The Annals of Probability
1976, Vol. 4, No. 3, 393-401

SUPERCRITICAL MULTITYPE BRANCHING PROCESSES[1]

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We show that there always exists a sequence of normalizing constants for the supercritical multitype Galton-Watson process so that the normalized sequence converges in probability to a limit which is proper and not identically zero. The Laplace-Stieltjes transform of the limit random variable is characterized as the unique solution under certain conditions of a vector Poincaré functional equation.

1. Introduction. Let $\{\mathbf{Z}_n = (Z_n^{(1)}, \cdots, Z_n^{(d)})\}$ denote a positively regular, supercritical, d-type Galton-Watson process. The matrix M of offspring expectations has maximal eigenvalue ρ, $1 < \rho < \infty$, with corresponding positive left and right eigenvectors $\mathbf{v}$ and $\mathbf{u}$ respectively. A generic point in $\mathbb{R}^d$ will be denoted by $\mathbf{a} = (a_1, \cdots, a_d)$, with $\mathbf{0}$ the zero point, $\mathbf{1} = (1, \cdots, 1)$ the unit element, and $\{\mathbf{e}_i\}_1^d$ the basis vectors, $\mathbf{e}_i$ having 1 in the ith place and zeros elsewhere. The following normalization which uniquely determines $\mathbf{v}$ and $\mathbf{u}$ will hold throughout the paper: $\mathbf{v} \cdot \mathbf{u} = 1$, $\mathbf{u} \cdot \mathbf{1} = 1$. An inequality or limit relation between two vectors or matrices is always to be interpreted as holding componentwise.

In what follows, $\mathbf{F}$ denotes the offspring probability generating function (p.g.f.), that is $\mathbf{F}(\mathbf{x}) = (F^{(1)}(\mathbf{x}), \cdots, F^{(d)}(\mathbf{x}))$ where $F^{(i)}$ is the p.g.f. of $\mathbf{Z}_1$ given that $\mathbf{Z}_0 = \mathbf{e}_i$. The equation $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ has two and only two roots in the unit cube $[0, 1]^d$, $\mathbf{x} = \mathbf{1}$ and $\mathbf{x} = \mathbf{q} < \mathbf{1}$. The root $\mathbf{q}$ is called the extinction probability vector because $q_i = \Pr[\mathbf{Z}_n \to \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i]$. R denotes the matrix with components $R_{ij} = u_i v_j$ and equals $\lim_{n\to\infty} \rho^{-n} M^n$.

By $\mathbf{F}_n$ we denote the nth functional iterate of $\mathbf{F}$. It is well-known that $F_n^{(i)}$ is the p.g.f. of $\mathbf{Z}_n$ given $\mathbf{Z}_0 = \mathbf{e}_i$. If $\mathbf{x} \neq \mathbf{1}$ then $\lim_{n\to\infty} \mathbf{F}_n(\mathbf{x}) = \mathbf{q}$. Proofs of all preceding assertions can be found in the treatise of Harris (1963), to which the reader is referred for supplementary background material.

We shall be making heavy use of an expansion for $\mathbf{F}$ due to Joffe and Spitzer (1967),

$$\mathbf{1} - \mathbf{F}(\mathbf{x}) = (M - E(\mathbf{x}))(\mathbf{1} - \mathbf{x}) \tag{1.1}$$

where $0 \leqq E(\mathbf{x}) \leqq M$, $E(\mathbf{x})$ is nonincreasing in $\mathbf{x}$ (with respect to the partial order induced by $\leqq$) and tends to zero as $\mathbf{x} \to \mathbf{1}$.

Received February 10, 1975; revised June 24, 1975.

[1] This work was supported by a Canada Council Doctoral and Travel Fellowship and was partly written in the Department of Statistics, S.G.S., Australian National University. This paper is taken from the author's doctoral dissertation at Princeton University, written under the direction of Dr. E. Seneta. The author is now at the University of Alberta, Edmonton.

AMS 1970 *subject classifications.* Primary 60J20; Secondary 60F15.

Key words and phrases. Multitype Galton-Watson process, supercritical, positively regular, normalizing constants, Poincaré functional equation, regular variation.

Kesten and Stigum (1966) have shown that the limit $\rho^{-n}\mathbf{Z}_n$ as $n \to \infty$ always exists and is not identically zero iff

$$E[Z_1^{(\alpha)} \log Z_1^{(\alpha)} \mid \mathbf{Z}_0 = \mathbf{e}_\beta] < \infty \qquad \text{for all} \quad 1 \leqq \alpha, \beta \leqq d\,. \tag{1.2}$$

In case (1.2) is satisfied then the limiting random variable is proper, has all its mass concentrated along the direction $\mathbf{v}$ and its mean is finite.

What then if (1.2) is broken? Seneta (1968) has shown for $d = 1$, that there always exists a positive sequence $\{c_n\}$, essentially unique, such that $\lim_{n\to\infty} c_n Z_n = W$ always exists in distribution, where W is proper and not identically zero. (This result was subsequently strengthened by Heyde (1970) to a.s. convergence.) The purpose of this paper is to show that analogous results hold for multitype processes. Our approach is considerably simpler than that of Kesten and Stigum who relied on a difficult truncation argument but we prove convergence in probability rather than a.s. convergence, although we do prove a.s. convergence for the associated scalar process $\{\mathbf{u} \cdot \mathbf{Z}_n\}$ properly normalized, and obtain the more general norming constants. The technique we use combines the functional equation approach of Seneta with the exponential martingale of Heyde.

2. Main results.

THEOREM 2.1. *There exist positive sequences* $\{\mathbf{c}_n\}$ *of vectors and related scalars* $\{\gamma_n\}$ *such that for each* i, *if* $\mathbf{Z}_0 = \mathbf{e}_i$, *then*:

$$\lim_{n\to\infty} \mathbf{c}_n \cdot \mathbf{Z}_n = W^{(i)} \quad \text{a.s.}\,; \tag{2.1}$$

$$\lim_{n\to\infty} \gamma_n/\gamma_{n+1} = \rho\,; \tag{2.2}$$

$$\lim_{n\to\infty} \mathbf{c}_n/\gamma_n = \mathbf{u}\,; \tag{2.3}$$

$$\lim_{n\to\infty} \gamma_n \mathbf{u} \cdot \mathbf{Z}_n = W^{(i)} \quad \text{a.s.}\,; \tag{2.4}$$

$$\lim_{n\to\infty} \gamma_n \mathbf{Z}_n = W^{(i)}\mathbf{v} \quad \textit{in probability; and} \tag{2.5}$$

$$\Pr[W^{(i)} < \infty] = 1\,, \qquad \Pr[W^{(i)} = 0] = q_i\,. \tag{2.6}$$

THEOREM 2.2.

$$\mathbf{c}_n \sim \rho^{-n} L(\rho^{-n})\mathbf{u} \qquad (n \to \infty) \tag{2.7}$$

where $L(s)$ *varies slowly as* $s \to 0$.

$$\lim_{n\to\infty} \rho^n \gamma_n = m \quad \textit{exists for some} \quad 0 < m \leqq \infty\,, \tag{2.8}$$

and $m < \infty$ iff (1.2) *holds, equivalently* iff $E[W^{(i)}] < \infty$ *for one (and then all)* i.

THEOREM 2.3. *Up to a scale factor, there is a unique strictly decreasing and convex (componentwise) solution with* $\boldsymbol{\phi}(0+) = \mathbf{1}$ *to the vector Poincaré equation*

$$\boldsymbol{\phi}(\rho s) = \mathbf{F}(\boldsymbol{\phi}(s))\,, \qquad s \in [0, \infty)\,, \tag{2.9}$$

given by $\boldsymbol{\phi}(s) = (\phi^{(1)}(s), \cdots, \phi^{(d)}(s))$, *where* $\phi^{(i)}(s) = E[\exp(-sW^{(i)})]$. *Furthermore*

$$\lim_{s\to 0} \frac{\mathbf{v} \cdot (\mathbf{1} - \boldsymbol{\phi}(\lambda s))}{\mathbf{v} \cdot (\mathbf{1} - \boldsymbol{\phi}(s))} = \lambda\,, \quad \text{all} \quad \lambda > 0\,, \tag{2.10}$$

and

$$\lim_{s\to\infty}(\mathbf{1}-\boldsymbol{\phi}(s))/\mathbf{v}\cdot(\mathbf{1}-\boldsymbol{\phi}(s))=\mathbf{u}\,. \tag{2.11}$$

3. Preliminary lemmas.

LEMMA 3.1. *There exists a sequence* $\{\mathbf{x}_n\}_{n=0}^{\infty}$ *with* $\mathbf{x}_n \neq \mathbf{q}$ *or* $\mathbf{1}$ *for all n, such that*

$$\mathbf{F}(\mathbf{x}_{n+1}) = \mathbf{x}_n\,, \qquad n = 0, 1, \cdots.$$

PROOF. Let Q denote the closed hypercube $\{\mathbf{x} : \mathbf{q} \leqq \mathbf{x} \leqq \mathbf{1}\}$. By the monotonicity of $\mathbf{F}(\cdot)$ and the fact that $\mathbf{q}$ is a fixed point it follows that the successive images $\mathbf{F}_n(Q)$ are nested, i.e., $Q \supseteqq \mathbf{F}_1(Q) \supseteqq \mathbf{F}_2(Q) \supseteqq \cdots \mathbf{F}_n(Q) \supseteqq \cdots$. Since $\mathbf{F}(\cdot)$ is continuous, the compactness of Q implies that of $\mathbf{F}(Q)$, and then by induction $\mathbf{F}_n(Q)$ is compact for all n. Similarly, since Q is connected, so is $\mathbf{F}_n(Q)$ for all n. Let $\{\mathbf{F}_{n_1}(Q), \cdots, \mathbf{F}_{n_k}(Q)\}$ be an arbitrary finite set of images. Set $n = \max\{n_1, \cdots, n_k\}$. Then by the nested property, $\bigcap_{j=1}^{k} \mathbf{F}_{n_j}(Q) = \mathbf{F}_n(Q)$ and hence arbitrary finite intersections of members of the sequence $\{Q, \mathbf{F}_1(Q), \mathbf{F}_2(Q), \cdots\}$ are connected. Let S denote $\bigcap_{n=1}^{\infty} \mathbf{F}_n(Q)$. Then by a result in Kelley ((1967), page 163) S must also be connected. But $\mathbf{q} \in S$ and $\mathbf{1} \in S$. Hence there exists a third point $\mathbf{x}_0$ in S. Consequently, for each integer $n \geqq 1$, there exists an $\mathbf{x}_n^{(n)}$ such that $\mathbf{x}_0 = \mathbf{F}(\mathbf{x}_n^{(n)})$. Now define for each j $(0 \leqq j \leqq n-1)$, $\mathbf{x}_j^{(n)} = \mathbf{F}(\mathbf{x}_{j+1}^{(n)})$. None of the $\mathbf{x}_j^{(n)}$ equals $\mathbf{1}$ or $\mathbf{q}$, for induction yields $\mathbf{x}_0 = \mathbf{F}_j(\mathbf{x}_j^{(n)})$ and clearly $\mathbf{x}_j^{(n)}$ equals $\mathbf{q}$ or $\mathbf{1}$ iff $\mathbf{x}_0$ equals $\mathbf{q}$ or $\mathbf{1}$ respectively, but our choice of $\mathbf{x}_0$ precludes this.

Next consider the lower triangular array $\{\mathbf{x}_i^{(n)}, n = 0, 1, 2 \cdots; i = 0, 1, 2, \cdots, n\}$. By the Bolzano–Weierstrass theorem we may extract from the second column a convergent subsequence, say $\{\mathbf{x}_1^{(n_1)}, \mathbf{x}_1^{(n_2)}, \cdots\}$, tending to a limit denoted by $\mathbf{x}_1$. Since $\mathbf{F}(\mathbf{x}_1^{(n_j)}) = \mathbf{x}_0$ for all superscripts (n_j) and $\mathbf{F}(\cdot)$ is continuous, it follows that $\mathbf{F}(\mathbf{x}_1) = \mathbf{x}_0$ and thus the assertion of the previous paragraph also applies, namely $\mathbf{x}_1$ equals neither $\mathbf{q}$ nor $\mathbf{1}$. Next from the third column extract a convergent subsequence $\{\mathbf{x}_2^{(m_1)}, \mathbf{x}_2^{(m_2)}, \cdots\}$ tending to a limit $\mathbf{x}_2$. (Note that the sequence of integers $\{m_1, m_2, \cdots\}$ is chosen from $\{n_1, n_2, \cdots\}$.) Thus $\mathbf{F}(\mathbf{x}_2) = \lim_{j\to\infty} \mathbf{F}(\mathbf{x}_2^{(m_j)}) = \lim_{j\to\infty} \mathbf{x}_1^{(m_j)} = \mathbf{x}_1$ and as before $\mathbf{x}_2$ equals neither $\mathbf{q}$ nor $\mathbf{1}$. By this method we inductively generate the required sequence. □

DEFINITION. We call a sequence such as $\{\mathbf{x}_n\}$ a sequence of *backward iterates* under $\mathbf{F}$.

REMARKS. (1) It is relevant to note that in case $d = 1$, the sequence of backward iterates may be chosen as the successive iterates of the inverse function of F for any initial point chosen in the open interval $(q, 1)$.

(2) We do not give a proof, but it will readily follow from Theorem 2.3 that the intersection set S is in fact, a simple curve, connecting the points $\mathbf{q}$ and $\mathbf{1}$, which is invariant under the mapping $\mathbf{F}$, and is also given as

$$S = \{\boldsymbol{\phi}(s) : s \geqq 0\} \cup \{\mathbf{q}\}$$

where $\boldsymbol{\phi}$ is defined by (2.9). Any point on S excepting the end points is a valid starting point for backward iteration and conversely all backward iterates must lie on S. Moreover S is the unique invariant curve for $\mathbf{F}$.

LEMMA 3.2. $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{1}$.

PROOF. Suppose the contrary and let $\{\mathbf{x}_{n_j}\}$ denote any subsequence remaining bounded away from $\mathbf{1}$. Thus we may find $\boldsymbol{\zeta} \neq \mathbf{1}$ such that $\mathbf{q} \leqq \mathbf{x}_{n_j} \leqq \boldsymbol{\zeta}$ for all j. Hence $\mathbf{q} \leqq \mathbf{F}_{n_j}(\mathbf{x}_{n_j}) \leqq \mathbf{F}_{n_j}(\boldsymbol{\zeta})$ and since $\mathbf{F}_{n_j}(\boldsymbol{\zeta}) \to \mathbf{q}$ as $j \to \infty$, we obtain $\mathbf{x}_0 = \mathbf{q}$, a contradiction. □

LEMMA 3.3. $\lim_{n\to\infty} \mathbf{v} \cdot (\mathbf{1} - \mathbf{x}_n)/\mathbf{v} \cdot (\mathbf{1} - \mathbf{x}_{n+1}) = \rho$.

PROOF. This follows from Lemma 3.2, since, from (1.1), $\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}(\mathbf{z}))/\mathbf{v} \cdot (\mathbf{1} - \mathbf{z}) \to \rho$ as $\mathbf{z} \to \mathbf{1}-$. □

LEMMA 3.4. *Let $\{E_j\}$ be a sequence of matrices satisfying $0 \leqq E_j \leqq M$ and $0 \leqq E_j \leqq \beta_j \rho R$ for all j, the inequalities holding componentwise, and $\{\beta_j\}$ being a positive sequence. Then there exists a positive null sequence $\{\delta_j\}$ such that for all $n \geqq 1$*

$$(1 - \delta_n - \textstyle\sum_{j=1}^{n} \beta_j)R \leqq \rho^{-n} \prod_{j=1}^{n} (M - E_j) \leqq (1 + \delta_n)R \,. \tag{3.1}$$

The reader may find a proof in Theorem 3.5 of Seneta ((1973a), page 75) where the result appears in a more general context. This lemma will play a fundamental role in the sequel in obtaining Perron–Frobénius type projection theorems for inhomogeneous products of nonnegative matrices.

Note that in (3.1) and in later occurrences of matrix products, the order of matrix multiplication is immaterial for the purpose at hand.

LEMMA 3.5. $\lim_{n\to\infty} (\mathbf{1} - \mathbf{x}_n)/\mathbf{v} \cdot (\mathbf{1} - \mathbf{x}_n) = \mathbf{u}$.

PROOF. Iteration of (1.1) gives us

$$\mathbf{1} - \mathbf{x}_n = (\textstyle\prod_{j=n+1}^{n+N} (M - E(\mathbf{x}_j)))(\mathbf{1} - \mathbf{x}_{n+N}) \,.$$

The sequence $\{E(\mathbf{x}_j)\}$ fulfills the conditions of Lemma 3.4, and the sequence $\{\beta_j\}$ may be chosen so as to decrease to zero. As a consequence

$$\frac{1 - \delta_N - \sum_{j=n+1}^{n+N} \beta_j}{1 + \delta_N} \mathbf{u} \leqq (\mathbf{1} - \mathbf{x}_n)/\mathbf{v} \cdot (\mathbf{1} - \mathbf{x}_n) \leqq \frac{1 + \delta_N}{1 - \delta_N - \sum_{j=n+1}^{n+N} \beta_j} \mathbf{u}$$

for all sufficiently large N and all sufficiently large n (depending on N), where we have used the fact that $R\mathbf{z}/\mathbf{v}R\mathbf{z} = \mathbf{u}$ for all $\mathbf{z} \neq \mathbf{0}$. By letting first n and then N tend to infinity we see that the lemma is true. □

4. The normalizing constants. We define the constants

$$\mathbf{c}_n = -\log \mathbf{x}_n \tag{4.1}$$

where by this we mean that $c_n^{(i)} = -\log x_n^{(i)}$ $(1 \leqq i \leqq d)$. Similarly $\mathbf{x}_n = \exp(-\mathbf{c}_n) \equiv (\exp(-c_n^{(1)}, \cdots, \exp(-c_n^{(d)})))$, and since $\mathbf{x}_n = \mathbf{F}(\mathbf{x}_{n+1})$ we conclude that

$$\mathbf{c}_n = -\log \mathbf{F}(\exp(-\mathbf{c}_{n+1})) \,. \tag{4.2}$$

In case $d = 1$ these constants reduce to those used by Seneta (1968). Employing a first order multivariate Taylor expansion for the logarithm, and Lemmas 3.3 and 3.5, it is easily verified that

$$\lim_{n\to\infty} \mathbf{v} \cdot \mathbf{c}_n/\mathbf{v} \cdot \mathbf{c}_{n+1} = \rho\,, \qquad \text{and} \tag{4.3}$$

$$\lim_{n\to\infty} \mathbf{c}_n/(\mathbf{v} \cdot \mathbf{c}_n) = \mathbf{u}\,. \tag{4.4}$$

5. A martingale. Let $\mathscr{F}_n$ denote the σ-algebra generated by the set $\{\mathbf{Z}_0, \mathbf{Z}_1, \cdots, \mathbf{Z}_n\}$. Define $W_n = \mathbf{c}_n \cdot \mathbf{Z}_n$ and $Y_n = \exp(-W_n)$. Let $\mathscr{K}$ denote all d-tuples of nonnegative integers.

THEOREM 5.1. *The sequence* $\{Y_n, \mathscr{F}_n\}$ *is a martingale.*

PROOF. $E[|Y_n|] < \infty$ obviously so that it only remains to show that $E[Y_{n+1} \mid \mathscr{F}_n] = Y_n$ a.s. The σ-algebra $\mathscr{F}_n$ is generated by sets of the form $\{\mathbf{Z}_0 = \mathbf{i}_0, \cdots, \mathbf{Z}_{n-1} = \mathbf{i}_{n-1}, \mathbf{Z}_n = \mathbf{k}\}$, $(\mathbf{i}_0, \cdots, \mathbf{i}_{n-1}, \mathbf{k} \in \mathscr{K})$, on each of which the conditional expectation is constant and assumes the value $E[Y_{n+1} \mid \mathbf{Z}_0 = \mathbf{i}_0, \cdots, \mathbf{Z}_n = \mathbf{k}]$, which by the Markov property equals $E[Y_{n+1} \mid \mathbf{Z}_n = \mathbf{k}]$. We decompose the $(n + 1)$th generation as

$$\mathbf{Z}_{n+1} = \sum_{j=1}^{Z_n^{(1)}} \mathbf{V}_j^{(1)} + \cdots + \sum_{j=1}^{Z_n^{(d)}} \mathbf{V}_j^{(d)}$$

where $\mathbf{V}_j^{(\alpha)}$ denotes the offspring in the $(n + 1)$th generation of the jth parent of type α in the nth generation (with appropriate convention for the summation sign if no parents of the type α exist). Recalling the independence of offspring reproduction we have

$$\begin{aligned} &E[Y_{n+1} \mid \mathbf{Z}_n = \mathbf{k}] \\ &\quad = E[\textstyle\prod_{\alpha=1}^{d} \prod_{j=1}^{k_\alpha} \exp(-\mathbf{c}_{n+1} \cdot \mathbf{V}_j^{(\alpha)})] = \prod_{\alpha=1}^{d} \prod_{j=1}^{k_\alpha} E[\exp(-\mathbf{c}_{n+1} \cdot \mathbf{V}_j^{(\alpha)})] \\ &\quad = \textstyle\prod_{\alpha=1}^{d} \prod_{j=1}^{k_\alpha} F^{(\alpha)}(\exp(-\mathbf{c}_{n+1})) = \exp(-\mathbf{c}_n \cdot \mathbf{k})\,. \end{aligned}$$

Thus $E[Y_{n+1} \mid \mathscr{F}_n] = \exp(-\mathbf{c}_n \cdot \mathbf{Z}_n) = Y_n$ a.s. □

6. Proofs.

PROOF OF THEOREM 2.1. Initiating the process with $\mathbf{Z}_0 = \mathbf{e}_i$, we then have by the martingale convergence theorem, the existence of

$$\lim_{n\to\infty} \mathbf{c}_n \cdot \mathbf{Z}_n = W^{(i)} \quad \text{a.s., proving (2.1).}$$

Defining $\gamma_n = \mathbf{v} \cdot \mathbf{c}_n$, the assertions (2.2) and (2.3) are just restatements of (4.3) and (4.4), and then (2.4) is immediate.

If $\phi_n^{(i)}(s) = E[\exp(-\mathbf{c}_n \cdot \mathbf{Z}_n s) \mid \mathbf{Z}_0 = \mathbf{e}_i]$ and $\phi^{(i)}(s) = E[\exp(-sW^{(i)})]$, then by the continuity theorem for Laplace–Stieltjes transforms,

$$\lim_{n\to\infty} \phi_n^{(i)}(s) = \phi^{(i)}(s)\,, \qquad s > 0\,.$$

Introduce the vectors

$$\begin{aligned} \boldsymbol{\phi}(s) &= (\phi^{(1)}(s), \cdots, \phi^{(d)}(s)) \qquad \text{and} \\ \boldsymbol{\phi}_n(s) &= (\phi_n^{(1)}(s), \cdots, \phi_n^{(d)}(s)), \quad n \geqq 1\,, \end{aligned} \tag{6.1}$$

so that

$$\phi_n(s) = \mathbf{F}_n(\exp(-s\mathbf{c}_n)) . \tag{6.2}$$

Given any $\varepsilon > 0$, for all sufficiently large n,

$$\mathbf{c}_{n-1}(1-\varepsilon)/\rho \leqq \mathbf{c}_n \leqq \mathbf{c}_{n-1}(1+\varepsilon)/\rho \quad \text{by (2.2) and (2.3)},$$

which implies by (6.2) that

$$\mathbf{F}(\phi_{n-1}(s(1+\varepsilon)/\rho)) \leqq \phi_n(s) \leqq \mathbf{F}(\phi_{n-1}(s(1-\varepsilon)/\rho)) ,$$

giving in the limit equation (2.9). Next, observe that $\phi(0+)$ and $\phi(\infty)$ are fixed points of $\mathbf{F}(\cdot)$ which we now show cannot be equal. For if so, then $\phi(1)$ is also a fixed point of $\mathbf{F}(\cdot)$, but $\phi(1) = \lim_{n\to\infty} \phi_n(1) = \lim_{n\to\infty} \mathbf{F}_n(\exp(-\mathbf{c}_n)) \equiv \mathbf{x}_0$, a contradiction. The remaining possibility is that $\phi(0+) = \mathbf{1}$ and $\phi(\infty) = \mathbf{q}$ which is exactly (2.6). Thus $q_i = \Pr[W^{(i)} = 0] \geqq \Pr[\mathbf{Z}_n \to \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i] = q_i$ so that the set of nonextinction of the process $\{\mathbf{Z}_n\}$ differs from the set where $W^{(i)} \neq 0$ by a null set. An easy application of Theorem V.6.3 of Athreya and Ney (1972) then proves (2.5) and we are done. □

PROOF OF THEOREM 2.3. Let $\phi(s)$ be any strictly decreasing, convex solution of (2.9) satisfying $\phi(0+) = \mathbf{1}$. The existence of at least one such solution is assured by the construction of $\phi(s)$ above.

For each $s > 0$, the sequence $\{\mathbf{x}_n(s) = \phi(s\rho^{-n})\}_{n=0}^{\infty}$ is a sequence of backward iterates for $\mathbf{F}(\cdot)$. Thus (2.3) and (2.4) hold if we replace $\mathbf{c}_n$ by $\mathbf{d}_n(s) = -\log \mathbf{x}_n(s)$ and γ_n by $\mathbf{v} \cdot \mathbf{d}_n(s)$. Applying Khinchine's theorem on positive types to the scalar sequence $\{\mathbf{u} \cdot \mathbf{Z}_n\}$ we deduce that

$$\mathbf{v} \cdot \mathbf{d}_n(s) \sim K(s)\mathbf{v} \cdot \mathbf{c}_n \quad \text{and} \quad \mathbf{d}_n(s) \sim K(s)\mathbf{c}_n \qquad (n \to \infty) \tag{6.3}$$

for some $K(s)$, $0 < K(s) < \infty$. We next show that $K(s)$ has the form Ks for some constant K.

Introducing $\mathbf{r}(s) = \mathbf{1} - \phi(s)$ and $\mathbf{f}(\mathbf{x}) = \mathbf{1} - \mathbf{F}(\mathbf{1} - \mathbf{x})$ gives us $\mathbf{r}(\rho s) = \mathbf{f}(\mathbf{r}(s))$. Letting $\mathbf{r}_+(s)$ denote the right-hand derivative of $\mathbf{r}(\cdot)$ (componentwise) and $M(\mathbf{x})$ the differential mapping of $\mathbf{F}(\cdot)$ at $\mathbf{x}$ we obtain

$$\rho\mathbf{r}_+(\rho s) = M(\mathbf{1} - \mathbf{r}(s))\mathbf{r}_+(s) . \tag{6.4}$$

For $1 \leqq \lambda \leqq \rho$, $\mathbf{r}_+(\rho s) \leqq \mathbf{r}_+(\lambda s) \leqq \mathbf{r}_+(s)$, and so $\rho\mathbf{v} \cdot \mathbf{r}_+(\rho s) \leqq \rho\mathbf{v} \cdot \mathbf{r}_+(\lambda s) \leqq \rho\mathbf{v} \cdot \mathbf{r}_+(s)$. Substituting (6.4) and noting that $M(\mathbf{x}) \to M$ as $\mathbf{x} \to \mathbf{1}$ and $\mathbf{v}M = \rho\mathbf{v}$ we find $\lim_{s\to 0} \mathbf{v} \cdot \mathbf{r}_+(\lambda s)/\mathbf{v} \cdot \mathbf{r}_+(s) = 1$ which result is easily extended to all $\lambda > 0$. Using the integral representation of a convex function and a standard result of Karamata on slowly varying functions,

$$\lim_{s\to 0} \mathbf{v} \cdot \mathbf{r}(\lambda s)/\mathbf{v} \cdot \mathbf{r}(s) = \lambda . \tag{6.5}$$

Let a, b be given positive numbers. A Taylor expansion shows that $\mathbf{r}(a\rho^{-n}) \equiv \mathbf{1} - \phi(a\rho^{-n}) \sim -\log \phi(a\rho^{-n})$ as $n \to \infty$, the latter equaling $\mathbf{d}_n(a)$. On putting $\lambda = b/a$ and letting s tend to 0 through the values $a\rho^{-n}$, we obtain from (6.5)

$$\lim_{n\to\infty} \mathbf{v} \cdot \mathbf{d}_n(b)/\mathbf{v} \cdot \mathbf{d}_n(a) = b/a \tag{6.6}$$

which together with (6.3) shows that $K(\cdot)$ is linear and the constant K is given by $K(1)$. Let $\varepsilon > 0$ be given. For all sufficiently large n

$$(1-\varepsilon)Ks\mathbf{c}_n \leqq \mathbf{d}_n(s) \leqq (1+\varepsilon)Ks\mathbf{c}_n\,.$$

Therefore

$$\mathbf{F}_n(\exp(-(1+\varepsilon)Ks\mathbf{c}_n)) \leqq \mathbf{F}_n(\exp(-\mathbf{d}_n(s))) \equiv \boldsymbol{\psi}(s) \leqq \mathbf{F}_n(\exp(-(1-\varepsilon)Ks\mathbf{c}_n))\,.$$

Upon taking limits we obtain

$$\boldsymbol{\phi}(Ks) = \boldsymbol{\psi}(s)$$

proving uniqueness up to scale factors. In addition, notice that (6.5) is precisely (2.10).

To conclude the proof of this theorem, it remains only to show (2.11). To this end write $\mathbf{1} - \boldsymbol{\phi}(s) = \mathbf{1} - \mathbf{F}(\boldsymbol{\phi}(s\rho^{-1}))$. Repeated application of (1.1) yields

$$\mathbf{1} - \boldsymbol{\phi}(s) = (\textstyle\prod_{j=1}^{N} [M - E(\boldsymbol{\phi}(s\rho^{-j}))])(\mathbf{1} - \boldsymbol{\phi}(s\rho^{-N}))\,.$$

Let $\{s_k\}$ be an arbitrary sequence decreasing to zero. Since $\boldsymbol{\phi}(0+) = \mathbf{1}$ and $E(\cdot)$ decreases to zero, we may find a positive null sequence $\{\alpha_k\}$ such that $0 \leqq E(\boldsymbol{\phi}(s_k)) \leqq \alpha_k \rho R$ for all k. Moreover, it is apparent by the monotonicity of $\boldsymbol{\phi}(\cdot)$ that $0 \leqq E(\boldsymbol{\phi}(s_k\rho^{-j})) \leqq \alpha_k\rho R$ uniformly in j for each k. Applying Lemma 3.4 we conclude that

$$(1 - \delta_N - N\alpha_k)R \leqq \rho^{-N} \textstyle\prod_{j=1}^{N} [M - E(\boldsymbol{\phi}(s_k\rho^{-j}))] \leqq (1 + \delta_N)R\,.$$

The same argument as in Lemma 3.5 shows that for each sufficiently large N and all sufficiently large k (depending on choice of N)

$$\frac{1-\delta_N - N\alpha_k}{1+\delta_N}\mathbf{u} \leqq \mathbf{1} - \boldsymbol{\phi}(s_k)/\mathbf{v}\cdot(\mathbf{1} - \boldsymbol{\phi}(s_k)) \leqq \frac{1+\delta_N}{1-\delta_N - N\alpha_k}\mathbf{u}\,.$$

Let $k \to \infty$ followed by $N \to \infty$; then since $\{s_k\}$ has been chosen arbitrarily we are done. □

PROOF OF THEOREM 2.2. A first order Taylor expansion together with (2.10) and (2.11) permits us to write $-\log \boldsymbol{\phi}(s) = s\mathscr{L}(s)$ where $\mathscr{L}(s)$ varies slowly in each component and $\mathscr{L}(s)/\mathbf{v}\cdot\mathscr{L}(s) \to \mathbf{u}$ as $s \to 0$. It can be shown that the backward iterates $\mathbf{x}_n$ have the form $\mathbf{x}_n = \boldsymbol{\phi}(\rho^{-n})$ for each n. This implies that $\mathbf{c}_n = \rho^{-n}\mathscr{L}(\rho^{-n}) \sim \rho^{-n}L(\rho^{-n})\mathbf{u}(n \to \infty)$ where we have put $L(s) = \mathbf{v}\cdot\mathscr{L}(s)$. This takes care of (2.7), and in addition shows that $\gamma_n = \rho^{-n}L(\rho^{-n})$.

Next, by (1.1), $\mathbf{1} - \mathbf{x}_n \equiv \mathbf{1} - \mathbf{F}(\mathbf{x}_{n+1}) = (M - E(\mathbf{x}_{n+1}))(\mathbf{1} - \mathbf{x}_{n+1})$ and so $\rho^n\mathbf{v}\cdot(\mathbf{1} - \mathbf{x}_n) = \rho^{n+1}\mathbf{v}\cdot(\mathbf{1} - \mathbf{x}_{n+1}) - \rho^n\mathbf{v}E(\mathbf{x}_{n+1})(\mathbf{1} - \mathbf{x}_{n+1}) \leqq \rho^{n+1}\mathbf{v}\cdot(\mathbf{1} - \mathbf{x}_{n+1})$. The monotone sequence $\{\rho^n\mathbf{v}\cdot(\mathbf{1} - \mathbf{x}_n)\}$ thus tends to a limit $0 < m \leqq \infty$, and therefore $-\rho^n \log \mathbf{v}\cdot\mathbf{x}_n = \rho^n\mathbf{v}\cdot\mathbf{c}_n = \rho^n\gamma_n$ tends to the same limit.

Finally it remains only to prove the circle of implications regarding the finiteness of m. Suppose $m < \infty$. Iterate (1.1) to obtain

$$\mathbf{1} - \boldsymbol{\phi}(1) = \textstyle\prod_{j=1}^{N} [\rho^{-1}M - \rho^{-1}E(\boldsymbol{\phi}(\rho^{-j}))][\rho^N(\mathbf{1} - \boldsymbol{\phi}(\rho^{-N}))]\,.$$

It follows that

$$\lim_{N\to\infty} \prod_{j=1}^{N} [\rho^{-1}M - \rho^{-1}E(\boldsymbol{\phi}(\rho^{-j}))]\mathbf{u} = (\mathbf{1} - \boldsymbol{\phi}(1))/m$$

and the right side being nonzero we may apply Lemma 1 of Joffe and Spitzer and mimic the proof of their Theorem 4 to conclude that (1.2) must hold.

If (1.2) holds then by Khinchine's theorem on positive types applied to the sequence $\{\mathbf{u} \cdot \mathbf{Z}_n\}$ and the result of Kesten and Stigum, it follows that $E[W^{(i)}] < \infty$ for all i. But since $\lim_{s\to 0} (1 - \phi^{(i)}(s))/s = E[W^{(i)}] < \infty$ this implies that $\mathbf{v} \cdot \mathscr{L}(0+) < \infty$ which implies that $\lim_{n\to\infty} \rho^n \gamma_n = \mathbf{v} \cdot \mathscr{L}(0+)$ is finite and so $m < \infty$. This latter argument, incidentally, also shows that if $EW^{(i)}$ is finite for some i then it is finite for all i. □

7. Concluding remarks. When (1.2) is satisfied, $E[W^{(i)}] = Cu_i$, $1 \leqq i \leqq d$, for some positive constant C. If (1.2) is not fulfilled, then while the expectations of the limit variables are infinite, the vector $\mathbf{u}$ still appears in a similar fashion since, from Section 6, $1 - \phi^{(i)}(s) \sim sL(s)u_i (s \to 0)$, so by Karamata's Tauberian theorem

$$\int_0^x \Pr [W^{(i)} > t]\, dt \sim u_i L(x^{-1}) \quad \text{as} \quad x \to \infty\,.$$

Thus $\lim_{x\to\infty} \int_0^x \Pr [W^{(i)} > t]\, dt / \int_0^x \Pr [W^{(j)} > t]\, dt = u_i/u_j$. (When (1.2) is satisfied the left side of this equation equals $E[W^{(i)}]/E[W^{(j)}]$.)

We may also apply a density version of this Tauberian theorem due to Seneta (1973b) to obtain

$$\Pr [W^{(i)} > x] = o(x^{-1}L(x^{-1})) \qquad (x \to \infty)\,,$$

and consequently

$$E[(W^{(i)})^\alpha] < \infty \qquad \text{for all} \quad 0 \leqq \alpha < 1\,.$$

In one dimension these results are contained in Seneta (1974).

It is also possible to show, along the lines of Theorem I.10.4 of Athreya and Ney, that the limit random variable $W^{(i)}$ has an absolutely continuous distribution on the positive reals but the details are too technical to be included here; a proof is contained in the author's doctoral dissertation.

Although we have proved the almost sure convergence of the normalized $\mathbf{u} \cdot \mathbf{Z}_n$ we have only managed to get convergence in probability of the individual components. It is inconceivable that a.s. convergence would fail, but a proof under no further assumptions appears formidable.

In a sequel to this paper we treat the supercritical immigration process by similar techniques, extending other work of Seneta (1970). The approach generalizes to the case where the immigration distribution is allowed to vary from generation to generation according to a (possibly infinite) ergodic Markov chain. We shall also consider decomposable Galton–Watson processes when the assumptions of Kesten and Stigum (1967) are not met.

Acknowledgments. I wish to thank my advisor, Dr. E. Seneta, for introducing me to the subject matter herein presented and for critical comments and assistance during various phases of the research. I also wish to record my appreciation to Miss Aida Simanjuntak, to whom this paper is fondly dedicated.

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