates $x_n \downarrow 0$, then*

(1) $\varphi \ll \mathscr{L}$,
(2) $S(\varphi) = [a, \infty), \ -\infty \leqq a < \infty$,
(3) $\mathscr{A}(\varphi) = R^1$ *or* $[0, \infty)$,
and
(4) $d\varphi/d\mathscr{L} > 0$ $\mathscr{L}$-a.e. *on* $S(\varphi)$.

In the second theorem a hypothesis is removed but now the conclusion is weakened.

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THEOREM 2. *If* φ *is a probability distribution on* $(R^1, \mathscr{B})$ *and if there is a Borel set* $E \subset \mathscr{A}(\varphi) \cap [0, \infty)$ *of positive Lebesgue measure, then*

(1) $\varphi \ll \mathscr{L}$,

(2) $\mathscr{A}(\varphi)$ *is closed and contains an infinite interval* $[\xi, \infty)$.

Furthermore, either

(3) $d\varphi/d\mathscr{L} > 0 \mathscr{L}$-a.e. *on* R^1

or there is a real number s *such that*

(4) $d\varphi/d\mathscr{L} = 0$ $\mathscr{L}$-a.e. *on* $(-\infty, s]$ *and* $d\varphi/d\mathscr{L} > 0$ $\mathscr{L}$-a.e. *on* $[s + \xi, \infty)$.

Questions involving admissible translates have been studied recently in connection with infinitely divisible laws. (See, for example, [2].) It has been shown that if φ is absolutely continuous and infinitely divisible, then $\mathscr{A}(\varphi)$ is one of three possibilities: $[0, \infty]$, $(-\infty, 0]$, or R^1. As a corollary to the results here, we see that if φ is not absolutely continuous, then the interior of $\mathscr{A}(\varphi)$ is empty.

To see that the second theorem does indeed represent a refinement of Skorokhod's result, consider the following example. Let $\{r_n\}$ denote the rationals in $(0, 1)$ and set $\mu_n(A) = \mathscr{L}[A \cap (r_n - \varepsilon_n, r_n + \varepsilon_n)]/2\varepsilon_n$ where ε_n is chosen so that for all n, $0 < r_n - \varepsilon_n < r_n + \varepsilon_n < 1$ and $\sum_1^\infty \varepsilon_n < \frac{1}{4}$ (say). Define $\mu = \sum_{n=1}^\infty 2^{-n-1}\mu_n + \mu_1'$ where $(d\mu'/d\mathscr{L})(x) = e^{-(x-2)}/2$ for $x \geqq 2$ and $(d\mu'/d\mathscr{L})(x) = 0$ for $x < 2$. Clearly $\mu \ll \mathscr{L}$ and it is easy to see that $\mathscr{A}(\mu) = \{0\} \cup [2, \infty)$. But μ is not equivalent to $\mathscr{L}$ on $[0, 1]$ and so $d\mu/d\mathscr{L}$ is not positive $\mathscr{L}$-a.e. on $[0, 1]$.

In order to prove these results some modifications were made in Skorokhod's techniques but the basic method is his. The proof of Lemma 1 uses an argument similar to one which shows that $E - E$ contains an interval whenever $\mathscr{L}(E) > 0$ (e.g., see [1], page 68).

2. Proofs.

LEMMA 1. *Suppose that there is a Borel set* $E \subset \mathscr{A}(\varphi) \cap [0, \infty)$ *of positive Lebesgue measure. Then* $\mathscr{A}(\varphi)$ *contains an infinite interval* $[\xi, \infty)$.

PROOF. First we will show that $\mathscr{A}(\varphi)$ contains a nonempty interval. Indeed, since $\mathscr{L}(E) > 0$ there is an open interval $I = (x - \varepsilon, x + \varepsilon)$ such that $\mathscr{L}(E \cap I) > \frac{3}{4}\mathscr{L}(I)$. Let $y \in (2x - \varepsilon, 2x + \varepsilon)$; we will show that $y \in \mathscr{A}(\varphi)$. If $y - E \cap I$ and $E \cap I$ were disjoint, then the Lebesgue measure of $y - (E \cap I) \cup (E \cap I)$ would be strictly greater than 3ε. But the union above is contained in $(y - I) \cup I$, a set of Lebesgue measure less than 3ε. It follows that $E \cap I$ and $y - E \cap I$ contain a common point z. But then for some $t \in E$, $z = y - t$ or $y = z + t \in E + E$. Now it is easy to see that $\mathscr{A}(\varphi)$ is a semigroup under addition and hence $y \in \mathscr{A}(\varphi)$.

The rest is easy; if $[a, b] \subset \mathscr{A}(\varphi) \cap [0, \infty)$, then again by the semigroup property $[ka, kb] \subset \mathscr{A}(\varphi) \cap [0, \infty]$ for $k = 1, 2, 3, \cdots$. Eventually these intervals will overlap. □

LEMMA 2. *If the condition of Lemma* 1 *holds, then* $\varphi \ll \mathscr{L}$ *and* $\mathscr{A}(\varphi)$ *is closed.*

PROOF. Write $\varphi = \alpha + \beta$ where $\alpha \ll \mathscr{L}$ and $\beta \perp \mathscr{L}$. In the first part of the proof we show that if $s \in \mathscr{A}(\varphi)$, then $s \in \mathscr{A}(\beta)$; that is if $\beta(A) = 0$, then $\beta_s(A) \equiv \beta(A - s) = 0$ for $s \in \mathscr{A}(\varphi)$. Suppose $\beta(A) = 0$ and that $N \in \mathscr{B}$ is such that $\mathscr{L}(N) = 0$ and $\beta(N^c) = 0$. Since $\beta \perp \mathscr{L}$, such a set N exists. Then $\beta_s(N^c + s) = 0$ and $\beta(A \cap (N + s)) = 0$. Also since $\alpha \ll \mathscr{L}$, $\alpha(A \cap (N + s)) = 0$. But $s \in \mathscr{A}(\varphi)$ so $\varphi_s \ll \varphi = \alpha + \beta$ and $\varphi_s(A \cap (N + s)) = 0$. Trivially, $\beta_s \ll \varphi_s$ and therefore $\beta_s(A) = \beta_s(A \cap (N + s)) = 0$ which shows that $\beta_s \ll \beta$ and $s \in \mathscr{A}(\beta)$.

Now define $\bar{\varphi}(A) = \int_E e^{-|s|}\beta_s(A)\, ds$; since the function $s \to \beta(A - s) = \int I_A(s + t)\beta(dt)$ is the integral of a product measurable function, it is measurable. Now

$$\begin{aligned}\bar{\varphi}(A) &= \int e^{-|s|}\left(\int_E I_A(s + t)\beta(dt)\right) ds \\ &= \int_E \left(\int e^{-|s|}I_A(s + t)\, ds\right)\beta(dt) \\ &\leqq \int \mathscr{L}(A - t)\beta(dt) \leqq \mathscr{L}(A - t) = \mathscr{L}(A)\,.\end{aligned}$$

The above inequality shows that $\bar{\varphi} \ll \mathscr{L}$. But since $\beta_s \ll \beta$ for $s \in E \subset \mathscr{A}(\varphi)$, $\bar{\varphi} \ll \beta$. Since $\mathscr{L} \perp \beta$, $\bar{\varphi} \equiv 0$. Thus

$$0 = \bar{\varphi}(R^1) = \int_E e^{-|s|}\beta(R^1)\, ds\,,$$

and hence $\beta(R^1) = 0$ which forces $\beta \equiv 0$ and $\varphi \equiv \alpha$ and so $\varphi \ll \mathscr{L}$.

In order to see that $\mathscr{A}(\varphi)$ is closed, note that since $\varphi \ll \mathscr{L}$, the function $x \to \varphi(A - x)$ is continuous. Let $x_n \in \mathscr{A}(\varphi)$ and suppose that $x_n \to x$. If $\varphi(A) = 0$, then $\varphi(A - x) = \lim_n \varphi(A - x_n) = 0$, and so $x \in \mathscr{A}(\varphi)$. □

PROOF OF THEOREM 1. From Lemma 2 it follows that $\varphi \ll \mathscr{L}$ and by hypothesis, there are admissible translates $x_n \downarrow 0$. An application of Theorem 1 of [2] shows that $S(\varphi) = [a, \infty)$ and $d\varphi/d\mathscr{L} > 0$ $\mathscr{L}$-a.e. on $S(\varphi)$. Since $\mathscr{A}(\varphi)$ is closed, $\mathscr{A}(\varphi) \supset [0, \infty)$. Now if there is a negative admissible translate, then the semigroup property forces $\mathscr{A}(\varphi) = R^1$. □

PROOF OF THEOREM 2. From Lemmas 1 and 2, conclusions (1) and (2) follow immediately. It remains to establish that if $f(t) = (d\varphi/d\mathscr{L})(t)$ and if $[\xi, \infty) \subset \mathscr{A}(\varphi)$, then there is an s (possibly equal to $-\infty$) such that $f = 0$ $\mathscr{L}$-a.e. on $(-\infty, s]$ and $f > 0$ $\mathscr{L}$-a.e. on $[s + \xi, \infty)$. Define

$$\tilde{\varphi}(A) = \int_\xi^\infty \varphi_s(A)e^{-s}\, ds$$

and

$$g(t) = \frac{d\tilde{\varphi}}{d\mathscr{L}}(t)\,.$$

Then $\tilde{\varphi} \ll \varphi$ since $[\xi, \infty] \subset \mathscr{A}(\varphi)$. Since $\varphi([f = 0]) = 0$, $\tilde{\varphi}([f = 0]) = 0$ and so $g = 0$ a.e. on $[f = 0]$. That is, $[f = 0] \subset [g = 0]$ a.e. From the definition of g it follows that

$$g(t) = \int_\xi^\infty f(t - s)e^{-s}\, ds \quad \text{a.e.}$$

Let $t_0 = \sup\{t: \int_\xi^\infty f(t-s)e^{-s}\,ds = 0\}$. Then if $t_0 = \infty$, there exist $t_n \nearrow +\infty$ for which $\int_\xi^\infty f(t_n - s)e^{-s}\,ds = 0$. But this implies that $f = 0$ a.e. on $(-\infty, t_n - \xi) \nearrow R^1$ which contradicts the fact that f is a probability density. If $t_0 = -\infty$, then $g > 0$ a.e. on R^1 and since $[f = 0] \subset [g = 0]$ a.e., $f > 0$ a.e. on R^1. Suppose $-\infty < t_0 < \infty$. Then $f = 0$ a.e. on $(-\infty, t_0 - \xi)$, and since $g > 0$ a.e. on $[t_0, \infty)$, $f > 0$ a.e. on $[t_0, \infty)$. □

We can identify the point t_0 in the above proof of Theorem 2. Let $\alpha = \inf S(\varphi)$; then for $t > \alpha + \xi$, $\int_\xi^\infty f(t-u)e^{-u}\,du > 0$ while for $t < \alpha + \xi$, $\int_\xi^\infty f(t-u)e^{-u}\,du = 0$. Hence $t_0 = \alpha + \xi$.

COROLLARY 1. *If $\mathscr{A}(\varphi)$ contains a Borel set of positive measure and both positive and negative numbers, then φ is equivalent to $\mathscr{L}$ over R^1.*

PROOF. By Lemma 1, $\mathscr{A}(\varphi)$ contains an infinite interval. From the semigroup property and the hypothesis that $\mathscr{A}(\varphi)$ contains both positive and negative numbers, it follows that $\mathscr{A}(\varphi) = R^1$. The corollary is now an easy consequence of Theorem 1. □

COROLLARY 2. *If φ is not absolutely continuous with respect to Lebesgue measure, then the interior of $\mathscr{A}(\varphi)$ is empty.*

PROOF. The interior is an open set and hence has Lebesgue measure zero iff it is empty. □

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