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MULTIPARAMETER SUBADDITIVE PROCESSES[1]

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Let N be the positive integers. We define a class of processes indexed by $N^r \times N^r$ which we call *subadditive* (when $r = 1$ our definition coincides with the usual one). Under a first moment condition we prove mean convergence of $x_{0\mathbf{t}}/|\mathbf{t}|$ as each coordinate of $\mathbf{t} \to \infty$, where $|\mathbf{t}| = t_1 t_2 \cdots t_r$. If the process is *strongly subadditive* (a more restrictive condition) then the same first moment condition gives a.s. sectorial convergence. We conjecture (and verify in several cases) that an $L(\log L)^{r-1}$ integrability condition is sufficient to give unrestricted a.s. convergence.

0. Introduction and notation. Subadditive processes were introduced by Hammersley and Welsh (1965) in the context of percolation theory. An excellent account of the properties and uses of subadditive processes is given in Kingman (1973) (although, as noted by Kingman, his stationarity postulate (S_2) is stronger than that used by Hammersley and Welsh).

To introduce the reader to the essential features of the theory, we will briefly outline some results in the one-dimensional case.

Consider a family $\{x_{st}, s < t\}$ of random variables, where s and t belong to the set N of nonnegative integers. In Kingman's formulation, the process $\{x_{st}\}$ is said to be *subadditive* if the following three conditions hold:

S_1. Whenever $s < t < u$, $x_{su} \leqq x_{st} + x_{tu}$.

S_2. The joint distributions of the process $\{x_{s+1,t+1}\}$ are the same as those of $\{x_{st}\}$.

S_3. $g_t \equiv E(x_{0t}) < \infty$, and $g_t \geqq -At$ for some constant A and all $t > 1$.

We first note that from the theory of subadditive functions and S_3 it follows that

$$\lim_{t\uparrow\infty} g_t/t = \gamma < \infty\,. \tag{0.1}$$

The constant γ is dubbed by Hammersley and Welsh the "time constant" of the process.

One of the principal results of the theory, due to Kingman (1968) is that

$$\xi \equiv \lim_{t\to\infty} x_{0t}/t \quad \text{exists a.s. and in } L^1\,, \quad \text{and} \quad E(\xi) = \gamma\,. \tag{0.2}$$

The proof of this result depends on the decomposition

$$x_{st} = y_{st} + z_{st} \tag{0.3}$$

where y is an additive process (meaning that there is equality in S_1) with

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$E(y_{01}) = \gamma$, and z is a nonnegative subadditive process with time constant equal to zero.

Our goal here is to generalize the theory of subadditive processes to the multidimensional case, i.e., we wish to consider processes $\{x_{\mathbf{st}}, \mathbf{s} < \mathbf{t}\}$ where $\mathbf{s}$ and $\mathbf{t}$ are vectors in N^r, equipped with the usual coordinatewise ordering. (We write $\mathbf{s} < \mathbf{t}$ if $\mathbf{s} \leqq \mathbf{t}$ and $s \neq t$, and $s \ll t$ if $s_i < t_i$, $i = 1, 2, \cdots, n$.)

First we must develop a suitable definition of the subadditive property analogous to S_1. Hammersley (1974) appears to suggest the following version of S_1 on page 671:

$$\text{(0.4)} \qquad \text{Whenever} \quad \mathbf{s} < \mathbf{t} < \mathbf{u}\,, \qquad x_{\mathbf{su}} < x_{\mathbf{st}} + x_{\mathbf{tu}}\,.$$

This might be an appropriate formulation in which to study, say, percolation processes on the square lattice, but it is not the geometrical analogue of S_1 (as Hammersley himself recognizes on the following page of [4]). In the one-dimensional case, regard x_{st} as a random set function assigning a value to the interval $[s, t)$; condition S_1 then expresses the subadditivity over intervals of this random set function, and the process is additive whenever x_{st} defines a finitely additive set function. Extending this approach to dimension 2 (say), the analogous condition would be

$\mathbf{S}_1$. Whenever $\mathbf{s} < \mathbf{u}$ and $s_2 < t_2 < u_2$, $x_{\mathbf{su}} \leqq x_{\mathbf{s}(u_1,t_2)} + x_{(s_1,t_2)\mathbf{u}}$; whenever $\mathbf{s} < \mathbf{u}$ and $s_1 < t_1 < u_1$, $x_{\mathbf{su}} \leqq x_{\mathbf{s}(t_1,u_2)} + x_{(t_1,s_2)\mathbf{u}}$, with the obvious generalization to $r > 2$.

In many respects this definition would seem to be the most natural extension of S_1. However, at least with respect to the present techniques, it appears to be too general to yield the strong form of our results. Consider the following reformulation of S_1:

S_1'. Whenever $s < t < u$, $x_{su} - x_{tu} \leqq x_{st}$.

Obviously S_1 and S_1' are identical, but consider the two-dimensional analogue of S_1':

$\mathbf{S}_1'$. Whenever $\mathbf{s} < \mathbf{t} < \mathbf{u}$,

$$x_{\mathbf{su}} - x_{(s_1,t_2)\mathbf{u}} - x_{(t_1,s_2)\mathbf{u}} + x_{\mathbf{tu}} \leqq x_{\mathbf{st}}\,.$$

It is easy to check that S_1 and S_1' are not equivalent, but if $x_{\mathbf{st}} = 0$ whenever $s_1 = t_1$ or $s_2 = t_2$, then S_1' will imply S_1.

Turning next to S_2, its extension is straightforward:

$\mathbf{S}_2$. The joint distributions of the processes $\{x_{(s_1+1,s_2)(t_1+1,t_2)}\}$ and $\{x_{(s_1,s_2+1)(t_1,t_2+1)}\}$ are the same as those of $\{x_{\mathbf{st}}\}$.

Now consider S_3. We have $x_{0t}^+ \leqq (x_{01} + \cdots + x_{t-1,t})^+ \leqq x_{01}^+ + \cdots + x_{t-1,t}^+$ (where we used S_1 in the first inequality). Thus, by S_2,

$$\text{(0.5)} \qquad E(x_{0t}^+) \leqq tE(x_{01}^+)$$

so that, under the condition $E(x_{0t}) < \infty$, we always have

$$\text{(0.6)} \qquad \sup_t E\left(\frac{x_{0t}^+}{t}\right) \leqq E\,(x_{01}^+) < \infty\,.$$

Condition S_3 is therefore equivalent to

$$\sup_t E\left|\frac{x_{0t}}{t}\right| < \infty . \tag{0.7}$$

Hence our final condition (where for $\mathbf{t} = (t_1, \cdots, t_r)$, $|\mathbf{t}| \equiv t_1 t_2 \cdots t_r$):

$\mathbf{S}_3$. $\sup_{\mathbf{t}} E|x_{0t}/|\mathbf{t}|| < \infty$.

DEFINITION 0.1. A process $\{x_{\mathbf{st}}, \mathbf{s} < \mathbf{t}\}$, where $\mathbf{s}, \mathbf{t} \in N^2$, is *subadditive* if $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ are satisfied. If $\mathbf{S}_1'$, $\mathbf{S}_2$, and $\mathbf{S}_3$ are satisfied, the process will be called *strongly subadditive.*

Definition 0.1 extends in the obvious way to processes $\{x_{\mathbf{st}}\}$ with $\mathbf{s}, \mathbf{t} \in N^r$. To keep the notation manageable we will always work in N^2, indicating where changes need to be made to give the corresponding results in N^r.

Here are some examples of subadditive processes.

EXAMPLE 1 (Modification of an example of Hammersley and Welsh (1965)). Let straight lines be distributed on the plane uniformly and independently at random (i.e., their directions are uniformly and independently distributed between 0 and 2π, and their perpendicular distances from the origin are the points of a Poisson process on the positive reals.) These lines divide the plane into convex polygons. Given $\mathbf{s}, \mathbf{t}$ in N^2 with $\mathbf{s} \ll \mathbf{t}$, let $R_{\mathbf{st}}$ be the rectangle with lower left corner at $\mathbf{s}$ and upper right corner at $\mathbf{t}$. Let $x_{\mathbf{st}}$ be the number of polygons of some given class (acute triangles, for example) which intersect $R_{\mathbf{st}}$. Then $\{x_{\mathbf{st}}\}$ is strongly subadditive.

EXAMPLE 2 (Hammersley (1974)). Let $d > 0$ be fixed. Distribute points in R^n according to a stationary point process (e.g., Poisson). Draw a sphere of radius d around each point and let $R_{\mathbf{st}}$ be defined as in Example 1. Let $x_{\mathbf{st}}$ be the area covered by the spheres whose interiors are wholly contained in $R_{\mathbf{st}}$. Then $\{-x_{\mathbf{st}}\}$ is strongly subadditive.

EXAMPLE 3. Let $\mathscr{S}$ be a family of sets in R^2, stable under finite union and intersection, which contains all the rectangles $R_{\mathbf{st}}$, $\mathbf{s} \ll \mathbf{t}$. Suppose that for each fixed $S \in \mathscr{S}$, $\omega \to C(S, \omega)$ is a random variable, and that for each fixed ω, $S \to C(S, \omega)$ is a subadditive Choquet capacity on $\mathscr{S}$ (cf. Meyer (1966), page 39), i.e., $C(\cdot, \omega)$ is a capacity such that $C(S \cup T, \omega) \leqq C(S, \omega) + C(T, \omega)$ for S, T disjoint. Using the relation

$$C(P \cup Q \cup R, \omega) - C(P \cup R, \omega) - C(Q \cup R, \omega) + C(R, \omega) \leqq 0 ,$$

it is easy to show that the process $x_{\mathbf{st}} \equiv C(R_{\mathbf{st}}, \omega)$ is strongly subadditive. In particular if C is nonnegative it will be subadditive in the sense given above, so that *every (measurable) nonnegative random capacity defines a strongly subadditive process.* Example 1 is of this type; but Example 2 shows that not every strongly subadditive process arises from a capacity, since the process $\{-x_{\mathbf{st}}\}$ is obviously not monotonic increasing on rectangles. The (unresolved) question then arises: which nonnegative, monotonic increasing, strongly subadditive functions on the rectangles R_{st} can be extended to a capacity on some class of sets containing the rectangles?

1. The L^1-convergence theorem. First we must address ourselves to the question of the existence of a "time constant" γ for our processes. When we write "$\mathbf{t} \to \infty$" we shall mean that each coordinate $t_i \to \infty$ independently.

PROPOSITION 1.1. *Let $\{x_{st}\}$ be subadditive, and let $g(\mathbf{t}) \equiv E(x_{0t})$. Then $\lim_{\mathbf{t}\to\infty} g(\mathbf{t})/|\mathbf{t}|$ exists.*

PROOF. We claim that the limit γ above is in fact given by $\gamma = \inf_{t_1 \geqq 1; t_2 \geqq 1} g(\mathbf{t})/|\mathbf{t}|$. As in the usual proof of the one-dimensional result (cf. for example, Chung (1967), page 131), start by picking $\mathbf{t}^\circ = (t_1{}^\circ, t_2{}^\circ)$ such that

$$g(\mathbf{t}^\circ)/|\mathbf{t}^\circ| < \gamma + \varepsilon, \qquad \text{where} \quad \varepsilon > 0 .$$

Given $\mathbf{t}$, write $\mathbf{t} = (n_1 t_1{}^\circ + \delta_1, n_2 t_2{}^\circ + \delta_2)$ where $0 \leqq \delta_1 < t_1{}^\circ$, $0 \leqq \delta_2 < t_2{}^\circ$, and n_1, n_2 are positive integers. Since the function $g(\mathbf{t})$ is subadditive in the sense of $\mathbf{S}_1$, we have $g(\mathbf{t})/|\mathbf{t}| \leqq 1/|\mathbf{t}|\{g(n_1 t_1{}^\circ, n_2 t_2{}^\circ) + g(n_1 t_1{}^\circ, \delta_2) + g(\delta_1, n_2 t_2{}^\circ) + g(\delta_1, \delta_2)\}$. But by stationarity $g(n_1 t_1{}^\circ, n_2 t_2{}^\circ) \leqq n_1 n_2 g(\mathbf{t}_0)$, so

$$\frac{g(\mathbf{t})}{|\mathbf{t}|} \leqq \frac{g(\mathbf{t}_0)}{|\mathbf{t}_0|} + \frac{1}{|\mathbf{t}|}\{g(n, t_1{}^\circ, \delta_2) + g(\delta_1, n_2 t_2{}^\circ) + g(\delta_1, \delta_2)\} . \tag{1.1}$$

Now for any fixed δ_1 with $1 \leqq \delta_1 < t_1{}^\circ$ and δ_2 with $1 \leqq \delta_2 < t_2{}^\circ$, the functions $g(\delta_1, s)$ and $g(u, \delta_2)$ are subadditive functions of s and u respectively, by $\mathbf{S}_1$. Therefore the limits $\lim_{s\to\infty} g(\delta_1, s)/s$ and $\lim_{u\to\infty} g(u, \delta_2)/u$ exist and are finite (Chung (1967), page 131). It follows that $\lim_{\mathbf{t}\to\infty} \sup_{0 \leqq \delta_1 < t_1{}^\circ} (1/|\mathbf{t}|) g(n_1 t_1{}^\circ, \delta_2)$ and $\lim_{\mathbf{t}\to\infty} \sup_{0 \leqq \delta_2 < t_2{}^\circ} (1/|\mathbf{t}|) g(\delta_1, n_2 t_2{}^\circ)$ are both zero, since $n_1 \to \infty$ and $n_2 \to \infty$ as $\mathbf{t} \to \infty$. Clearly $\lim_{\mathbf{t}\to\infty} g(\delta_1, \delta_2)/|\mathbf{t}| = 0$, so from (1.1) we have

$$\limsup_{\mathbf{t}\to\infty} g(\mathbf{t})/|\mathbf{t}| \leqq \gamma + \varepsilon . \tag{1.2}$$

Since ε is arbitrary, it follows by definition of γ that $g(\mathbf{t})/|\mathbf{t}| \to \gamma$ as $\mathbf{t} \to \infty$.

In the remainder of this section we will investigate the convergence in L^1 of $z_{\mathbf{t}}$ as $\mathbf{t} \to \infty$. Here and later in Section 2 we shall need to appeal to several classical multiparameter ergodic theorems. For convenience we will state and reference all of them at this point.

THEOREM A (Riesz–Dunford). *Let $\{X_{\mathbf{j}}\}_{\mathbf{j} \in N^r}$ be an array of random variables stationary under each shift θ_i, $1 \leqq i \leqq r$, where θ_i: $X_{j_1,\cdots,j_i,\cdots,j_r} \to X_{j_1,\cdots,j_i+1,\cdots,j_r}$. Let $S_{\mathbf{n}} = \sum_{\mathbf{j} \leqq \mathbf{n}} X_{\mathbf{j}}$ and let $\mathbf{1} = (1, 1, \cdots, 1)$.*

If $X_{\mathbf{1}}$ is integrable, then $S_{\mathbf{n}}/|\mathbf{n}|$ converges in L^1.

PROOF. As observed by Dunford (1951), this theorem is essentially a consequence of a mean ergodic theorem of F. Riesz (1938); the proof is implicit in the proof of the main theorem in [3].

THEOREM B (Dunford–Zygmund). *Let $\{X_{\mathbf{j}}\}_{\mathbf{j} \in N^r}$ and $S_{\mathbf{n}}$ be as in Theorem* A.

If $E(|X_{\mathbf{1}}|(\log^+|X_{\mathbf{1}}|)^{r-1}) < \infty$, then $S_{\mathbf{n}}/|\mathbf{n}|$ converges a.s. *and in* L^1.

PROOF. Dunford (1951) for $r = 2$; Zygmund (1951) for $r > 2$, in the continuous case. It was shown by the author in [11] that the integrability condition in Theorem B cannot in general be weakened.

THEOREM C (Zygmund). *Let* $\{x_{\mathbf{j}}\}_{\mathbf{j}\in N^r}$ *and* $S_{\mathbf{n}}$ *be as in Theorem* A. *Let* T_r^α *be the sector of r-space defined by*

$$T_r^\alpha = \{(i_1, \cdots, i_r) \in N^r \,|\, \alpha i_k < i_l \leqq \alpha^{-1} i_k;\ 1 \neq k,\ 1, k = 1, 2, \cdots, r\}\,.$$

If X_1 *is integrable, then* $\lim_{\mathbf{n}\to\infty;\mathbf{n}\in T_r^\alpha} S_{\mathbf{n}}/|\mathbf{n}|$ *exists and is finite* a.s.

PROOF. Zygmud (1951).

Here is our principal result on L^1-convergence of $z_{\mathbf{t}}$.

THEOREM 1.1. *Let* $\{x_{\mathbf{st}}\}$ *be subadditive with time constant* γ. *Then* $z_{\mathbf{t}}$ *converges in* L^1 *to a limit* ξ, *were* $E(\xi) = \gamma$.

If in addition $E(|x_{\mathbf{0t}}| \log^+ |x_{\mathbf{0t}}|) < \infty$ *for all* $\mathbf{t} \in N^2$, *then* $\limsup_{\mathbf{t}\to\infty} z_{\mathbf{t}} = \xi$ a.s. *The corresponding result holds in* N^r *if* $\log^+ |x_{\mathbf{0t}}|$ *above is replaced by* $(\log^+ |x_{\mathbf{0t}}|)^{r-1}$.

PROOF. To show that $z_{\mathbf{t}}$ converges in L^1 it is enough to show first that, given any increasing sequence $\{\mathbf{t}_k\}_{k\in N}$ of points in N^2 such that $t_k^1 \to \infty$ and $t_k^2 \to \infty$ as $k \to \infty$, there exists a subsequence on which $z_{\mathbf{t}}$ is L^1-convergent; and second, that one obtains the same limit no matter which sequence is selected.

Let such a sequence $\{\mathbf{t}_k\}$ be chosen and fix n for the moment. Then

$$\mathbf{t}_k = (ni_k + \delta_1, nj_k + \delta_2)\,, \qquad \text{where} \quad 0 \leqq \delta_1 < i_k \quad \text{and} \quad 0 \leqq \delta_2 < j_k\,.$$

We have

$$(1.3) \qquad x_{\mathbf{0t}_k} \leqq \sum_{r_1=1}^{i_k} \sum_{r_2=1}^{j_k} x_{([r_1-1]n,[r_2-1]n)(r_1 n, r_2 n)} + x_{(i_k n, j_k n)(t_k^1, t_k^2)} + x_{(0, j_k n)(i_k n, t_k^2)} + x_{(i_k n, 0)(t_k^1, j_k n)}\,.$$

Consider the first term on the right-hand side of (1.3). It is a partial sum associated with a stationary process which is integrable; by Theorem A, the limit

$$(1.4) \qquad u_n \equiv \lim_{k\to\infty} (i_k j_k)^{-1} \sum_{r_1=1}^{i_k} \sum_{r_2=1}^{j_k} x_{([r_1-1]n,[r_2-1]n)(r_1 n, r_2 n)}$$

exists in L^1. Hence we can choose a subsequence from $\{(i_k, j_k)\}_{k\in N}$ (which we continue to denote for typographical convenience as $\{(i_k, j_k)\}$) such that $i_k \to \infty$ and $j_k \to \infty$ as $k \to \infty$ and the limit in (1.4) exists a.s.

Now consider the remaining terms on the right-hand side of (1.3), where k is restricted to the subsequence chosen above. Define

$$S_{\mathbf{t}_k} \equiv \sum_{l_1=1}^{t_k^1-1} \sum_{l_2=1}^{t_k^2-1} x_{(l_1, l_2)(l_1+1, l_2+1)}$$

and note that the $S_{\mathbf{t}_k}$ are partial sums of an integrable stationary process. Now

$$(1.5) \qquad x_{(i_k n, j_k n)(t_k^1, t_k^2)} + x_{(0, j_k n)(i_k n, t_k^2)} + x_{(i_k n, 0)(t_k^1, j_k n)} \leqq S_{\mathbf{t}_k} - S_{i_k n, j_k n}$$

by subadditivity. By Theorem A, $S_{\mathbf{t}_k}/|\mathbf{t}_k|$ and $S_{i_k n, j_k n}/i_k j_k n^2$ are L^1-convergent to the same limit as $k \to \infty$; since $|\mathbf{t}_k|/i_k j_k n^2 \to 1$ as $k \to \infty$ it follows that $(S_{\mathbf{t}_k} - S_{i_k n, j_k n})/|\mathbf{t}_k| \to 0$ in L^1 as $k \to \infty$. Hence we can pick a further subsequence, which we once again denote $\{(i_k, j_k)\}$, such that

(1.6) $\limsup_{k\to\infty} (i_k j_k)^{-1}\{x_{(i_k n, j_k n)(t_k^1, t_k^2)} + x_{(0, j_k n)(i_k n, t_k^2)} + x_{(i_k n, 0)(t_k^1, j_k n)}\} \leqq 0$ a.s.

We thus have, for each n, a subsequence $\{(i_k, j_k)\}$ satisfying (1.4) (a.s.) and (1.6). Using a standard diagonal argument, we may assume that the same subsequence satisfies (1.4) (a.s.) and (1.6) for *all n*. From (1.3), (1.4) and (1.6) we then deduce that

$$\xi \equiv \limsup_{k\to\infty} x_{\mathbf{0}\mathbf{t}_k}/|\mathbf{t}_k| \leqq u_n/n^2 . \tag{1.7}$$

Hence $E(\xi) \leqq E(u_n/n^2)$ for each n; but if $\mathbf{n} = (n, n)$, $E(u_n/n^2) = g(\mathbf{n})/|\mathbf{n}|$, and since $E(\xi) \leqq g(\mathbf{n})/|\mathbf{n}|$ for each n, we have $E(\xi) \leqq \lim_{n\to\infty} g(\mathbf{n})/|\mathbf{n} = \gamma$, by Proposition 1.1. The proof that $E(\xi) \geqq \gamma$ and that $z_{\mathbf{t}_k} \to \xi$ in L^1 as $k \to \infty$ can now be completed as in [6], page 502.

It remains to be shown that the limit ξ obtained above is the same no matter which sequence $\{\mathbf{t}_k\}$ we start with. This is done much as in [6], page 504. Let $\mathscr{I}$ be the σ-field of events generated by the process $\{x_{\mathbf{s}\mathbf{t}}\}$ and invariant under both the shifts $\theta_1 : x_{(s_1,s_2)(t_1,t_2)} \to x_{(s_1+1,s_2)(t_1+1,t_2)}$ and $\theta_2 : x_{(s_1,s_2)(t_1,t_2)} \to x_{(s_1,s_2+1)(t_1,t_2+1)}$. Let $\Phi_{\mathbf{t}}$ be a version of the conditional expectation $E(x_{\mathbf{0}\mathbf{t}} \mid \mathscr{I})$. By stationarity,

$$E(x_{\mathbf{s}\mathbf{t}} \mid \mathscr{I}) = \Phi_{\mathbf{t}-\mathbf{s}}$$

and by subadditivity it follows that, if $\mathbf{0} \ll \mathbf{t} \ll \mathbf{u}$,

$$\Phi_{\mathbf{u}} \leqq \Phi_{\mathbf{t}} + \Phi_{(t_1, u_2 - t_2)} + \Phi_{(u_1 - t_1, t_2)} + \Phi_{(u_1 - t_1, u_2 - t_2)} \tag{1.8}$$

for all ω in a set of probability one (which can be chosen independent of $\mathbf{t}$ and $\mathbf{u}$). Hence by Proposition 1.1,

$$\phi = \lim_{\mathbf{t}\to\infty} \Phi_{\mathbf{t}}/|\mathbf{t}|$$

exists a.s.; we claim that the limit ξ found above is a.s. equal to ϕ, and hence independent of the sequence $\{\mathbf{t}_k\}$. To see this note that since $x_{\mathbf{0}\mathbf{t}_k}/|\mathbf{t}_k| \to \xi$ in L^1, then $E(x_{\mathbf{0}\mathbf{t}_k}/|\mathbf{t}_k| \mid \mathscr{I}) \to E(\xi \mid \mathscr{I})$ in L^1. But it is easy to show that ξ is an invariant random variable, so that $E(\xi \mid \mathscr{I}) = \xi$ a.s.; thus $\phi = \lim_{k\to\infty} E(x_{\mathbf{0}\mathbf{t}_k}/|\mathbf{t}_k| \mid \mathscr{I}) = \xi$ a.s., completing the proof of the first statement of Theorem 1.1.

The proof of the second statement follows by applying the same argument to arbitrary t that was used on the sequence $\{\mathbf{t}_k\}$ and using Theorem B instead of Theorem A. We write

$$\mathbf{t} = (in + \delta_1, jn + \delta_2) \qquad \text{where} \quad 0 \leqq \delta_1 < i , \quad 0 \leqq \delta_2 < j .$$

If $E(|x_{\mathbf{0}\mathbf{t}}| \log^+ |x_{\mathbf{0}\mathbf{t}}|) < \infty$ for all $\mathbf{t} \in N^2$ we apply Theorem B to get (1.4) (a.s.) without passing to a subsequence. Instead of (1.7) we then arrive at the statement

$$\eta \equiv \limsup_{\mathbf{t}\to\infty} x_{\mathbf{0}\mathbf{t}}/|\mathbf{t}| \leqq n^{-2} \lim_{i,j\to\infty} \sum_{r_1=1}^{i} \sum_{r_2=1}^{j} x_{([r_1-1]n,[r_2-1]n)(r_1 n, r_2 n)} . \tag{1.9}$$

This gives $\eta < \infty$ a.s., $E(\eta) \leqq g(\mathbf{n})/|\mathbf{n}|$ as before; the proof that $z_{\mathbf{t}} \to \eta$ in L^1 and hence that $\eta = \xi = \phi$ a.s. is virtually the same as before.

COROLLARY 1.1. *Let $\{x_{st}\}$ be subadditive with time constant γ, and let S_r^α be a sector of r-space as in Theorem* C. *Then* $\limsup_{t\to\infty; t S_r^\alpha} z_t = \xi$ a.s., *where ξ is given in Theorem* 1.1.

PROOF. This is proved just like the second statement of Theorem 1.1, except that here we use Theorem C instead of Theorem B.

2. Convergence a.s. To get a.s. convergence of z_t using the present techniques we need to assume that $\{x_{st}\}$ is strongly subadditive. This will enable us to decompose $\{x_{st}\}$ in a manner analogous to that of (0.3). The proof of this basic result is modeled on that given by Burkholder (1973) in the one-dimensional case.

PROPOSITION 2.1 *Let $\{X_{st}\}$ be strongly subadditive. Then $x_{st} = y_{st} + w_{st}$, where $\{y_{st}\}$ is an additive process with $E(y_{01}) = \gamma$ and $\{w_{st}\}$ is a nonnegative subadditive process with time constant zero.*

PROOF. For each $\mathbf{k}$ and $\mathbf{n} \in N^2$, define

$$f_{\mathbf{kn}} = \frac{1}{|\mathbf{n}|} \sum_{\mathbf{1}\le\mathbf{r}\le\mathbf{n}} (x_{\mathbf{k},\mathbf{k}+\mathbf{r}} - x_{(k_1+1,k_2)\mathbf{k}+\mathbf{r}} - x_{(k_1,k_2+1)\mathbf{k}+\mathbf{r}} + x_{\mathbf{k}+1,\mathbf{k}+\mathbf{r}}) . \tag{2.1}$$

For each fixed $\mathbf{n}$, the array $\{f_{\mathbf{kn}}\}_{\mathbf{k}\in N^2}$ is stationary by $\mathbf{S}_2$. Note that under the given hypothesis, each summand in (2.1) is dominated by $x_{\mathbf{k},\mathbf{k}+1}$.

Take $\mathbf{t}$ such that $\mathbf{t} \geqq \mathbf{k}+\mathbf{1} \geqq \mathbf{s}+\mathbf{1}$, and $\mathbf{n} > \mathbf{t}$. Then

$$\begin{aligned} |\mathbf{n}| f_{\mathbf{kn}} &= \textstyle\sum_{\mathbf{k}+1\le\mathbf{r}\le\mathbf{k}+\mathbf{n}} (x_{\mathbf{kr}} - x_{(k_1+1,k_2)\mathbf{r}} - x_{(k_1,k_2+1)\mathbf{r}} + x_{\mathbf{k}+1,\mathbf{r}}) \\ &\leqq \textstyle\sum_{\mathbf{n}\ge\mathbf{r}\ge\mathbf{t}} (x_{\mathbf{kr}} - x_{(k_1+1,k_2)\mathbf{r}} - x_{(k_1,k_2+1)\mathbf{r}} + x_{\mathbf{k}+1,\mathbf{r})} \\ &\quad + (n_1 n_2 - (n - t_1)(n_2 - t_2)) x_{\mathbf{k},\mathbf{k}+1} . \end{aligned}$$

Hence

$$\begin{aligned} &\mathbf{n} \textstyle\sum_{k_1=s_1}^{t_1-1} \sum_{k_2=s_2}^{t_2-1} f_{\mathbf{kn}} \\ &\quad\leqq \textstyle\sum_{\mathbf{n}\ge\mathbf{r}\ge\mathbf{t}} (x_{\mathbf{sr}} - x_{(s_1,t_2)\mathbf{r}} - x_{(t_1,s_2)\mathbf{r}} + x_{\mathbf{tr}}) \\ &\qquad + (n_1 n_2 - (n_1 - t_1)(n_2 - t_2)) \textstyle\sum_{k_1=s_2}^{t_1-1} \sum_{k_2=s_2}^{t_2-1} x_{\mathbf{k},\mathbf{k}+1} \\ &\quad\leqq n_1 n_2 x_{\mathbf{st}} - (n_1 n_2 - (n_1 - t_1)(n_2 - t_2))\{x_{\mathbf{st}} - \textstyle\sum_{k_1=s_1}^{t_1-1} \sum_{k_2=s_2}^{t_2-1} x_{\mathbf{k},\mathbf{k}+1}\} \end{aligned} \tag{2.2}$$

so that

$$\textstyle\sum_{k_1=s_1}^{t_1-1} \sum_{k_2=s_2}^{t_2-1} f_{\mathbf{kn}} \leqq x_{\mathbf{st}} + \dfrac{(n_1 n_2 - (n_1 - t_1)(n_2 - t_2))}{n_1 n_2} v_{\mathbf{st}} \qquad (\mathbf{n} > \mathbf{t}) \tag{2.3}$$

where $v_{\mathbf{st}} \equiv x_{\mathbf{st}} - \sum_{k_1=1}^{t_1-1} \sum_{k_2=1}^{t_2-1} x_{\mathbf{k},\mathbf{k}+1}$. In particular we have $f_{\mathbf{0n}} \leqq x_{\mathbf{01}}$ and by (2.1) and stationarity,

$$\begin{aligned} E(f_{\mathbf{0n}}) &= \frac{1}{|\mathbf{n}|} \textstyle\sum_{\mathbf{1}\le\mathbf{r}\le\mathbf{n}} [g(\mathbf{r}) - g((r_1 - 1, r_2)) - g((r_2, r_1 - 1)) \\ &\qquad + g((r_1 - 1, r_2 - 1))] \\ &= \frac{g(\mathbf{n})}{|\mathbf{n}|} \geqq \gamma . \end{aligned} \tag{2.4}$$

Therefore $E|f_{0n}| \leqq E|x_{01}| + E(x_{01} - f_{0n}) \leqq E|x_{01}| + (g(1) - \gamma)$ and the array $\{f_{0n}\}_{n \in N^2}$ is uniformly bounded in L^1. We can then apply the theorem of Komlós (1967) to pull out a sequence $n_1 < n_2 < \cdots$ of positive integers and an integrable function f_0 such that

$$(2.5) \qquad A_0{}^j = \frac{1}{j} \sum_{i=1}^{j} f_{0(n_i, n_i)} \to f_0 \quad \text{a.s.} \quad j \to \infty .$$

For convenience let $\mathbf{n}_i$ denote the point (n_i, n_i). Then by stationarity,

$$A_{\mathbf{k}}{}^j = \frac{1}{j} \sum_{i=1}^{j} f_{\mathbf{k}\mathbf{n}_i} \to f_{\mathbf{k}} \quad \text{a.s. as} \quad j \to \infty ,$$

and $\{f_{\mathbf{k}}\}_{\mathbf{k} \in N^2}$ is a stationary array.

Given $\mathbf{t} \in N^2$, let $i_{\mathbf{t}} = \inf\{i : \mathbf{t} < \mathbf{n}_i\}$ and define

$$A_0{}^j(\mathbf{t}) = \frac{1}{j} \sum_{i=i_{\mathbf{t}}}^{i_{\mathbf{t}}+j-1} f_{0\mathbf{n}_i}$$

with a similar definition of $A_{\mathbf{k}}{}^j(t)$. Clearly $A_{\mathbf{k}}{}^j(\mathbf{t}) \to f_{\mathbf{k}}$ a.s. as $j \to \infty$, for any $\mathbf{t}$ and $\mathbf{k} \in N^2$.

Recalling (2.3), we have

$$(2.6) \qquad \sum_{k_1=s_1}^{t_1-1} \sum_{k_2=s_2}^{t_2-1} A_{\mathbf{k}}{}^j(\mathbf{t}) \leqq x_{\mathbf{st}} + v_{\mathbf{st}} \left\{ \frac{1}{j} \sum_{i_{\mathbf{t}}}^{i_{\mathbf{t}}+j-1} \frac{n_i^2 - (n_i - t_1)(n_i - t_2)}{n_i^2} \right\} .$$

Passing to the limit as $j \to \infty$, we have

$$(2.7) \qquad \sum_{\mathbf{s} \leqq \mathbf{k} \ll \mathbf{t}} f_{\mathbf{k}} \leqq x_{\mathbf{st}} .$$

Let $y_{\mathbf{st}} = \sum_{\mathbf{s} \leqq \mathbf{k} \ll \mathbf{t}} f_{\mathbf{k}}$; $y_{\mathbf{st}}$ is then an additive process, and it follows as in Burkholder's arguments in [1] that $E(f_0) = \gamma$.

Let $w_{\mathbf{st}} = x_{\mathbf{st}} - y_{\mathbf{st}}$. $w_{\mathbf{st}}$ is clearly a nonnegative subadditive process, and

$$E\left(\frac{w_{0\mathbf{t}}}{|\mathbf{t}|}\right) = E(z_{\mathbf{t}}) - E(y_{0\mathbf{t}}/|\mathbf{t}|) = g_{\mathbf{t}}/|\mathbf{t}| - \gamma \to 0$$

as $\mathbf{t} \to \infty$ by Proposition 1.1; this completes the proof of Proposition 2.1.

As a start we can use the the decomposition of Proposition 2.1 and Corollary 1.1 to show a.s. sectorial convergence of any strongly subadditive process.

THEOREM 2.1. *Let $\{x_{\mathbf{st}}\}$ be strongly subadditive with time constant γ, and let $S_r{}^\alpha$ be any sector of r-space. Then* $\lim_{\mathbf{t} \to \infty; \mathbf{t} \in S_r{}^\alpha} z_{\mathbf{t}} = \xi$ *exists and is finite* a.s., *and* $E(\xi) = \gamma$.

PROOF. By Proposition 2.1 we can write $x_{0\mathbf{t}} = y_{0\mathbf{t}} + w_{0\mathbf{t}}$. The process $\{y_{0\mathbf{t}}\}$ is the partial sum process of a stationary integrable process (Proposition 2.1) and therefore by Theorem C, $y_{0\mathbf{t}}/|\mathbf{t}|$ converges, for $\mathbf{t} \in S_r{}^\alpha$, to a finite limit ξ as $\mathbf{t} \to \infty$, with $E(\xi) = \gamma$. Now $w_{0\mathbf{t}}$ is a nonnegative subadditive process with time constant zero. Applying Corollary 1.1 to it we have

$$\limsup_{\mathbf{t} \to \infty; \mathbf{t} \in S_r{}^\alpha} w_{0\mathbf{t}}/|\mathbf{t}| = \eta \quad \text{a.s.}$$

where $E(\eta) = 0$. Since w_{0t} is nonnegative this implies that $w_{0t}/|\mathbf{t}|$ converges sectorially a.s. to zero, completing the proof.

Finally we consider the more difficult problem of unrestricted convergence a.s. Retaining the notation of Proposition 2.1, the remark in the "proof" of Theorem B in Section 1 shows that the condition $E(|f_0| \log^+ |f_0|) < \infty$ is necessary in order to conclude that $y_{0t}/|\mathbf{t}|$ converges a.s. as $\mathbf{t} \to \infty$.

THEOREM 2.2. *Let* $\{x_{st}\}$ *be strongly subadditive with time constant* γ *and assume that*

$$(2.8) \qquad \textit{for each} \quad \mathbf{n} \in N^2, \qquad E(|x_{0\mathbf{n}}| \log^+ |x_{0\mathbf{n}}|) < \infty ;$$

$$(2.9) \qquad E(|f_0| \log^+ |f_0|) < \infty .$$

Then $z_t \to \xi$ a.s. *as* $\mathbf{t} \to \infty$, *where* $E(\xi) = \gamma$. *The corresponding result holds in* N^r *if* $\log^+$ *is replaced by* $(\log^+)^{r-1}$.

PROOF. Again write $x_{0t} = y_{0t} + w_{0t}$ as in Proposition 2.1. By Theorem B, $y_{0t}/|\mathbf{t}|$ converges a.s. to a limit ξ as $\mathbf{t} \to \infty$, where $E(\xi) = \gamma$. Also, since both y_{0t} and x_{0t} are $X \log^+ X$-integrable by hypothesis, so is w_{0t}; applying Theorem 1.1 we have

$$\limsup_{\mathbf{t}\to\infty} w_{0t}/|\mathbf{t}| = \eta < \infty \quad \text{a.s.},$$

with $E(\eta) = 0$, the time constant of the w-process. It follows as in the preceding theorem that $\lim_{\mathbf{t}\to\infty} w_{0t}/|\mathbf{t}| = 0$ a.s.

Theorem 2.2 is not of much use as it stands, since it gives no clue concerning how to prove the $X \log^+ X$-integrability of f_0. Consider again the decomposition of Proposition 2.1. Using (2.5), the convexity of the function $x \to |x| \log^+ |x|$, and Fatou's lemma, it is easily seen that the condition

$$(2.10) \qquad \sup_i E(|f_{0\mathbf{n}_i}| \log^+ |f_{0\mathbf{n}_i}|) < \infty$$

is sufficient to give $E(|f_0| \log^+ |f_0|) < \infty$. To shed some light on this condition consider again the one-dimensional case under the hypothesis that $E(x_{0t}) < \infty$ for all t. It is easy to show that the condition $g_t \geqq -At$ (S_3, Section 0) is equivalent to

$$(2.11) \qquad \sup_n E\left|\frac{1}{n} \sum_{j=1}^n (x_{01} - (x_{0j} - x_{1j}))\right| < \infty .$$

Relation (2.11) makes explicit what is already intuitively clear. By subadditivity, $x_{0j} - x_{1j} \leqq x_{01}$ for any j, with equality if $\{x_{st}\}$ is additive, and so $(1/n) \sum_{j=1}^n (x_{01} - (x_{0j} - x_{1j}))$ is a (local) measure of the departure from additivity of $\{x_{st}\}$. If the departure is not too severe, we can get an ergodic theorem.

Returning to the two-dimensional case, define, for $\mathbf{r} > \mathbf{1}$, $\Delta_r \equiv x_{0\mathbf{r}} - x_{(0,1)\mathbf{r}} - x_{(1,0)\mathbf{r}} + x_{1\mathbf{r}}$. From Theorem 2.2 and (2.10) we get a corollary which makes explicit the analogy with (2.11):

COROLLARY 2.1. *Let* $\{x_{st}\}$ *be strongly subadditive and suppose that* (2.8) *holds,* *as well as* (2.12) *below*:

$$\left\{\frac{1}{|\mathbf{n}_i|}\sum_{\mathbf{r}\leq\mathbf{n}_i}(x_{01}-\Delta_{\mathbf{r}})\right\}_{i\in N} \quad \textit{is } L\log^+ L\textit{-bounded.} \tag{2.12}$$

Then $z_{\mathbf{t}}\to\xi$ a.s. *as* $\mathbf{t}\to\infty$, *where* $E(\xi)=\gamma$.

The corresponding result holds in N^r *if* $\log^+$ *is replaced by* $(\log^+)^{r-1}$.

Again this result seems unsatisfactory since the condition (2.12) is apt to be difficult to verify in practice. We make the following conjecture:

CONJECTURE. Conditions (2.8) and (2.12) can be replaced by the single condition

$$\sup_{\mathbf{n}} E(|z_{\mathbf{n}}|\log^+|z_{\mathbf{n}}|)<\infty\,. \tag{2.13}$$

In a number of special cases, described below, the conjecture is verifiable. The difficulty in establishing the general conjecture is that the linearity of the first moment, which makes things work in the one-dimensional case, does not extend to linearity of the $X\log^+ X$ moment.

CASE 1. Suppose that $\{x_{st}\}$ has positive increments in the sense that $\Delta_{\mathbf{r}}\geqq 0$ for all $\mathbf{r}\in N^2$. Then $0\leqq x_{01}-\Delta_{\mathbf{r}}\leqq x_{01}$ and if $E(|x_{01}|\log^+|x_{01}|)<\infty$, (2.12) is clearly satisfied. (Example 1 of Section 0 is of this type.)

CASE 2. Suppose that $\{x_{st}\}$ has "invariant increments" in the sense that $\Delta_{\mathbf{r}}$ is independent of $\mathbf{r}$. Under (2.13), $\Delta_{\mathbf{r}}$ is $X\log^+ X$-integrable so that (2.12) holds. (Example 2 of Section 0 is of this type.)

CASE 3. Let $\mathscr{A}$ be the σ-field generated by the process $\{x_{st}\}$ and let θ be the shift operator which sends x_{st} to $x_{s+1,t+1}$. Let f be a random variable measurable in $\mathscr{A}$. If the distribution of $f(\omega)-f(\theta\omega)$ is symmetric and (2.13) holds, it can be shown without difficulty that (2.12) is valid. In particular the conclusion of Corollary 2.1 will hold if the process $\{x_{st}\}$ is subadditive over the entire integer grid (with coordinates positive or negative) and the distributions of $\{x_{st}\}$ are *isotropic*, i.e., invariant not only under linear shifts but also under rotations.

CASE 4. We saw in Proposition 1.1 that $g(\mathbf{n})/|\mathbf{n}|\to\gamma$ as $\mathbf{n}\to\infty$. If (2.13) holds and this convergence is rapid enough, (2.12) will hold. The specific rate of convergence needed is that

$$\sup_i\left\{\frac{1}{|\mathbf{n}_i|}\sum_{\mathbf{j}\leq\mathbf{n}_i}(g(\mathbf{j})-\gamma|j|)\right\}<\infty\,.$$

This can be shown by a series of manipulations of a fairly elementary nature, which we will omit. Thus in particular if $g(\mathbf{j})/|j|-\gamma$ is $O(|\mathbf{j}|^{-1})$, the conclusion of Corollary 2.1 is valid.

In closing we remark that the identification of the limiting process and even of the time constant γ continues to be a problem even in the one-dimensional case; we have nothing new to add. From the proof of Theorem 1.1 we can of course deduce that the limiting process ξ may be characterized as $\lim_{t\to\infty} E(x_{0\mathbf{t}}/|\mathbf{t}| \,|\, \mathscr{I})$, where $\mathscr{I}$ is the invariant σ-field. Hence if $\mathscr{I}$ is trivial we have $\xi = \gamma$ a.s.; as shown in [5], this will be the case if $\{x_{\mathbf{st}}\}$ is an independent process (i.e., if the retangles $\{R_{\mathbf{s}_i\mathbf{t}_i}\}_{i=1,2,\cdots,n}$ are disjoint, then the random variables $\{x_{\mathbf{s}_i\mathbf{t}_i}\}_{i=1,2,\cdots,n}$ are independent).

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