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ON THE UNIMODALITY OF STABLE DENSITIES

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The current proof of the unimodality of all stable densities uses general harmonic function arguments to interpolate unimodality between the symmetric case and the totally asymmetric case. It is here shown that the proof is invalid.

1. Introduction. The problem of proving the unimodality of all stable densities has attracted the attention of many researchers. A false proof was given in [6]. In [1] Chernin and Ibragimov give another proof. In fact the proof of Chernin and Ibragimov contains an essential gap.

2. Background material. It is well known that the set of all stable densities can be parametrized by the family of densities $p(x; \alpha, \beta)$ where the parameters α and β are restricted to lie in $(0, 2]$ and $[-1, 1]$ respectively and where the characteristic function $f(t; \alpha, \beta)$ of $p(x; \alpha, \beta)$ is given by

$$(2.1) \qquad \ln f(t; \alpha, \beta) = -|t|^\alpha \exp(-1/2\pi i\beta k(\alpha)\,\mathrm{sgn}\,(t)) \qquad \text{for} \quad \alpha \neq 1$$
$$= -|t|\left(1 + \frac{2}{\pi} i\beta \ln|t|\,\mathrm{sgn}\,(t)\right) \qquad \text{if} \quad \alpha = 1$$

where $k(\alpha) = 1 - |1 - \alpha|$ (see e.g., Feller [2], page 548).

3. Comment on the proof of Chernin and Ibragimov. In this section we shall explain why the argument due to Chernin and Ibragimov for the proof of unimodality of stable random variables is invalid. We follow the proof as presented in [4, page 75].

The argument of Chernin and Ibragimov goes roughly as follows. Consider the density $p(x; \alpha, \beta)$. First prove unimodality when $\beta = 1$ and $\beta = 0$; then shown unimodality for $0 < \beta < 1$ by using some general properties of harmonic functions.

The details are as follows. Fix α. Define $x_0(\beta)$ to be smallest zero of $p_x'(x; \alpha, \beta)$ (which exists because $p_x'(x; \alpha, \beta) > 0$ for $x \leqq 0$) and define the set D by

$$D = \{(x, \beta); x > x_0(\beta), 0 < \beta < 1, p_x'(x; \alpha, \beta) > 0\}.$$

Unimodality of $p(x, \alpha, \beta)$ is established if it can be shown that D is empty. To do this it is first shown that the function

$$A(t, \mu) = x^2 p_x'(x; \alpha, \beta)$$

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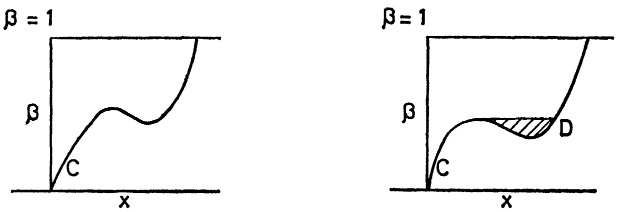
where $x = e^{-t}$ and $\beta = 2\mu/\pi$, is harmonic in $-\infty < t < \infty$, $0 < \mu < \frac{1}{2}\pi$. It is then asserted that $A(t, \mu) = 0$ on the boundary of $\tilde{D}$ (where $\tilde{D}$ is the image of D under the map $(x, \beta) \to (t, \mu)$.) This last assertion is unjustified. Consider the following diagrams. The diagrams are not supposed to indicate the general situation but only a special case not outlawed by the reasoning in [4]. In particular we are assuming that $p_x'(x; \alpha, \beta)$ has exactly three zero derivatives (this contradicts unimodality of course, but that is the point at issue.)



(a) (b)

FIG. 2.1

We suppose that the set $\{(x, \beta); p_x'(x; \alpha, \beta) = 0\}$ can be represented by the curve C. In diagram (b) the set D corresponding to the curve C is shaded. It is clear that there is no a priori reason for asserting that $A(t, \mu)$ vanishes on the boundary of $\tilde{D}$ as this boundary will contain the image under the map $(x, \beta) \to (t, \mu)$ of the horizontal segment bounding D from above. Also, being zero on the boundary is equivalent to the result of unimodality being proved.

Briefly, the general line of argument which depends on properties of harmonic functions cannot be successful for the following reason.

It is necessary to show that $\tilde{C}$ intersects every horizontal line between $\mu = 0$ and $\mu = \pi/2$ only once, where $\tilde{C}$ is the image of C under the map $(x, \beta) \to (t, \mu)$. If this argument is true in general then it must be applicable to any bounded harmonic function $A(t, \mu)$ on the strip $\{(t, \mu); a < \mu < b\}$ with $A(\cdot, a)$ and $A(\cdot, b)$ each having at most one change of sign in $(-\infty, +\infty)$, and $\tilde{C} = \{(t, \mu); A(t, \mu) = 0\}$. It would then follow that any harmonic function $A(t, \mu)$ with at most one change of sign on the edge lines of an infinite strip $\{(t, \mu); a < \mu < b\}$ has at most one change in sign in any interior line $\mu = c$.

It would then follow (the width of the strip being inessential) that this "theorem" would continue to hold when b is set equal to ∞ and a is set equal to 0. In that case

$$A(t, \mu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{uA(s, 0)}{(t - s)^2 + \mu^2} \, ds$$

as is well known.

We now conclude that $A(t, \mu)$ has at most one change of sign for fixed μ, given $A(t, 0)$ has at most one change of sign. By [5, page 21] this implies that the kernel $K(x, y) = 1/(1 + (x - y)^2)$ is totally positive of index 2, which contradicts the theorem of Ibragimov [3] since $-\log(1 + x^2)$ is not concave.

Another approach is to construct a specific counterexample to the above "theorem." Recently, Larry Shepp sent me such a counterexample due to Ben Logan.

Let $f(t, u) = \arctan(\tanh(t) \tan(u)) + \varepsilon \sin(t) \cosh(u)$. Then $f(t, u)$ is harmonic on the strip $|u| \leqq \pi/4$ since it is the imaginary part of $\log \cosh(t + iu) + \varepsilon \sinh(u + it)$. We now note that for $u = \pm \pi/4$ (and ε small) the function $f(t, u)$ has only one change of sign, while for $u = 0$ the function $f(t, u)$ has infinitely many changes of sign.

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